PIE Tech

POLLACHI INSTITUTE OF ENGINEERING AND TECHNOLOGY

(Approved by AICTE and Affiliated to Anna University)

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DEPARTMENT OF SCIENCE AND HUMANITIES

REGULATION 2021

I YEAR /I SEM

MA3151 MATRICES & CALCULUS

MATRICES

1) Find the eigenvalues & eigenvectors of the matrix $A = \begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$. [A/M 2018] [N/D 2016] [N/D 2011]

Sol. Given $A = \begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$

Characteristic equation: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 =$ Sum of the main diagonal elements $= 11 - 2 - 6 = 3$

$S_2 =$ Sum of the minors of main diagonal elements

$= \begin{vmatrix} -2 & -5 \\ -4 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -7 \\ 10 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -4 \\ 7 & -2 \end{vmatrix} = (12-20) + (-66+70) + (-22+28)$

$= -8 + 4 + 6 = 2$

$S_3 = |A| = 11(12-20) + 4(-42+50) - 7(-28+20) = 11(-8) + 4(8) - 7(-8)$

$= -88 + 32 + 56 = 0$

$\therefore$ The characteristic eqn. is $\lambda^3 - 3\lambda^2 + 2\lambda = 0$

×	+
2	-3
-1	-2
$\lambda - 1$	$\lambda - 2$

$\lambda(\lambda^2 - 3\lambda + 2) = 0 \Rightarrow \lambda = 0,\ \lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\therefore \lambda = 0, 1, 2$

Hence the eigenvalues are 0, 1, 2.

Eigenvectors: $(A - \lambda I)X = 0$

$$\begin{pmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\lambda = 0$: $\begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -4 & -7 & 11 & -4 \\ -2 & -5 & 7 & -2 \end{array}$$

$11x_1 - 4x_2 - 7x_3 = 0$

$7x_1 - 2x_2 - 5x_3 = 0$

$10x_1 - 4x_2 - 6x_3 = 0$

$$\frac{x_1}{20-14} = \frac{x_2}{-49+55} = \frac{x_3}{-22+28} \Rightarrow \frac{x_1}{6} = \frac{x_2}{6} = \frac{x_3}{6} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = 1$: $\begin{pmatrix} 10 & -4 & -7 \\ 7 & -3 & -5 \\ 10 & -4 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$10x_1-4x_2-7x_3=0$$
$$7x_1-3x_2-5x_3=0$$
$$10x_1-4x_2-7x_3=0$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -4 & -7 & 10 & -4 \\ -3 & -5 & 7 & -3 \end{array}$$

$$\frac{x_1}{20-21}=\frac{x_2}{-49+50}=\frac{x_3}{-30+28} \Rightarrow \frac{x_1}{-1}=\frac{x_2}{1}=\frac{x_3}{-2}$$

$$\therefore X_2=\begin{pmatrix}-1\\1\\-2\end{pmatrix}$$

$\underline{\lambda=2}$
$$\begin{pmatrix}9 & -4 & -7\\ 7 & -4 & -5\\ 10 & -4 & -8\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -4 & -7 & 9 & -4 \\ -4 & -5 & 7 & -4 \end{array}$$

$$9x_1-4x_2-7x_3=0$$
$$7x_1-4x_2-5x_3=0$$
$$10x_1-4x_2-8x_3=0$$

$$\frac{x_1}{20-28}=\frac{x_2}{-49+45}=\frac{x_3}{-36+28} \Rightarrow \frac{x_1}{-8}=\frac{x_2}{-4}=\frac{x_3}{-8} \Rightarrow \frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{2}$$

$$\therefore X_3=\begin{pmatrix}2\\1\\2\end{pmatrix}$$

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(2) Find the eigenvalues & eigenvectors of the matrix $\begin{pmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{pmatrix}$ [M/J-2014] [N/D-2014] [M/J-2009] [Jan-2010]

Sol: Let $A=\begin{pmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{pmatrix}$

Characteristic equation: $\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$

S_1 = Sum of the main diagonal elements $=-2+1+0=-1$

S_2 = Sum of the minors of main diagonal elements

$$=\begin{vmatrix}1 & -6\\ -2 & 0\end{vmatrix}+\begin{vmatrix}-2 & -3\\ -1 & 0\end{vmatrix}+\begin{vmatrix}-2 & 2\\ 2 & 1\end{vmatrix}=(0-12)+(0-3)+(-2-4)$$

$$=-12-3-6=-21$$

$$S_3=|A|=-2(0-12)-2(0-6)-3(-4+1)=24+12+9=45$$

Hence the characteristic eqn. is $\lambda^3+\lambda^2-21\lambda-45=0$

$$\begin{array}{c|cccc} \lambda=5 & 1 & 1 & -21 & -45 \\ & & 5 & 30 & 45 \\ \hline & 1 & 6 & 9 & 0 \end{array}$$

$$\begin{array}{c|c} + & \times \\ 6 & 9 \\ \hline +3 & 3 \\ \lambda+3 & \lambda+3 \end{array}$$

$$\lambda^2+6\lambda+9=0$$
$$(\lambda+3)(\lambda+3)=0$$
$$\therefore \lambda=5,-3,-3$$

Hence the eigenvalues are $-3,-3,5$.

Eigenvectors: $(A-\lambda I)X = 0$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\underline{\underline{\lambda = -3}}$ $$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$x_1 + 2x_2 - 3x_3 = 0$

$2x_1 + 4x_2 - 6x_3 = 0$

$-x_1 - 2x_2 + 3x_3 = 0$

Take $x_1 + 2x_2 - 3x_3 = 0$

Put $x_1 = 0 \Rightarrow 2x_2 - 3x_3 = 0 \Rightarrow 2x_2 = 3x_3 \Rightarrow \dfrac{x_2}{3} = \dfrac{x_3}{2}$

$\therefore X_1 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$

Put $x_2 = 0 \rightarrow x_1 - 3x_3 = 0 \Rightarrow x_1 = 3x_3 \Rightarrow \dfrac{x_1}{3} = \dfrac{x_3}{1}$

$\therefore X_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\underline{\lambda = 5}}$ $$\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -4 & -6 & 2 & -4 \\ -2 & -5 & -1 & -2 \end{array}$$

$-7x_1 + 2x_2 - 3x_3 = 0$

$2x_1 - 4x_2 - 6x_3 = 0$

$-x_1 - 2x_2 - 5x_3 = 0$

$$\frac{x_1}{20-12} = \frac{x_2}{6+10} = \frac{x_3}{-4-4} \Rightarrow \frac{x_1}{8} = \frac{x_2}{16} = \frac{x_3}{-8} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

$\therefore X_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

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(3) Find the eigenvalues & eigenvectors of the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ [N/D-2015] [M/J-2013]

Sol: Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

Characteristic equation: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 2+2+2 = 6$

S_2 = Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = (4-0)+(4-1)+(4-0) = 4+3+4 = 11$$

$S_3 = |A| = 2(4-0) - 0(0-0) + 1(0-2) = 8-2 = 6$

Hence the characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$$\begin{array}{r|rrrr} \lambda = 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \lfloor 0 \end{array}$$

$$\begin{array}{c|c} \times & + \\ 6 & -5 \\ \hline -3 & -2 \\ \lambda-3 & \lambda-2 \end{array}$$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda-3)(\lambda-2) = 0$

$\therefore \lambda = 1, 2, 3$

Hence the eigenvalues are 1, 2 & 3.

Eigenvectors: $(A - \lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\underline{\underline{\lambda = 1}}$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}$$

$x_1 + 0x_2 + x_3 = 0$

$0x_1 + x_2 + 0x_3 = 0$

$x_1 + 0x_2 + x_3 = 0$

$$\frac{x_1}{0-1} = \frac{x_2}{0-0} = \frac{x_3}{1-0} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$\therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\underline{\lambda = 2}}$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

$0x_1 + 0x_2 + x_3 = 0$

$0x_1 + 0x_2 + 0x_3 = 0$

$x_1 + 0x_2 + 0x_3 = 0$

$$\frac{x_1}{0-0} = \frac{x_2}{1-0} = \frac{x_3}{0-0} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{0}$$

$\therefore X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\underline{\underline{\lambda = 3}}$ $\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$-x_1 + 0x_2 + x_3 = 0$

$0x_1 - x_2 + 0x_3 = 0$

$x_1 + 0x_2 - x_3 = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{array}$$

$$\frac{x_1}{0+1} = \frac{x_2}{0-0} = \frac{x_3}{1-0} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

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(4) Find the eigenvalues & eigenvectors of the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

[A/M - 2015]
[N/D - 2015]

Sol: Let $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

Characteristic equation: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 6+3+3 = 12$

S_2 = Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = (9-1) + (18-4) + (18-4)$$

$$= 8 + 14 + 14 = 36$$

$$S_3 = |A| = 6(9-1) + 2(-6+2) + 2(2-6) = 6(8) + 2(-4) + 2(-4)$$

$$= 48 - 8 - 8 = 32$$

Hence the characteristic eqn. is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$$\begin{array}{r|rrrr} \lambda = 2 & 1 & -12 & 36 & -32 \\ & & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$$\begin{array}{c|c} \times & + \\ 16 & -10 \\ \hline -8 & -2 \\ \lambda-8 & \lambda-2 \end{array}$$

$\lambda^2 - 10\lambda + 16 = 0$

$(\lambda-8)(\lambda-2) = 0 \Rightarrow \lambda = 8, 2$

$\therefore \lambda = 2, 2, 8$

Hence the eigenvalues are 2, 2, 8.

Eigenvectors: $(A - \lambda I)X = 0$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\underline{\lambda = 2}$ $\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{array}{l} 4x_1 - 2x_2 + 2x_3 = 0 \\ -2x_1 + x_2 - x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \end{array}\right\} \Rightarrow 2x_1 - x_2 + x_3 = 0$$

Put $x_1 = 0 \Rightarrow -x_2 + x_3 = 0 \Rightarrow x_2 = x_3 \Rightarrow \frac{x_2}{1} = \frac{x_3}{1}$

$$\therefore X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 8}$ $\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$-2x_1 - 2x_2 + 2x_3 = 0$
$-2x_1 - 5x_2 - x_3 = 0$
$2x_1 - x_2 - 5x_3 = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -2 & 2 & -2 & -2 \\ -5 & -1 & -2 & -5 \end{array}$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4} \Rightarrow \frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\therefore X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

To find the third eigenvector orthogonal to X_1 & X_2 since the matrix A is symmetric.

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ as X_3 is orthogonal to X_1 & X_2.

$X_1^T X_3 = 0 \Rightarrow (0 \ 1 \ 1)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow 0a + b + c = 0$

$X_2^T X_3 = 0 \Rightarrow (2 \ -1 \ 1)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow 2a - b + c = 0$

$$\begin{array}{cccc} a & b & c & \\ 1 & 1 & 0 & 1 \\ -1 & 1 & 2 & -1 \end{array}$$

$$\frac{a}{1+1} = \frac{b}{2-0} = \frac{c}{0-2} \Rightarrow \frac{a}{2} = \frac{b}{2} = \frac{c}{-2} \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{-1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

(5) Find the eigenvalues & eigenvectors of the matrix $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$. [Jan 2014] [N/D-2010] [M/J-2010] [Jan-2012]

Sol: Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

Characteristic equation: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 2+3+2 = 7$

S_2 = Sum of the minors of main diagonal elements

$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = (6-2) + (4-1) + (6-2) = 4+3+4 = 11$

$S_3 = |A| = 2(6-2) - 2(2-1) + 1(2-3) = 8-2-1 = 5$

Hence the characteristic eqn. is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$.

$$\begin{array}{r|rrrr} \lambda = 1 & 1 & -7 & 11 & -5 \\ & & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & 0 \end{array}$$

$$\begin{array}{c|c} \times & + \\ 5 & -6 \\ \hline -3 & -2 \\ \lambda - 3 & \lambda - 2 \end{array}$$

$\lambda^2 - 6\lambda + 5 = 0$

$(\lambda - 3)(\lambda - 2) = 0 \Rightarrow \lambda = 2, 3$

$\therefore \lambda = 1, 2, 3$

Hence the eigenvalues are $1, 2, 3$.

Eigenvectors: $(A - \lambda I)X = 0$

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$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\underline{\underline{\lambda=1}}$ $$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$

Put $x_1 = 0 \Rightarrow 2x_2 + x_3 = 0 \Rightarrow 2x_2 = -x_3 \Rightarrow \frac{x_2}{-1} = \frac{x_3}{2}$

$$\therefore X_1 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$\underline{\underline{\lambda=2}}$ $$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$0x_1 + 2x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 0$$
$$x_1 + 2x_2 + 0x_3 = 0$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{array}$$

$$\frac{x_1}{2-1} = \frac{x_2}{1-0} = \frac{x_3}{0-2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$\underline{\underline{\lambda=3}}$ $$\begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-x_1 + 2x_1 + x_3 = 0$$
$$x_1 + 0x_2 + x_3 = 0$$
$$x_1 + 2x_2 - x_3 = 0$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \end{array}$$

$$\frac{x_1}{2-0} = \frac{x_2}{1+1} = \frac{x_3}{0-2} \Rightarrow \frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{-2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Cayley – Hamilton theorem:

Every square matrix satisfies its own characteristic equation.

Uses of Cayley - Hamilton theorem:

To calculate (i) the positive integral powers of A &

(ii) the inverse of a non-singular square matrix A.

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(6) Verify Cayley-Hamilton thm. for the matrix $A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$. Hence using it find A^{-1} & A^4. [N/D-2014] [A/M-2017] [M/J-2013] [M/J-2010]

Sol: Given $A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

Characteristic equation: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 2+2+2 = 6$

S_2 = Sum of the minors of main diagonal elements

$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = (4-1)+(4-2)+(4-1) = 3+2+3 = 8$

$S_3 = |A| = 2(4-1)+1(-2+1)+2(1-2) = 2(3)+1(-1)+2(-1) = 6-1-2 = 3$

Hence the characteristic eqn. is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$.

By Cayley-Hamilton thm., every square matrix satisfies its own characteristic equation. $\therefore A^3 - 6A^2 + 8A - 3I = 0$ —(1)

Verification:

$$A^2 = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix}$$

$$A^3 - 6A^2 + 8A - 3I = \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - 6\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + 8\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - \begin{pmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{pmatrix} + \begin{pmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence Cayley-Hamilton thm. verified.

(1) $\Rightarrow A^3 - 6A^2 + 8A - 3I = 0 \Rightarrow A^2 - 6A + 8I - 3A^{-1} = 0 \Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$$\therefore A^{-1} = \frac{1}{3}(A^2 - 6A + 8I)$$

$$A^2 - 6A + 8I = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} - \begin{pmatrix} 12 & -6 & 12 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1/3 & 2/3 & 0 \\ -1/3 & 1/3 & 1 \end{pmatrix}$$

$① \Rightarrow A^3 - 6A^2 + 8A - 3I = 0 \rightarrow A^4 - 6A^3 + 8A^2 - 3A = 0 \Rightarrow A^4 = 6A^3 - 8A^2 + 3A$

$$A^4 = 6\begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - 8\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + 3\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 174 & -168 & 228 \\ -132 & 138 & -168 \\ 132 & -132 & 174 \end{pmatrix} - \begin{pmatrix} 56 & -48 & 72 \\ -40 & 48 & -48 \\ 40 & -40 & 56 \end{pmatrix} + \begin{pmatrix} 6 & -3 & 6 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{pmatrix} = \begin{pmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{pmatrix}$$

⑦ Verify Cayley-Hamilton thm. for the matrix $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix}$. Hence using it find A^{-1}. [N/D-2015]

<u>Sol:</u> Given $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix}$

Characteristic <u>eqn.</u>: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 1 + 5 - 5 = 1$

S_2 = Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 5 & -4 \\ 7 & -5 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 3 & -5 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = (-25+28) + (-5+6) + (5-4)$$

$= 3 + 1 + 1 = 5$

$S_3 = |A| = 1(-25+28) - 2(-10+12) - 2(14-15) = 1(3) - 2(2) - 2(-1)$

$= 3 - 4 + 2 = 1$

Hence the characteristic eqn. is $\lambda^3 - \lambda^2 + 5\lambda - 1 = 0$

<u>Verification:</u> By C-H thm., every square matrix satisfies its own characteristic eqn.. $\therefore A^3 - A^2 + 5A - I = 0$ — ①

$$A^2 = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{pmatrix} \qquad A^3 = \begin{pmatrix} -5 & -12 & 10 \\ -10 & -23 & 16 \\ -13 & -29 & 17 \end{pmatrix}$$

$$\therefore A^3 - A^2 + 5A - I = \begin{pmatrix} -5 & -12 & 10 \\ -10 & -23 & 16 \\ -13 & -29 & 17 \end{pmatrix} - \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{pmatrix} + 5\begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -12 & 10 \\ -10 & -23 & 16 \\ -13 & -29 & 17 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 4 \\ -2 & -6 & 9 \end{pmatrix} + \begin{pmatrix} 5 & 10 & -10 \\ 10 & 25 & -20 \\ 15 & 35 & -25 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence C-H thm. verified.

$① \Rightarrow A^3 - A^2 + 5A - I = 0 \Rightarrow A^2 - A + 5I - A^{-1} = 0 \Rightarrow A^{-1} = A^2 - A + 5I$

$$\therefore A^{-1} = A^2 - A + 5I = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{pmatrix} - \begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{pmatrix} + \begin{pmatrix} -1 & -2 & 2 \\ -2 & -5 & 4 \\ -3 & -7 & 5 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

⑧ Verify C-H thm. for the matrix $A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$. Hence using it find A^{-1}.

[A/M-2015]
[N/D-2011]

Sol: Given $A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$

Characteristic eqn. : $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 1+2+1 = 4$

S_2 = Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (2-6) + (1-7) + (2-12) = -4-6-10 = -20$$

$S_3 = |A| = 1(2-6) - 3(4-3) + 7(8-2) = -4 - 3(1) + 7(6) = -4-3+42 = 35$

Hence the characteristic eqn. is $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

By C-H thm., every square matrix satisfies its own characteristic eqn..

$\therefore A^3 - 4A^2 - 20A - 35I = 0$ — ①

Verification:

$$A^2 = \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix}$$

$$A^3 - 4A^2 - 20A - 35I = \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix} - 4\begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} - 20\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} - 35\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix} - \begin{pmatrix} 80 & 92 & 92 \\ 60 & 88 & 148 \\ 40 & 36 & 56 \end{pmatrix} - \begin{pmatrix} 20 & 60 & 140 \\ 80 & 40 & 60 \\ 20 & 40 & 20 \end{pmatrix} - \begin{pmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$. Hence Cayley-Hamilton thm. verified.

$(1) \Rightarrow A^3 - 4A^2 - 20A - 35I = 0 \Rightarrow A^2 - 4A - 20I - 35A^{-1} = 0$

$\Rightarrow 35A^{-1} = A^2 - 4A - 20I \rightarrow A^{-1} = \frac{1}{35}(A^2 - 4A - 20I)$

$$\therefore A^2 - 4A - 20I = \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} - 4\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} - 20\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} - \begin{pmatrix} 4 & 12 & 28 \\ 16 & 8 & 12 \\ 4 & 8 & 4 \end{pmatrix} - \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix} = \begin{pmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{35}\begin{pmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{pmatrix}$$

⑨ Verify Cayley-Hamilton thm. for $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$. Hence using it find A^{-1} & A^4.
[Jan-2011]

Sol: Given $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$

Characteristic eqn.: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 1 + 1 + 3 = 5$

S_2 = Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = (3-0) + (3-2) + (1-0) = 3 + 1 + 1 = 5$$

$S_3 = |A| = 1(3-0) + 1(0-0) + 1(0-2) = 3 - 2 = 1$

Hence the characteristic eqn. is $\lambda^3 - 6\lambda^2 + 5\lambda - 1 = 0$.

By C-H thm., every square matrix satisfies its own characteristic eqn..

$\therefore A^3 - 5A^2 + 5A - I = 0.$ — (1)

Verification:

$$A^2 = \begin{pmatrix} 3 & -2 & 4 \\ 0 & 1 & 0 \\ 8 & -2 & 11 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 11 & -5 & 15 \\ 0 & 1 & 0 \\ 30 & -10 & 41 \end{pmatrix}$$

$$A^3 - 5A^2 + 5A - I = \begin{pmatrix} 11 & -5 & 15 \\ 0 & 1 & 0 \\ 30 & -10 & 41 \end{pmatrix} - 5\begin{pmatrix} 3 & -2 & 4 \\ 0 & 1 & 0 \\ 8 & -2 & 11 \end{pmatrix} + 5\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -5 & 15 \\ 0 & 1 & 0 \\ 30 & -10 & 41 \end{pmatrix} - \begin{pmatrix} 15 & -10 & 20 \\ 0 & 5 & 0 \\ 40 & -10 & 55 \end{pmatrix} + \begin{pmatrix} 5 & -5 & 5 \\ 0 & 5 & 0 \\ 10 & 0 & 15 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence Cayley-Hamilton thm. verified.

(1) $\Rightarrow A^3 - 5A^2 + 5A - I = 0 \Rightarrow A^2 - 5A + 5I - A^{-1} = 0 \Rightarrow A^{-1} = A^2 - 5A + 5I$.

$$A^{-1} = A^2 - 5A + 5I = \begin{pmatrix} 3 & -2 & 4 \\ 0 & 1 & 0 \\ 8 & -2 & 11 \end{pmatrix} - 5\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & 4 \\ 0 & 1 & 0 \\ 8 & -2 & 11 \end{pmatrix} - \begin{pmatrix} 5 & -5 & 5 \\ 0 & 5 & 0 \\ 10 & 0 & 15 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 3 & 3 & -1 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix}$$

From (1), $A^4 - 5A^3 + 5A^2 - A = 0 \Rightarrow A^4 = 5A^3 - 5A^2 + A$

$$A^4 = 5A^3 - 5A^2 + A = 5\begin{pmatrix} 11 & -5 & 15 \\ 0 & 1 & 0 \\ 30 & -10 & 41 \end{pmatrix} - 5\begin{pmatrix} 3 & -2 & 4 \\ 0 & 1 & 0 \\ 8 & -2 & 11 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 55 & -25 & 75 \\ 0 & 5 & 0 \\ 150 & -50 & 205 \end{pmatrix} - \begin{pmatrix} 15 & -10 & 20 \\ 0 & 5 & 0 \\ 40 & -10 & 55 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

$$\therefore A^4 = \begin{pmatrix} 41 & -16 & 56 \\ 0 & 1 & 0 \\ 112 & -40 & 153 \end{pmatrix}$$

(10) Using Cayley-Hamilton thm. find the inverse of the given matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}. \quad [A/M\ 2018]$$

Sol: Given $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

Characteristic eqn.: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 1 + 2 + 3 = 6$

S_2 = Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = (6-1) + (3-1) + (2-4) = 5 + 2 - 2 = 5$$

$S_3 = |A| = 1(6-1) - 2(6-1) + 1(2-2) = 5 - 10 = -5$.

Hence the characteristic eqn. is $\lambda^3 - 6\lambda^2 + 5\lambda + 5 = 0$

Using C-H thm., we get $A^3 - 6A^2 + 5A + 5I = 0$

$A^2 - 6A + 5I + 5A^{-1} = 0 \Rightarrow 5A^{-1} = -A^2 + 6A - 5I \Rightarrow A^{-1} = \frac{1}{5}(-A^2 + 6A - 5I)$

$$\therefore A^{-1} = \frac{1}{5}\left[-\begin{pmatrix} 6 & 7 & 6 \\ 7 & 9 & 7 \\ 6 & 7 & 11 \end{pmatrix} + 6\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$A^{-1} = \frac{1}{5}\left[\begin{pmatrix}-6 & -7 & -6\\ -7 & -9 & -7\\ -6 & -7 & -11\end{pmatrix} + \begin{pmatrix}6 & 12 & 6\\ 12 & 12 & 6\\ 6 & 6 & 18\end{pmatrix} + \begin{pmatrix}-5 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & -5\end{pmatrix}\right]$$

$$\therefore A^{-1} = \frac{1}{5}\begin{pmatrix}-5 & 5 & 0\\ 5 & -2 & -1\\ 0 & -1 & 2\end{pmatrix}$$

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(11) Use C-H thm. to find the value of the matrix given by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$, if the matrix $A = \begin{pmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{pmatrix}$.

[M/J-2009]

Sol: Given $A = \begin{pmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{pmatrix}$

Characteristic eqn.: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 2+1+2 = 5$

S_2 = Sum of the minors of main diagonal elements

$$= \begin{vmatrix}1 & 0\\ 1 & 2\end{vmatrix} + \begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix} + \begin{vmatrix}2 & 1\\ 0 & 1\end{vmatrix} = (2-0)+(4-1)+(2-0) = 2+3+2 = 7$$

$S_3 = |A| = 2(2-0) - 1(0-0) + 1(0-1) = 4-1 = 3$

Hence the characteristic eqn. is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

Using C-H thm. we get, $A^3 - 5A^2 + 7A - 3I = 0$ — (1)

$$
\begin{array}{r|l}
 & A^5 + A \\
A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\
 & A^8 - 5A^7 + 7A^6 - 3A^5 \\
 & (-)\ (+)\ \ (-)\ \ (+) \\
\hline
 & \qquad A^4 - 5A^3 + 8A^2 - 2A \\
 & \qquad A^4 - 5A^3 + 7A^2 - 3A \\
 & \qquad (-)\ (+)\ \ (-)\ \ (+) \\
\hline
 & \qquad\qquad A^2 + A + I
\end{array}
$$

$$\therefore A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + A^2 + A + I$$

$$= (0)(A^5 + A) + A^2 + A + I \quad (\because \text{by } (1))$$

$$= A^2 + A + I$$

$$= \begin{pmatrix}5 & 4 & 4\\ 0 & 1 & 0\\ 4 & 4 & 5\end{pmatrix} + \begin{pmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{pmatrix} + \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}8 & 5 & 5\\ 0 & 3 & 0\\ 5 & 5 & 8\end{pmatrix}$$

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(12) Find A^n using C-H thm. , taking $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Hence find A^3.

[Jan-2012]

Sol: Given $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

Characteristic eqn.: $\lambda^2 - S_1\lambda + S_2 = 0$

S_1 = Sum of the main diagonal elements $= 1+3 = 4$

$S_2 = |A| = 3 - 8 = -5$

Hence the characteristic eqn. is $\lambda^2 - 4\lambda - 5 = 0$

Using C-H thm. we get, $A^2 - 4A - 5I = 0$. — (1)

$A^n = (A^2 - 4A - 5I)\,Q(A) + aA + bI$ where $Q(A)$ is the quotient & $aA + bI$ is the remainder.

$\therefore A^n = (0)\,Q(A) + aA + bI$ ($\because$ by (1))

$A^n = aA + bI \Rightarrow \lambda^n = a\lambda + b$ — (2)

Eigenvalues: $\lambda^2 - 4\lambda - 5 = 0$

$(\lambda+1)(\lambda-5) = 0$

$\therefore \lambda = -1, 5$

+	×
-5	-4
$+1$	-5
$\lambda+1$	$\lambda-5$

Subs. $\lambda = -1$ & 5 in (2),

$(-1)^n = a(-1) + b \Rightarrow (-1)^n = -a + b$ — (3)

$5^n = a(5) + b \Rightarrow 5^n = 5a + b$ — (4)

(3) − (4) $\Rightarrow (-1)^n - 5^n = -6a \Rightarrow a = \frac{-1}{6}\left[(-1)^n - 5^n\right] = \frac{1}{6}\left[5^n - (-1)^n\right]$

Subs. a value in (3), $(-1)^n = -\frac{1}{6}\left[5^n - (-1)^n\right] + b$

$\Rightarrow b = (-1)^n + \frac{1}{6}\left[5^n - (-1)^n\right] = \frac{6(-1)^n + 5^n - (-1)^n}{6} = \frac{5(-1)^n + 5^n}{6}$

$\therefore b = \frac{1}{6}\left[5(-1)^n + 5^n\right]$

$\therefore A^n = \frac{1}{6}\left[5^n - (-1)^n\right]A + \frac{1}{6}\left[5(-1)^n + 5^n\right]I$

$A^3 = \frac{1}{6}\left[5^3 - (-1)^3\right]A + \frac{1}{6}\left[5(-1)^3 + 5^3\right]I$

$= \frac{1}{6}\left[125 + 1\right]A + \frac{1}{6}\left[-5 + 125\right]I = \frac{126}{6}A + \frac{120}{6}I$

$= 21A + 20I = 21\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} + 20\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 21 & 84 \\ 42 & 63 \end{pmatrix} + \begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix} = \begin{pmatrix} 41 & 84 \\ 42 & 83 \end{pmatrix}$

(13) Reduce the matrix $\begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$ to diagonal form. [A/M-2017]

Sol:

Let $A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$

Characteristic eqn.: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 10+2+5 = 17$

S_2 = Sum of the minors of main diagonal elements

$= \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ -5 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix} = (10-9)+(50-25)+(20-4) = 1+25+16 = 42$

$S_3 = |A| = 10(10-9)+2(-10+15)-5(-6+10) = 10+10-20 = 0$

Hence the characteristic eqn. is $\lambda^3 - 17\lambda^2 + 42\lambda = 0$

$\lambda(\lambda^2 - 17\lambda + 42) = 0$.

$\lambda = 0, \quad \lambda^2 - 17\lambda + 42 = 0$

$(\lambda - 14)(\lambda - 3) = 0$

$\begin{array}{c|c} \times & + \\ 42 & -17 \\ \hline -14 & -3 \\ \lambda-14 & \lambda-3 \end{array}$

$\therefore \lambda = 0, 3, 14$

Hence the eigenvalues are $0, 3$ & 14.

Eigenvectors: $(A - \lambda I)X = 0$

$\begin{pmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\underline{\lambda = 0}$ $\begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$10x_1 - 2x_2 - 5x_3 = 0$

$-2x_1 + 2x_2 + 3x_3 = 0$

$-5x_1 + 3x_2 + 5x_3 = 0$

$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -2 & -5 & 10 & -2 \\ 2 & 3 & -2 & 2 \end{array}$

$\dfrac{x_1}{-6+10} = \dfrac{x_2}{10-30} = \dfrac{x_3}{20-4} \Rightarrow \dfrac{x_1}{4} = \dfrac{x_2}{-20} = \dfrac{x_3}{16} \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{-5} = \dfrac{x_3}{4}$

$\therefore X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$

$\underline{\lambda = 3}$ $\begin{pmatrix} 7 & -2 & -5 \\ -2 & -1 & 3 \\ -5 & 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$7x_1 - 2x_2 - 5x_3 = 0$

$-2x_1 - x_2 + 3x_3 = 0$

$-5x_1 + 3x_2 + 2x_3 = 0$

$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -2 & -5 & 7 & -2 \\ -1 & 3 & -2 & -1 \end{array}$

$\dfrac{x_1}{-6-5} = \dfrac{x_2}{10-21} = \dfrac{x_3}{-7-4} \Rightarrow \dfrac{x_1}{-11} = \dfrac{x_2}{-11} = \dfrac{x_3}{-11} \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{1} = \dfrac{x_3}{1}$

$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\underline{\lambda = 14}}$ $\begin{pmatrix} -4 & -2 & -5 \\ -2 & -12 & 3 \\ -5 & 3 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ $\qquad \begin{matrix} -4x_1 - 2x_2 - 5x_3 = 0 \\ -2x_1 - 12x_2 + 3x_3 = 0 \\ -5x_1 + 3x_2 - 9x_3 = 0 \end{matrix}$ $\qquad \begin{array}{cccc} & x_1 & x_2 & x_3 \\ -2 & -5 & -4 & -2 \\ -12 & 3 & -2 & -12 \end{array}$

$$\frac{x_1}{-6-60} = \frac{x_2}{10+12} = \frac{x_3}{48-4} \Rightarrow \frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44} \Rightarrow \frac{x_1}{-6} = \frac{x_2}{2} = \frac{x_3}{4}$$

$$\Rightarrow \frac{x_1}{-3} = \frac{x_2}{1} = \frac{x_3}{2} \qquad \therefore X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

$$X_1^T X_2 = \begin{pmatrix} 1 & -5 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - 5 + 4 = 0$$

$$X_1^T X_3 = \begin{pmatrix} 1 & -5 & 4 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3 - 5 + 8 = 0$$

$$X_2^T X_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3 + 1 + 2 = 0$$

Hence the eigenvectors are orthogonal to each other.

$$N = \begin{pmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{pmatrix} \qquad N^T = \begin{pmatrix} 1/\sqrt{42} & -5/\sqrt{42} & 4/\sqrt{42} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -3/\sqrt{14} & 1/\sqrt{14} & 2/\sqrt{14} \end{pmatrix}$$

$$AN = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3/\sqrt{3} & -42/\sqrt{14} \\ 0 & 3/\sqrt{3} & 14/\sqrt{14} \\ 0 & 3/\sqrt{3} & 28/\sqrt{14} \end{pmatrix}$$

$$D = N^T A N = \begin{pmatrix} 1/\sqrt{42} & -5/\sqrt{42} & 4/\sqrt{42} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -3/\sqrt{14} & 1/\sqrt{14} & 2/\sqrt{14} \end{pmatrix} \begin{pmatrix} 0 & 3/\sqrt{3} & -42/\sqrt{14} \\ 0 & 3/\sqrt{3} & 14/\sqrt{14} \\ 0 & 3/\sqrt{3} & 28/\sqrt{14} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

(4) Diagonalize the matrix $A=\begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$. [M/J-2014]

Sol: Given $A=\begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$

Characteristic eqn.: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 2+6+2=10$

S_2 = Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix} = (12-0)+(4-16)+(12-0) = 12-12+12 = 12$$

$S_3 = |A| = 2(12-0) - 0(0-0) + 4(0-24) = 24-96 = -72$

Hence the characteristic eqn. is $\lambda^3 - 10\lambda^2 + 12\lambda + 72 = 0$

$$\begin{array}{r|rrrr} \lambda=-2 & 1 & -10 & 12 & 72 \\ & & -2 & 24 & -72 \\ \hline & 1 & -12 & 36 & \boxed{0} \end{array}$$

$\lambda^2 - 12\lambda + 36 = 0$

$(\lambda-6)(\lambda-6)=0$

$$\begin{array}{c|c} 36 & -12 \\ \hline -6 & -6 \\ \lambda-6 & \lambda-6 \end{array}$$

$\therefore \lambda = -2, 6, 6$

Hence the eigenvalues are $-2, 6$ & 6.

Eigenvectors: $(A-\lambda I)X=0$

$$\begin{pmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\underline{\lambda=-2}$ $\begin{pmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$4x_1 + 0x_2 + 4x_3 = 0$

$0x_1 + 8x_2 + 0x_3 = 0$

$4x_1 + 0x_2 + 4x_3 = 0$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 0 & 4 & 4 & 0 \\ 8 & 0 & 0 & 8 \end{array}$$

$$\frac{x_1}{0-32} = \frac{x_2}{0-0} = \frac{x_3}{32-0} \Rightarrow \frac{x_1}{-32} = \frac{x_2}{0} = \frac{x_3}{32} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$\therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda=6}$ $\begin{pmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$-4x_1 + 0x_2 + 4x_3 = 0$

$0x_1 + 0x_2 + 0x_3 = 0$

$4x_1 + 0x_2 - 4x_3 = 0$

Consider, $4x_1 + 0x_2 - 4x_3 = 0$

Put $x_1 = 0$, $0x_2 - 4x_3 = 0 \Rightarrow 4x_3 = 0x_2 \Rightarrow \frac{x_3}{0} = \frac{x_2}{4} \Rightarrow \frac{x_3}{0} = \frac{x_2}{1}$

$\therefore X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Let $x_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{array}$$

$x_1^T x_3 = (-1 \quad 0 \quad 1)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -x_1 + 0x_2 + x_3$

$x_2^T x_3 = (0 \quad 1 \quad 0)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0x_1 + x_2 + 0x_3$

$$\frac{x_1}{0-1} = \frac{x_2}{0-0} = \frac{x_3}{-1-0} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{-1} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$\therefore x_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$x_1^T x_2 = (-1 \quad 0 \quad 1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$; $x_1^T x_3 = (-1 \quad 0 \quad 1)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -1+0+1 = 0$

$x_2^T x_3 = (0 \quad 1 \quad 0)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$

Hence the eigenvectors are orthogonal to each other.

$$N = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}, \quad N^T = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$AN = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 2/\sqrt{2} & 0 & 6/\sqrt{2} \\ 0 & 6 & 0 \\ -2/\sqrt{2} & 0 & 6/\sqrt{2} \end{pmatrix}$$

$$D = N^T AN = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 2/\sqrt{2} & 0 & 6/\sqrt{2} \\ 0 & 6 & 0 \\ -2/\sqrt{2} & 0 & 6/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -4/2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12/2 \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

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(15) Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the canonical form through orthogonal transformation. [N/D-2014] [Jan-2011] [M/J-2013]

Sol: Given: Quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$

$$A = \begin{pmatrix} \text{coeff. } x^2 & \frac{1}{2}\text{coeff. } xy & \frac{1}{2}\text{coeff. } xz \\ \frac{1}{2}\text{coeff. } xy & \text{coeff. } y^2 & \frac{1}{2}\text{coeff. } yz \\ \frac{1}{2}\text{coeff. } xz & \frac{1}{2}\text{coeff. } yz & \text{coeff. } z^2 \end{pmatrix} = \begin{pmatrix} 3 & -2/2 & 2/2 \\ -2/2 & 5 & -2/2 \\ 2/2 & -2/2 & 3 \end{pmatrix}$$

$\therefore A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

Characteristic eqn.: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 3+5+3 = 11$

S_2 = Sum of the minors of main diagonal elements

$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix} = (15-1)+(9-1)+(15-1) = 14+8+14 = 36$

$S_3 = |A| = 3(15-1) + 1(-3+1) + 1(1-5) = 3(14) + 1(-2) + 1(-4) = 42-2-4 = 36.$

Hence the characteristic eqn. is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$$\begin{array}{r|rrrr} \lambda = 2 & 1 & -11 & 36 & -36 \\ & & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\begin{array}{c|c} \times & + \\ 18 & -9 \\ \hline -6 & -3 \\ \lambda-6 & \lambda-3 \end{array}$$

$\lambda^2 - 9\lambda + 18 = 0$

$(\lambda-6)(\lambda-3) = 0$

$\therefore \lambda = 2, 3, 6$

Hence the eigenvalues are 2, 3 & 6.

Eigenvectors: $(A - \lambda I)X = 0$

$\begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & 1 & 1 & -1 \\ 3 & -1 & -1 & 3 \end{array}$$

$\underline{\underline{\lambda = 2}}$ $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$x_1 - x_2 + x_3 = 0$

$-x_1 + 3x_2 - x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$\dfrac{x_1}{1-3} = \dfrac{x_2}{-1+1} = \dfrac{x_3}{3-1} \Rightarrow \dfrac{x_1}{-2} = \dfrac{x_2}{0} = \dfrac{x_3}{2} \Rightarrow \dfrac{x_1}{-1} = \dfrac{x_2}{0} = \dfrac{x_3}{1}$

$\therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & 1 & 0 & -1 \\ 2 & -1 & -1 & 2 \end{array}$$

$\underline{\underline{\lambda = 3}}$ $\begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$0x_1 - x_2 + x_3 = 0$

$-x_1 + 2x_2 - x_3 = 0$

$x_1 - x_2 + 0x_3 = 0$

$\dfrac{x_1}{1-2} = \dfrac{x_2}{-1+0} = \dfrac{x_3}{0-1} \Rightarrow \dfrac{x_1}{-1} = \dfrac{x_2}{-1} = \dfrac{x_3}{-1} \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{1} = \dfrac{x_3}{1}$

$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\lambda = 6}$ $\begin{pmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ $\quad$ $\begin{aligned} -3x_1 - x_2 + x_3 &= 0 \\ -x_1 - x_2 - x_3 &= 0 \\ x_1 - x_2 - 3x_3 &= 0 \end{aligned}$ $\quad$ $\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -1 & 1 & -3 & -1 \\ -1 & -1 & -1 & -1 \end{array}$

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$X_1^T X_2 = (-1 \;\; 0 \;\; 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -1+0+1 = 0 \quad , \quad X_1^T X_3 = (-1 \;\; 0 \;\; 1)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = -1+0+1 = 0$$

$$X_2^T X_3 = (1 \;\; 1 \;\; 1)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 1-2+1 = 0$$

Hence the eigenvectors are orthogonal to each other.

$$N = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} \quad , \quad N^T = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$$

$$AN = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} = \begin{pmatrix} -2/\sqrt{2} & 3/\sqrt{3} & 6/\sqrt{6} \\ 0 & 3/\sqrt{3} & -12/\sqrt{6} \\ 2/\sqrt{2} & 3/\sqrt{3} & 6/\sqrt{6} \end{pmatrix}$$

$$D = N^T A N = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}\begin{pmatrix} -2/\sqrt{2} & 3/\sqrt{3} & 6/\sqrt{6} \\ 0 & 3/\sqrt{3} & -12/\sqrt{6} \\ 2/\sqrt{2} & 3/\sqrt{3} & 6/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 4/2 & 0 & 0 \\ 0 & 9/3 & 0 \\ 0 & 0 & 36/6 \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Canonical form:

$$(y_1 \;\; y_2 \;\; y_3)\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = (2y_1 \;\; 3y_2 \;\; 6y_3)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 2y_1^2 + 3y_2^2 + 6y_3^2$$

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(16) Reduce the quadratic form $2x^2+5y^2+3z^2+4xy$ to a canonical form through an orthogonal transformation. Find also its nature. [A/M 2018] [M/J-2010] [Jan-2012]

Sol: Given: Quadratic form $2x^2+5y^2+3z^2+4xy$

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Characteristic eqn.: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 2+5+3 = 10$

S_2 = Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = (15-0) + (6-0) + (10-4) = 15+6+6 = 27$$

$S_3 = |A| = 2(15-0) - 2(6-0) + 0(0-0) = 30 - 12 = 18$

Hence the characteristic eqn. is $\lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$

$$\begin{array}{c|cccc} \lambda=1 & 1 & -10 & 27 & -18 \\ & & 1 & -9 & 18 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$\lambda^2 - 9\lambda + 18 = 0$

$(\lambda-6)(\lambda-3) = 0$

$$\begin{array}{c|c} \times & + \\ 18 & -9 \\ \hline -6 & -3 \\ \lambda-6 & \lambda-3 \end{array}$$

$\therefore \lambda = 1, 3, 6$

Hence the eigenvalues are 1, 3 & 6.

Eigenvectors: $(A - \lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\underline{\underline{\lambda=1}}$ $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$x_1 + 2x_2 + 0x_3 = 0$

$2x_1 + 4x_2 + 0x_3 = 0$

$0x_1 + 0x_2 + 2x_3 = 0$

① & ③

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \end{array}$$

$$\frac{x_1}{4-0} = \frac{x_2}{0-2} = \frac{x_3}{0-0} \Rightarrow \frac{x_1}{4} = \frac{x_2}{-2} = \frac{x_3}{0} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$\therefore X_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$\underline{\underline{\lambda=3}}$ $\begin{pmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$-x_1 + 2x_2 + 0x_3 = 0$

$2x_1 + 2x_2 + 0x_3 = 0$

$0x_1 + 0x_2 + 0x_3 = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2 & 0 & -1 & 2 \\ 2 & 0 & 2 & 2 \end{array}$$

$$\frac{x_1}{0-0} = \frac{x_2}{0-0} = \frac{x_3}{-2-4} \Rightarrow \frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-6} \Rightarrow \frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

② & ③

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & 0 & 2 & -1 \\ 0 & -3 & 0 & 0 \end{array}$$

$\underline{\underline{\lambda=6}}$ $\begin{pmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$-4x_1 + 2x_2 + 0x_3 = 0$

$2x_1 - x_2 + 0x_3 = 0$

$0x_1 + 0x_2 - 3x_3 = 0$

$$\frac{x_1}{3-0} = \frac{x_2}{0+6} = \frac{x_3}{0-0} \Rightarrow \frac{x_1}{3} = \frac{x_2}{6} = \frac{x_3}{0} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{0}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$x_1^T x_2 = (2\ -1\ \ 0)\begin{pmatrix}0\\0\\-1\end{pmatrix} = 0$, $x_1^T x_3 = (2\ -1\ \ 0)\begin{pmatrix}1\\2\\0\end{pmatrix} = 2-2+0=0$

$x_2^T x_3 = (0\ \ 0\ -1)\begin{pmatrix}1\\2\\0\end{pmatrix} = 0$

Hence the eigenvectors are orthogonal to each other.

$$N = \begin{pmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & -1 & 0 \end{pmatrix}, \quad N^T = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{pmatrix}$$

$$AN = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} & 0 & 6/\sqrt{5} \\ -1/\sqrt{5} & 0 & 12/\sqrt{5} \\ 0 & -3 & 0 \end{pmatrix}$$

$$D = N^TAN = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} & 0 & 6/\sqrt{5} \\ -1/\sqrt{5} & 0 & 12/\sqrt{5} \\ 0 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Canonical form:

$$(y_1\ y_2\ y_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = (y_1\ \ 3y_2\ \ 6y_3)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = y_1^2+3y_2^2+6y_3^2$$

Canonical form contains only +ve terms. ∴ Quadratic form is said to be positive definite.

(17) Reduce the quadratic form $6x_1^2+3x_2^2+3x_3^2-4x_1x_2-2x_2x_3+4x_3x_1$ to canonical form. Hence find its rank, signature, index & nature.

[N/D-2016]
[M/J-2016]
[N/D-2015]
[Jan-2014]

Sol: Given: Q.F $6x_1^2+3x_2^2+3x_3^2-4x_1x_2-2x_2x_3+4x_3x_1$

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

Characteristic eqn.: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements $= 6+3+3 = 12$

S_2 = Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = (9-1)+(18-4)+(18-4) = 8+14+14 = 36$$

$S_3 = |A| = 6(9-1)+2(-6+2)+2(2-6) = 6(8)+2(-4)+2(-4) = 48-8-8 = 32$

Hence the characteristic eqn. is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$$\begin{array}{c|cccc} \lambda=2 & 1 & -12 & 36 & -32 \\ & & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$\lambda^2-10\lambda+16=0$

$(\lambda-8)(\lambda-2)=0$

$\therefore \lambda=2,2,8$

$$\begin{array}{c|c} \times & + \\ 16 & -10 \\ \hline -8 & -2 \\ \lambda-8 & \lambda-2 \end{array}$$

Hence the eigenvalues are 2, 2 & 8.

Eigenvectors: $(A-\lambda I)X=0$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$$

$\lambda=2$ $\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$

$4x_1-2x_2+2x_3=0$

$-2x_1+x_2-x_3=0$

$2x_1-x_2+x_3=0$

Consider, $2x_1-x_2+x_3=0$

Put $x_1=0 \Rightarrow -x_2+x_3=0 \Rightarrow x_2=x_3$

$$\therefore X_1=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda=8$ $\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$

$-2x_1-2x_2+2x_3=0$

$-2x_1-5x_2-x_3=0$

$2x_1-x_2-5x_3=0$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -2 & 2 & -2 & -2 \\ -5 & -1 & -2 & -5 \end{array}$$

$$\frac{x_1}{2+10}=\frac{x_2}{-4-2}=\frac{x_3}{10-4} \Rightarrow \frac{x_1}{12}=\frac{x_2}{-6}=\frac{x_3}{6} \Rightarrow \frac{x_1}{2}=\frac{x_2}{-1}=\frac{x_3}{1}$$

$$\therefore X_2=\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Let $X_3=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 1 & 0 & 1 \\ -1 & 1 & 2 & -1 \end{array}$$

$X_1^TX_3=\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0x_1+x_2+x_3$

$X_2^TX_3=\begin{pmatrix} 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=2x_1-x_2+x_3$

$$\frac{x_1}{1+1}=\frac{x_2}{2-0}=\frac{x_3}{0-2} \Rightarrow \frac{x_1}{2}=\frac{x_2}{2}=\frac{x_3}{-2} \Rightarrow \frac{x_1}{1}=\frac{x_2}{1}=\frac{x_3}{-1}$$

$$\therefore X_3=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$X_1^TX_2=\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}=0-1+1=0$, $X_1^TX_3=\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}=0+1-1=0$

$X_2^TX_3=\begin{pmatrix} 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}=2-1-1=0$

Hence the eigenvectors are orthogonal to each other.

$$N=\begin{pmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{pmatrix} \qquad N^T=\begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

$$AN=\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{pmatrix}=\begin{pmatrix} 0 & 16/\sqrt{6} & 2/\sqrt{3} \\ 2/\sqrt{2} & -8/\sqrt{6} & 2/\sqrt{3} \\ 2/\sqrt{2} & 8/\sqrt{6} & -2/\sqrt{3} \end{pmatrix}$$

$$D=N^TAN=\begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}\begin{pmatrix} 0 & 16/\sqrt{6} & 2/\sqrt{3} \\ 2/\sqrt{2} & -8/\sqrt{6} & 2/\sqrt{3} \\ 2/\sqrt{2} & 8/\sqrt{6} & -2/\sqrt{3} \end{pmatrix}=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 48/6 & 0 \\ 0 & 0 & 6/3 \end{pmatrix}$$

$$D=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Canonical form:

$$(y_1\ y_2\ y_3)\begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}=2y_1^2+8y_2^2+2y_3^2$$

Canonical form contains only +ve terms. ∴ Quadratic form is said to be positive definite.

Rank = No/. of non-zero terms in C.F = 3

Index = No/. of +ve terms in C.F = 3

Signature = (No/. of +ve terms − No/. of −ve terms) in C.F = 3 − 0 = 3

(18) Reduce the quadratic form $x_1^2+5x_2^2+x_3^2+2x_1x_2+2x_2x_3+6x_3x_1$ to the canonical form. Hence find its nature & rank. [M/J-2014] [A/M-2015] [N/D-2014]

Given: Q.F : $x_1^2+5x_2^2+x_3^2+2x_1x_2+2x_2x_3+6x_3x_1$

$$A=\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

Characteristic eqn/.: $\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$

S_1 = Sum of the main diagonal elements $=1+5+1=7$

S_2 = Sum of the minors of main diagonal elements

$$=\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}=(5-1)+(1-9)+(5-1)=4-8+4=0$$

$S_3=|A|=1(5-1)-1(1-3)+3(1-15)=4+2-42=-36$

Hence the characteristic eqn/. is $\lambda^3-7\lambda^2+0\lambda+36=0 \Rightarrow \lambda^3-7\lambda^2+36=0$

$$\lambda = 3 \;\begin{array}{|cccc} 1 & -7 & 0 & 36 \\ & 3 & -12 & -36 \\ \hline 1 & -4 & -12 & \boxed{0} \end{array}$$

$$\begin{array}{c|c} \times & + \\ -12 & -4 \\ \hline -6 & +2 \\ \lambda-6 & \lambda+2 \end{array}$$

$\lambda^2 - 4\lambda - 12 = 0$

$(\lambda - 6)(\lambda + 2) = 0$

$\therefore \lambda = -2, 3, 6$

Hence the eigenvalues are -2, 3 & 6.

Eigenvectors: $(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\underline{\underline{\lambda = -2}}$

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$3x_1 + x_2 + 3x_3 = 0$

$x_1 + 7x_2 + x_3 = 0$

$3x_1 + x_2 + 3x_3 = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & 3 & 1 \\ 7 & 1 & 1 & 7 \end{array}$$

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1} \Rightarrow \frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{\underline{\lambda = 3}}$

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$-2x_1 + x_2 + 3x_3 = 0$

$x_1 + 2x_2 + x_3 = 0$

$3x_1 + x_2 - 2x_3 = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & -2 & 1 \\ 2 & 1 & 1 & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1} \Rightarrow \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$\underline{\underline{\lambda = 6}}$

$$\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$-5x_1 + x_2 + 3x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$3x_1 + x_2 - 5x_3 = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & -5 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1} \Rightarrow \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$X_1^T X_2 = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 1 + 0 - 1 = 0, \quad X_1^T X_3 = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = -1 + 0 + 1 = 0$$

$$X_2^T X_3 = \begin{pmatrix} -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = -1 + 2 - 1 = 0$$

Hence the eigenvectors are orthogonal to each other.

$$N = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} \qquad N^T = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$$

$$AN=\begin{pmatrix}1&1&3\\1&5&1\\3&1&1\end{pmatrix}\begin{pmatrix}-1/\sqrt2&-1/\sqrt3&1/\sqrt6\\0&1/\sqrt3&2/\sqrt6\\1/\sqrt2&-1/\sqrt3&1/\sqrt6\end{pmatrix}=\begin{pmatrix}2/\sqrt2&-3/\sqrt3&6/\sqrt6\\0&3/\sqrt3&12/\sqrt6\\-2/\sqrt2&-3/\sqrt3&6/\sqrt6\end{pmatrix}$$

$$D=N^TAN=\begin{pmatrix}-1/\sqrt2&0&1/\sqrt2\\-1/\sqrt3&1/\sqrt3&-1/\sqrt3\\1/\sqrt6&2/\sqrt6&1/\sqrt6\end{pmatrix}\begin{pmatrix}2/\sqrt2&-3/\sqrt3&6/\sqrt6\\0&3/\sqrt3&12/\sqrt6\\-2/\sqrt2&-3/\sqrt3&6/\sqrt6\end{pmatrix}=\begin{pmatrix}-2&0&0\\0&3&0\\0&0&6\end{pmatrix}$$

Canonical form:

$$(y_1\ y_2\ y_3)\begin{pmatrix}-2&0&0\\0&3&0\\0&0&6\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=-2y_1^2+3y_2^2+6y_3^2$$

Canonical form contains 2 +ve terms & one -ve term. ∴ Quadratic form is said to be indefinite.

Rank = No/. of non-zero terms in C.F = 3

(19) Reduce the quadratic form $2x^2+y^2+z^2+2xy-2xz-4yz$ to the canonical form. Hence find its nature, rank, index & signature. [A/M-2015] [N/D-2010]

Sol: Q.F : $2x^2+y^2+z^2+2xy-2xz-4yz$

$$A=\begin{pmatrix}2&1&-1\\1&1&-2\\-1&-2&1\end{pmatrix}$$

Characteristic eqn/. : $\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$

S_1 = Sum of the main diagonal elements = 2+1+1=4

S_2 = Sum of the minors of main diagonal elements

$$=\begin{vmatrix}1&-2\\-2&1\end{vmatrix}+\begin{vmatrix}2&-1\\-1&1\end{vmatrix}+\begin{vmatrix}2&1\\1&1\end{vmatrix}=(1-4)+(2-1)+(2-1)=-3+1+1=-1$$

$S_3=|A|=2(1-4)-1(1-2)-1(-2+1)=-6+1+1=-4$

Hence the characteristic eqn/. is $\lambda^3-4\lambda^2-\lambda+4=0$

$$\begin{array}{r|rrrr}\lambda=1 & 1 & -4 & -1 & 4\\ & & 1 & -3 & -4\\\hline & 1 & -3 & -4 & 0\end{array}$$

$$\begin{array}{c|c}\times & +\\ -4 & -3\\\hline +1 & -4\\ \lambda+1 & \lambda-4\end{array}$$

$\lambda^2-3\lambda-4=0$

$(\lambda+1)(\lambda-4)=0$

$\therefore \lambda=-1,1,4$

Hence the eigenvalues are $-1, 1$ & 4.

Eigenvectors: $(A-\lambda I)X=0$

$$\begin{pmatrix}2-\lambda&1&-1\\1&1-\lambda&-2\\-1&-2&1-\lambda\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

$\underline{\lambda = -1}$ $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ $\quad$ $\begin{aligned} 3x_1 + x_2 - x_3 &= 0 \\ x_1 + 2x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}$ $\quad$ $\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & -1 & 3 & 1 \\ 2 & -2 & 1 & 2 \end{array}$

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1} \Rightarrow \frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 1}$ $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ $\quad$ $\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ x_1 + 0x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 + 0x_3 &= 0 \end{aligned}$ $\quad$ $\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & -1 & 1 & 1 \\ 0 & -2 & 1 & 0 \end{array}$

$$\frac{x_1}{-2+0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1} \Rightarrow \frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

$\underline{\lambda = 4}$ $\begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ $\quad$ $\begin{aligned} -2x_1 + x_2 - x_3 &= 0 \\ x_1 - 3x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 - 3x_3 &= 0 \end{aligned}$ $\quad$ $\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & -1 & -2 & 1 \\ -3 & -2 & 1 & -3 \end{array}$

$$\frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1} \Rightarrow \frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$X_1^T X_2 = (0 \;\; 1 \;\; 1)\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 0 + 1 - 1 = 0$, $\quad X_1^T X_3 = (0 \;\; 1 \;\; 1)\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0$

$X_2^T X_3 = (-2 \;\; 1 \;\; -1)\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = 2 - 1 - 1 = 0$

Hence the eigenvectors are orthogonal to each other.

$$N = \begin{pmatrix} 0 & -2/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \qquad N^T = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

$$AN = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & -2/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 & -2/\sqrt{6} & -4/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & -4/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 4/\sqrt{3} \end{pmatrix}$$

$$D = N^T A N = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}\begin{pmatrix} 0 & -2/\sqrt{6} & -4/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & -4/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 4/\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Canonical form: $(y_1 \;\; y_2 \;\; y_3)\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = -y_1^2 + y_2^2 + 4y_3^2$

Canonical form contains 2 +ve terms & one -ve term. ∴ Quadratic form is said to be indefinite.

Rank = No. of non-zero terms in C.F = 3

Index = No. of +ve terms in C.F = 2

Signature = (No. of +ve terms - No. of -ve terms) in C.F = 2 - 1 = 1

(20) Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to the canonical form through an orthogonal transformation, & hence show that is +ve semi-definite. Also given a non-zero set of values (x_1, x_2, x_3) which makes this quadratic form zero. [M/J-2009]

Sol: Given: Q.F $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Characteristic eqn.: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 1 + 2 + 1 = 4$

$$S_2 = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = (2-1) + (1-0) + (2-1) = 1+1+1 = 3$$

$S_3 = |A| = 1(2-1) + 1(-1-0) + 0(-1-0) = 1 - 1 = 0$

Hence the characteristic eqn. is $\lambda^3 - 4\lambda^2 + 3\lambda = 0$

$$\lambda(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda = 0, \ (\lambda - 1)(\lambda - 3) = 0$$

×	+
3	-4
-1	-3
$\lambda - 1$	$\lambda - 3$

$$\therefore \lambda = 0, 1, 3$$

Hence the eigenvalues are 0, 1 & 3.

Eigenvectors: $(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\lambda = 0$: $$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$x_1 - x_2 + 0x_3 = 0$

$-x_1 + 2x_2 + x_3 = 0$

$0x_1 + x_2 + x_3 = 0$

x_1		x_2	x_3
-1	0	1	-1
2	1	-1	2

$$\frac{x_1}{-1-0} = \frac{x_2}{0-1} = \frac{x_3}{2-1} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1} \qquad \therefore X_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{\underline{\lambda=1}} \quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{aligned} 0x_1 - x_2 + 0x_3 &= 0 \\ -x_1 + x_2 + x_3 &= 0 \\ 0x_1 + x_2 + 0x_3 &= 0 \end{aligned}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & 1 \end{array}$$

$$\frac{x_1}{-1-0} = \frac{x_2}{0-0} = \frac{x_3}{0-1} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{-1} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$\therefore X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -1 & 0 & -2 & -1 \\ -1 & 1 & -1 & -1 \end{array}$$

$$\underline{\underline{\lambda=3}} \quad \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{aligned} -2x_1 - x_2 + 0x_3 &= 0 \\ -x_1 - x_2 + x_3 &= 0 \\ 0x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

$$\frac{x_1}{-1+0} = \frac{x_2}{0+2} = \frac{x_3}{2-1} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{1} \qquad \therefore X_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$X_1^T X_2 = (-1 \ -1 \ 1)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -1+0+1 = 0, \quad X_1^T X_3 = (-1 \ -1 \ 1)\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 1-2+1 = 0$$

$$X_2^T X_3 = (1 \ 0 \ 1)\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = -1+0+1 = 0$$

Hence the eigenvectors are orthogonal to each other.

$$N = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} \qquad N^T = \begin{pmatrix} -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$$

$$AN = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{2} & -3/\sqrt{6} \\ 0 & 0 & 6/\sqrt{6} \\ 0 & 1/\sqrt{2} & 3/\sqrt{6} \end{pmatrix}$$

$$D = N^T A N = \begin{pmatrix} -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}\begin{pmatrix} 0 & 1/\sqrt{2} & -3/\sqrt{6} \\ 0 & 0 & 6/\sqrt{6} \\ 0 & 1/\sqrt{2} & 3/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

<u>Canonical form</u>:

$$(y_1 \ y_2 \ y_3)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0y_1^2 + y_2^2 + 3y_3^2$$

Canonical form contains 2 +ve terms & one zero.

∴ Quadratic form is said to be positive semi-definite.

The orthogonal transformation is $X = NY$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x_1 = -\frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 - \frac{1}{\sqrt{6}} y_3$$

$$x_2 = \frac{-1}{\sqrt{3}} y_1 + 0 y_2 + \frac{2}{\sqrt{6}} y_3$$

$$x_3 = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3$$

Verification : $x_1 = -1,\ x_2 = -1,\ x_3 = 1$

$$Q.F = x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$$
$$= 1 + 2 + 1 - 2 - 2 = 0$$

Take $y_1 = \sqrt{3}$, $y_2 = 0$ & $y_3 = 0$

$x_1 = -1,\ x_2 = -1,\ x_3 = 1$

These values x_1, x_2, x_3 make the Q.F. zero.

Properties:

(21) Prove that the eigenvalues of a real symmetric matrix are real. [M/J-2014]

Proof: Let λ be an eigenvalue of the real symmetric matrix A. Let the corresponding eigenvector be X. Let A^T denote the transpose of A.

We have $AX = \lambda X$

Pre-multiplying this eqn. by $1 \times n$ matrix $\bar{X}^T$, where the bar denotes the complex conjugate of X^T, we get

$$\bar{X}^T A X = \lambda \bar{X}^T X \quad \text{--- (1)}$$

Taking complex conjugate, we get

$$X^T \bar{A} \bar{X} = \bar{\lambda} X^T \bar{X}$$

$$X^T A \bar{X} = \bar{\lambda} X^T \bar{X} \quad (\because A \text{ is real})$$

Taking transpose on both sides, we get

$$(X^T A \bar{X})^T = (\bar{\lambda} X^T \bar{X})^T$$

$$\bar{X}^T A^T X = \bar{\lambda} \bar{X}^T X$$

$$\bar{X}^T A X = \bar{\lambda} \bar{X}^T X \quad \text{--- (2)} \quad (\because A \text{ is symmetric})$$

From (1) & (2), $\lambda \bar{X}^T X = \bar{\lambda} \bar{X}^T X \Rightarrow \lambda = \bar{\lambda}$. Hence λ is real

(22) If λ is an eigenvalue of a matrix A, then $\frac{1}{\lambda}$ $(\lambda \neq 0)$ is the eigenvalue of A^{-1}. [N/D-2014]

Proof: Given λ is an eigenvalue of a matrix A. Let the corresponding [M/J-2012] eigenvector be x. Then we have $Ax = \lambda x$

Pre-multiplying both sides by A^{-1}, we get

$$A^{-1}Ax = A^{-1}\lambda x$$

$$Ix = \lambda A^{-1}x$$

$$x = \lambda A^{-1}x$$

$$\div \lambda \Rightarrow \frac{1}{\lambda}x = A^{-1}x$$

From this we get, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

(23) If λ_i for $(i=1,2,\dots,n)$ are the non-zero eigenvalues of A, then prove that $k\lambda_i$ are the eigenvalues of KA, where K being a non-zero scalar. [M/J-2012]

Proof: Given λ_i $(i=1,2,\dots,n)$ are the non-zero eigenvalues of A. Let the corresponding eigenvectors be x_i $(i=1,2,\dots,n)$. Then we have

$$Ax_i = \lambda_i x_i \quad (i=1,2,\dots,n)$$

Pre-multiplying both sides by K, we get

$$KAx_i = K\lambda_i x_i$$

From this we get $K\lambda_i$ $(i=1,2,\dots,n)$ are the eigenvalues of KA.

(24) If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of a matrix A, then A^m has the eigenvalues $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$. ($m$ being a +ve integer)

Proof: Given λ_i $(i=1,2,\dots,n)$ are the eigenvalues of A. Let the corresponding eigenvectors be x_i $(i=1,2,\dots,n)$. Then we have

$$Ax_i = \lambda_i x_i \rightarrow (1) \quad (i=1,2,\dots,n)$$

$$A^2x_i = A\lambda_i x_i = \lambda_i Ax_i = \lambda_i(\lambda_i x_i) \quad (\because \text{by } (1))$$

$$A^2x_i = \lambda_i^2 x_i$$

Similarly we get, $A^3x_i = \lambda_i^3 x_i$

In general, $A^m x_i = \lambda_i^m x_i$

From this we get, $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$ are the eigenvalues of A^m.

(25) Find the sum & product of the eigenvalues of the matrix $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$.

Sol:

Sum of the eigenvalues = Sum of the main diagonal elements $= -2+1+0 = -1$

Product of the eigenvalue $= |A| = -2(0-12) - 2(0-6) - 3(-4+1) = 24+12+9 = 45$

(26) The product of 2 eigenvalues of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. Find the third eigenvalue.

Sol: Given $\lambda_1 \lambda_2 = 16$ — (1)

Product of eigenvalues $= |A| = 6(9-1) + 2(-6+2) + 2(2-6) = 48 - 8 - 8 = 32$

$$\lambda_1 \lambda_2 \lambda_3 = 32$$
$$16 \lambda_3 = 32 \quad (\because \text{by } (1))$$
$$\lambda_3 = \frac{32}{16} = 2 \qquad \therefore \lambda_3 = 2$$

(27) Two of the eigenvalues of $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ are 3 & 6. Find the eigenvalues of A^{-1}.

Sol: Given $\lambda_1 = 3$ & $\lambda_2 = 6$

Sum of the eigenvalues = Sum of the main diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 3 + 5 + 3 = 11$$
$$3 + 6 + \lambda_3 = 11 \Rightarrow 9 + \lambda_3 = 11 \Rightarrow \lambda_3 = 11 - 9 = 2$$

Hence the eigenvalues of A^{-1} are $\frac{1}{3}, \frac{1}{6}$ & $\frac{1}{2}$.

(28) Find the eigenvalues of A^3 given $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{pmatrix}$.

Sol: Given matrix A is a upper triangular matrix.

$\therefore$ Eigenvalues of A are 1, 2 & 3. ($\because$ Entries of main diagonal elements)

Hence the eigenvalues of A^3 are $1^3, 2^3$ & 3^3 (ie) 1, 8 & 27.

(29) The eigenvectors of a 3×3 real symmetric matrix A corresponding to the eigenvalues 2, 3, 6 are $[1, 0, -1]^T$, $[1, 1, 1]^T$ & $[-1, 2, -1]^T$ respectively, find the matrix A.

Sol:

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$N = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{pmatrix}$$

$$A = NDN^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 2/\sqrt{2} & 0 & -2/\sqrt{2} \\ 3/\sqrt{3} & 3/\sqrt{3} & 3/\sqrt{3} \\ -6/\sqrt{6} & 12/\sqrt{6} & -6/\sqrt{6} \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} 1+1+1 & 1-2 & -1+1+1 \\ 1-2 & 1+4 & 1-2 \\ -1+1+1 & 1-2 & 1+1+1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

Diagonalisation of non-symmetric matrix:

(30) Diagonalise the matrix $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$.

Sol: Let $A = \begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$

Characteristic eqn.: $\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 =$ Sum of the main diagonal elements $= 1+4 = 5$

$S_2 = |A| = 4 - 10 = -6$

Hence the characteristic eqn. is $\lambda^2 - 5\lambda - 6 = 0$

$$(\lambda - 6)(\lambda + 1) = 0$$

$$\therefore \lambda = -1, 6$$

×	+
−6	−5
−6	+1
λ−6	λ+1

Hence the eigenvalues are -1 & 6.

Eigenvectors: $(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$\underline{\lambda = -1}$ $\begin{pmatrix} 2 & -2 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$2x_1 - 2x_2 = 0 \Rightarrow x_1 - x_2 = 0 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$

$-5x_1 + 5x_2 = 0 \Rightarrow x_1 - x_2 = 0$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 6}$ $\begin{pmatrix} -5 & -2 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$-5x_1 - 2x_2 = 0 \Rightarrow -5x_1 = 2x_2 \Rightarrow \frac{x_1}{2} = \frac{x_2}{-5}$

$-5x_1 - 2x_2 = 0$

$$\therefore X_2 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

Eigenvector matrix: $P = \begin{pmatrix} 1 & 2 \\ 1 & -5 \end{pmatrix}$

$$P^{-1} = \frac{1}{|P|} \text{Adj} P = \frac{1}{-5-2} \begin{pmatrix} -5 & -2 \\ -1 & 1 \end{pmatrix} = \frac{1}{-7} \begin{pmatrix} -5 & -2 \\ -1 & 1 \end{pmatrix}$$

$$AP = \begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} -1 & 12 \\ -1 & -30 \end{pmatrix}$$

$$D = P^{-1}AP = \frac{-1}{7}\begin{pmatrix} -5 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 12 \\ -1 & -30 \end{pmatrix} = \frac{-1}{7}\begin{pmatrix} 7 & 0 \\ 0 & -42 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}$$

(31) Reduce the matrix $\begin{pmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ to the diagonal form.

Sol:

Let $A = \begin{pmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$

Characteristic eqn.: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = -1 + 2 + 0 = 1$

$S_2 = \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -1 & -2 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = (0+1) + (0-2) + (-2-2) = 1 - 2 - 4 = -5$

$S_3 = |A| = -1(0+1) - 2(0+1) - 2(-1+2) = -1 - 2 - 2 = -5$

Hence the characteristic eqn. is $\lambda^3 - \lambda^2 - 5\lambda + 5 = 0$

$$\begin{array}{c|cccc} \lambda = 1 & 1 & -1 & -5 & 5 \\ & & 1 & 0 & -5 \\ \hline & 1 & 0 & -5 & \boxed{0} \end{array}$$

$\lambda^2 + 0\lambda - 5 = 0 \Rightarrow \lambda^2 = 5 \Rightarrow \lambda = \pm\sqrt{5}$

$\therefore \lambda = -\sqrt{5}, 1, \sqrt{5}$

Hence the eigenvalues are $-\sqrt{5}$, 1 & $\sqrt{5}$.

Eigenvectors: $(A - \lambda I)X = 0$

$$\begin{pmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\underline{\lambda = -\sqrt{5}}$

$$\begin{pmatrix} -1+\sqrt{5} & 2 & -2 \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -1 & \sqrt{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$(-1+\sqrt{5})x_1 + 2x_2 - 2x_3 = 0$

$x_1 + (2+\sqrt{5})x_2 + x_3 = 0$

$-x_1 - x_2 + \sqrt{5}x_3 = 0$

(2) & (3)

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 2+\sqrt{5} & 1 & 1 & 2+\sqrt{5} \\ -1 & \sqrt{5} & -1 & -1 \end{array}$$

$$\frac{x_1}{2\sqrt{5}+5+1} = \frac{x_2}{-1-\sqrt{5}} = \frac{x_3}{-1+2+\sqrt{5}}$$

$$\frac{x_1}{2\sqrt{5}+6} = \frac{x_2}{-1-\sqrt{5}} = \frac{x_3}{1+\sqrt{5}}$$

$$\therefore X_1 = \begin{pmatrix} 2\sqrt{5}+6 \\ -1-\sqrt{5} \\ 1+\sqrt{5} \end{pmatrix}$$

$\underline{\lambda = 1}$ $\begin{pmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$-2x_1 + 2x_2 - 2x_3 = 0$
$x_1 + x_2 + x_3 = 0$
$-x_1 - x_2 - x_3 = 0$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2 & -2 & -2 & 2 \\ 1 & 1 & 1 & 1 \end{array}$$

$$\frac{x_1}{2+2} = \frac{x_2}{-2+2} = \frac{x_3}{-2-2} \Rightarrow \frac{x_1}{4} = \frac{x_2}{0} = \frac{x_3}{-4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\underline{\lambda = \sqrt{5}}$ $\begin{pmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & -\sqrt{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$(-1-\sqrt{5})x_1 + 2x_2 - 2x_3 = 0$
$x_1 + (2-\sqrt{5})x_2 + x_3 = 0$
$-x_1 - x_2 - \sqrt{5}x_3 = 0$

$\underline{②\ \&\ ③}$
$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2-\sqrt{5} & 1 & 1 & 2-\sqrt{5} \\ -1 & -\sqrt{5} & -1 & -1 \end{array}$$

$$\frac{x_1}{-2\sqrt{5}+5+1} = \frac{x_2}{-1+\sqrt{5}} = \frac{x_3}{-1+2-\sqrt{5}}$$

$$\frac{x_1}{6-2\sqrt{5}} = \frac{x_2}{-1+\sqrt{5}} = \frac{x_3}{1-\sqrt{5}}$$

$$\therefore X_3 = \begin{pmatrix} 6-2\sqrt{5} \\ -1+\sqrt{5} \\ 1-\sqrt{5} \end{pmatrix}$$

Eigenvector matrix: $P = \begin{pmatrix} 2\sqrt{5}+6 & 1 & 6-2\sqrt{5} \\ -1-\sqrt{5} & 0 & -1+\sqrt{5} \\ 1+\sqrt{5} & -1 & 1-\sqrt{5} \end{pmatrix}$

$$AP = \begin{pmatrix} -23.4164 & 1 & 3.4164 \\ 7.2361 & 0 & 2.7639 \\ -7.2361 & -1 & -2.7639 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 0.0691 & -0.0163 & 0.0691 \\ 0 & -1 & -1 \\ 0.1809 & 0.7663 & 0.1809 \end{pmatrix}$$

$$\therefore D = P^{-1}AP = \begin{pmatrix} -2.236 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2.236 \end{pmatrix} = \begin{pmatrix} -\sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix}$$

(32) Diagonalise the matrix $A = \begin{pmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{pmatrix}$

Sol:

Given $A = \begin{pmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{pmatrix}$

Characteristic eqn.: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 0+1+2 = 3$

$S_2 = \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} = (2+2)+(0-2)+(0-2) = 4-2-2 = 0$

$S_3 = 0 + 2(-2+2) - 2(1+1) = -4$

Hence the characteristic eqn. is $\lambda^3 - 3\lambda^2 + 4 = 0$

$$\begin{array}{r|rrrr} \lambda = 2 & 1 & -3 & 0 & 4 \\ & & 2 & -2 & -4 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$$\begin{array}{c|c} \times & + \\ -2 & -1 \\ \hline +1 & -2 \\ \lambda+1 & \lambda-2 \end{array}$$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda+1)(\lambda-2) = 0$

$\therefore \lambda = -1, 2, 2$

Hence the eigenvalues are $-1, 2$ & 2.

Eigenvectors: $(A - \lambda I)X = 0$

$$\begin{pmatrix} -\lambda & -2 & -2 \\ -1 & 1-\lambda & 2 \\ -1 & -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\lambda = -1$

$$\begin{pmatrix} 1 & -2 & -2 \\ -1 & 2 & 2 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{aligned} x_1 - 2x_2 - 2x_3 &= 0 \\ -x_1 + 2x_2 + 2x_3 &= 0 \\ -x_1 - x_2 + 3x_3 &= 0 \end{aligned}$$

(2) & (3)

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 2 & 2 & -1 & 2 \\ -1 & 3 & -1 & -1 \end{array}$$

$$\frac{x_1}{6+2} = \frac{x_2}{-2+3} = \frac{x_3}{1+2} \Rightarrow \frac{x_1}{8} = \frac{x_2}{1} = \frac{x_3}{3}$$

$\therefore X_1 = \begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix}$

$\lambda = 2$

$$\begin{pmatrix} -2 & -2 & -2 \\ -1 & -1 & 2 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{aligned} -2x_1 - 2x_2 - 2x_3 &= 0 \\ -x_1 - x_2 + 2x_3 &= 0 \\ -x_1 - x_2 + 0x_3 &= 0 \end{aligned}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -2 & -2 & -2 & -2 \\ -1 & 2 & -1 & -1 \end{array}$$

$$\frac{x_1}{-4-2} = \frac{x_2}{2+2} = \frac{x_3}{2-2} \Rightarrow \frac{x_1}{-6} = \frac{x_2}{4} = \frac{x_3}{0} \Rightarrow \frac{x_1}{-3} = \frac{x_2}{2} = \frac{x_3}{0}$$

$\therefore X_2 = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$

$\therefore$ We get one eigenvector corresponding to the repeated root 2.

$\therefore$ Diagonalisation not possible.

DIFFERENTIAL CALCULUS

Representation of function:

(i) Verbally (by a description in words)
(ii) Visually (by a graph)
(iii) Numerically (by a table of values)
(iv) Algebraically (by an explicit formula)

Definition: (Real-valued functions)

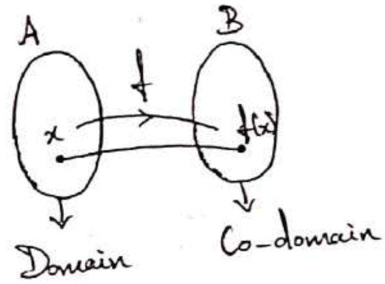
A function, whose domain & co-domain are subsets of the set of all real numbers, is known as real-valued function.

Definition:

Let $f: A \to B$, then set A is called the domain of the function & set B is called the co-domain of the function.

The set of all the images of all the elements of A under the function f is called the range of f is denoted by $f(A)$.

Thus the range of f is $f(A) = \{f(x) : x \in A\}$.



Definition: (Explicit function)

If x & y be so related that y can be expressed explicitly in terms of x, then y is called explicit function of x.

E.g: $y = x^2 - 4x + 2$

Definition: (Implicit function)

If x & y be so related that y cannot be expressed explicitly in terms of x, then y is called implicit function of x.

Eg: $x^3 + y^3 - 3xy = 0$.

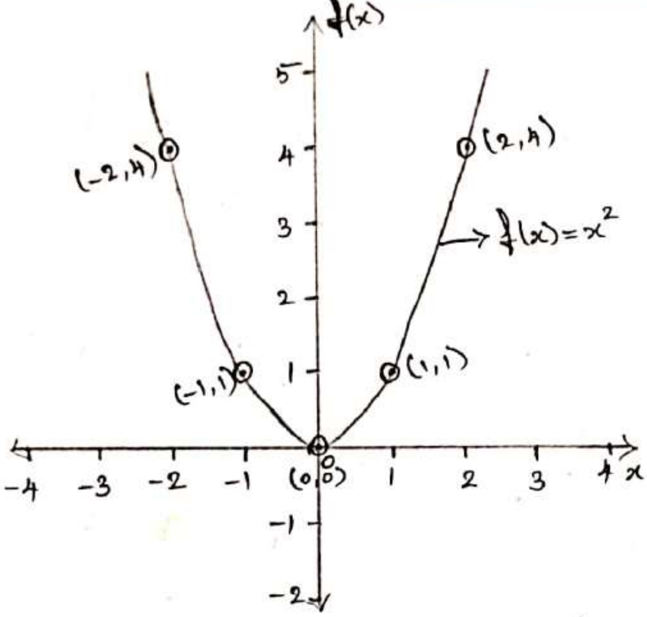
Problems:

① Find the domain & range & sketch the graph of the function $f(x)=x^2$.

Sol:

Given $f(x)=x^2$

Domain (x)	$-\infty$	....	-2	-1	0	1	2	....	∞
Range $(f(x))$	∞	....	4	1	0	1	4	....	∞



Domain $= (-\infty, \infty)$

Range $= [0, \infty)$

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② Find the domain & range of $f(x)=\sqrt{5x+10}$.

Sol: Given $f(x)=\sqrt{5x+10}$

Here $5x+10 \geq 0$

$\Rightarrow 5x \geq -10$

$\Rightarrow x \geq \frac{-10}{5}$

$\Rightarrow x \geq -2$

Domain (x)	-2	-1	0	1	2	....	∞
Range $(f(x))$	0	$\sqrt{5}$	$\sqrt{10}$	$\sqrt{15}$	$\sqrt{20}$	....	∞

Domain $= [-2, \infty)$

Range $= [0, \infty)$

---

③ Find the domain of $f(x) = \frac{x+4}{x^2-9}$.

Sol: Given $f(x) = \frac{x+4}{x^2-9}$.

$x^2 - 9 = 0 \Rightarrow x^2 = 9 \Rightarrow x = \sqrt{9} = \pm 3$

Domain $= (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

④ Find the domain of $f(x) = \frac{1}{\sqrt[4]{x^2-5x}}$.

Sol: Given $f(x) = \frac{1}{\sqrt[4]{x^2-5x}}$

For $x=0$, we get $x^2 - 5x = 0 - 0 = 0$

For $x=5$, we get $x^2 - 5x = 25 - 25 = 0$

Domain $= (-\infty, 0) \cup (5, \infty)$.

⑤ Find the domain & sketch the graph of the function

$$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0 \end{cases}$$

Sol: Given $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0 \end{cases}$

$x < 0$

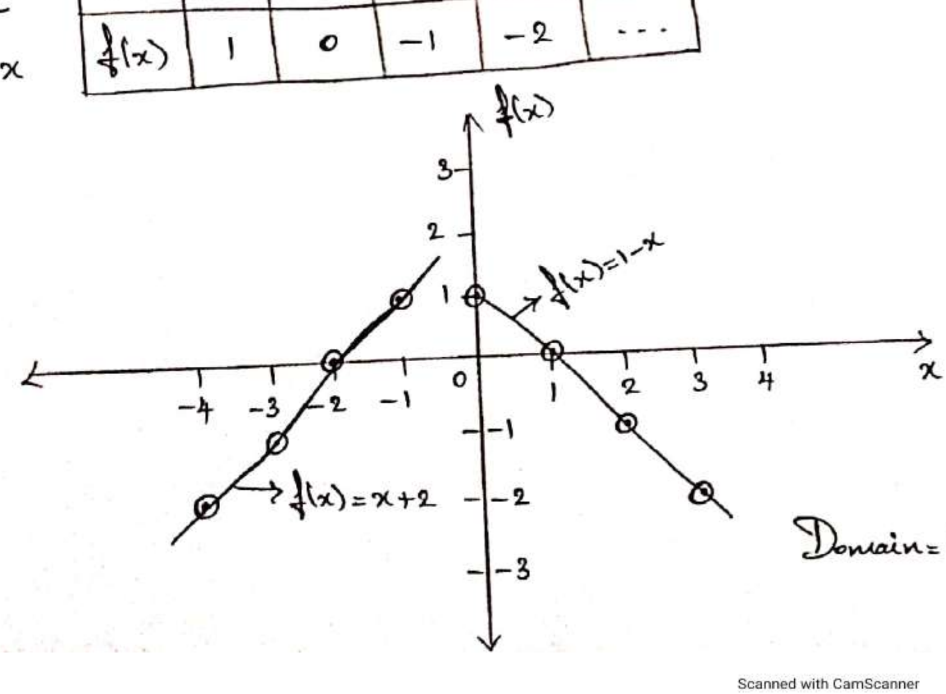
$f(x) = x+2$

x	-1	-2	-3	-4	...
$f(x)$	1	0	-1	-2	...

$x \geq 0$

$f(x) = 1-x$

x	0	1	2	3	...
$f(x)$	1	0	-1	-2	...



Domain $= (-\infty, \infty)$

AU

6 Find the domain of $f(x) = \sqrt{3-x} - \sqrt{2+x}$.

Sol: Given $f(x) = \sqrt{3-x} - \sqrt{2+x}$

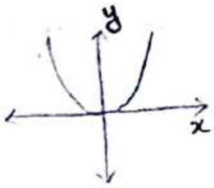
Here $3 - x \geq 0$ & $2 + x \geq 0$

$\Rightarrow 3 \geq x$ & $x \geq -2$

$\Rightarrow -2 \leq x \leq 3$

Domain $= [-2, 3]$

Definition:



Even function: $f(-x) = f(x)$ [(or) symmetric about the y-axis]

E.g: ① $f(x) = 1 - x^4$

$f(-x) = 1 - (-x)^4 = 1 - x^4 = f(x)$

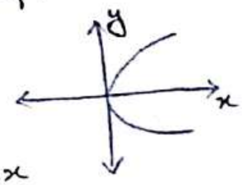
$\therefore f(-x) = f(x)$

Hence $f(x) = 1 - x^4$ is an even function.

② $f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x = f(x)$

$\therefore f(x) = \cos x$ is an even function.



Odd function: $f(-x) = -f(x)$ [(or) symmetric about the x-axis]

E.g: ① $f(x) = x^5 + x$

$f(-x) = (-x)^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$

$\therefore f(-x) = -f(x)$

Hence $f(x) = x^5 + x$ is an odd function.

② $f(x) = \sin x$

$f(-x) = \sin(-x)$

$= -\sin x = -f(x)$

Hence $f(x) = \sin x$ is an odd function.

Example for neither even nor odd function:

① $f(x) = \dfrac{1}{x-1}$

$f(-x) = \dfrac{1}{-x-1} \neq f(x) \neq -f(x)$

Hence the given function is neither even nor odd.

② $f(x) = e^x$

$f(-x) = e^{-x} \neq f(x) \neq -f(x)$.

Hence $f(x) = e^x$ is neither even nor odd function.

H.W

① Find the domain of $f(x) = \sqrt{x+2}$.

② Find the domain of $f(x) = \dfrac{1}{x^2 - x}$.

Limit of a function:

$\lim_{x \to a} f(x) = l$ is $f(x) \to l$ as $x \to a$ (or) $f(x)$ approaches l as x approaches a.

Left-hand limit:

$\lim_{x \to a^-} f(x) = l$

Here $x \to a^-$ means $x < a$.

Right-hand limit:

$\lim_{x \to a^+} f(x) = l$

Here $x \to a^+$ means $x > a$.

Definition:

$\lim_{x \to a} f(x) = l$ if & only if $\lim_{x \to a^-} f(x) = l$ & $\lim_{x \to a^+} f(x) = l$.

Problems:

(7) Guess the value of $\lim_{x \to 1} \frac{x-1}{x^2-1}$.

Sol: Here $f(x) = \frac{x-1}{x^2-1}$.

$x < 1$

x	$f(x)$
0.5	0.66667
0.6	0.625
0.7	0.58824
0.8	0.55556
0.9	0.52632
0.99	0.50251
0.999	0.50025

$x > 1$

x	$f(x)$
1.5	0.4
1.4	0.41667
1.3	0.43478
1.2	0.45455
1.1	0.47619
1.01	0.49751
1.001	0.49975

$$\therefore \lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$$

H.W

(1) Guess the value of $\lim_{x \to 0} \frac{\sin x}{x}$.

(AU) (8) Guess the value of the limit (if it exists) for the function $\lim_{x \to 0} \frac{e^{5x}-1}{x}$ by evaluating the function at the given numbers $x = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$ (correct to 6 decimal places)

Sol: Here $f(x) = \frac{e^{5x}-1}{x}$

x	f(x)
−0.5	1.83583
−0.1	3.934693
−0.01	4.877058
−0.001	4.987521
−0.0001	4.99875

x	f(x)
0.5	22.364988
0.1	6.487213
0.01	5.12711
0.001	5.012521
0.0001	5.00125

$$\therefore \lim_{x \to 0} \frac{e^{5x}-1}{x} = 5$$

(AU) (9) Evaluate $\lim_{t \to 1} \frac{t^4-1}{t^3-1}$.

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(c) = 0$ where c is a constant

Sol: $\lim_{t \to 1} \frac{t^4-1}{t^3-1} = \lim_{t \to 1} \frac{4t^3}{3t^2} = \frac{4}{3}$

(AU) (10) Given that $\lim_{x \to 2} f(x) = 4$ & $\lim_{x \to 2} g(x) = -2$. Find the limit that exists for $\lim_{x \to 2} \left[\frac{3f(x)}{g(x)}\right]$.

Sol: Given $\lim_{x \to 2} f(x) = 4$ & $\lim_{x \to 2} g(x) = -2$.

$$\therefore \lim_{x \to 2} \left[\frac{3f(x)}{g(x)}\right] = \frac{3(4)}{-2} = -6$$

(AU) (11) Sketch the graph of the function $f(x) = \begin{cases} 1+x, & x < -1 \\ x^2, & -1 \le x \le 1 \\ 2-x, & x \ge 1 \end{cases}$ & use it to determine the value of 'a' for which $\lim_{x \to a} f(x)$ exists?

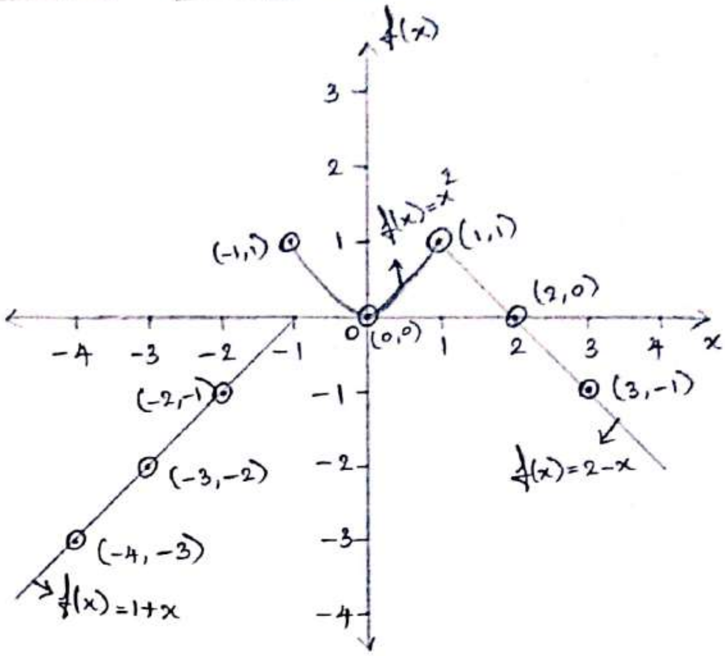
Sol:

Sol: $f(x)=1+x,\ x<-1$ $\quad$ $f(x)=x^2,\ -1\le x\le 1$ $\quad$ $f(x)=2-x,\ x\ge 1$

x	-2	-3	-4
$f(x)$	-1	-2	-3

x	-1	0	1
$f(x)$	1	0	1

x	1	2	3
$f(x)$	1	0	-1



At $x=-1$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (1+x) = 1+(-1) = 0$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x^2 = (-1)^2 = 1$$

$\therefore \lim_{x\to -1} f(x)$ doesn't exist.

At $x=1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 2-1 = 1$$

$\therefore \lim_{x\to 1} f(x)$ exists.

Hence $\lim_{x\to a} f(x)$ exists for all 'a' except at $a=-1$.

(12) Sketch the graph of the function $f(x)=\begin{cases} 1+\sin x & \text{if } x<0 \\ \cos x & \text{if } 0\le x\le \pi \\ \sin x & \text{if } x>\pi \end{cases}$ & use it to determine the value of 'a' for which $\lim_{x\to a} f(x)$ exists.

Sol:

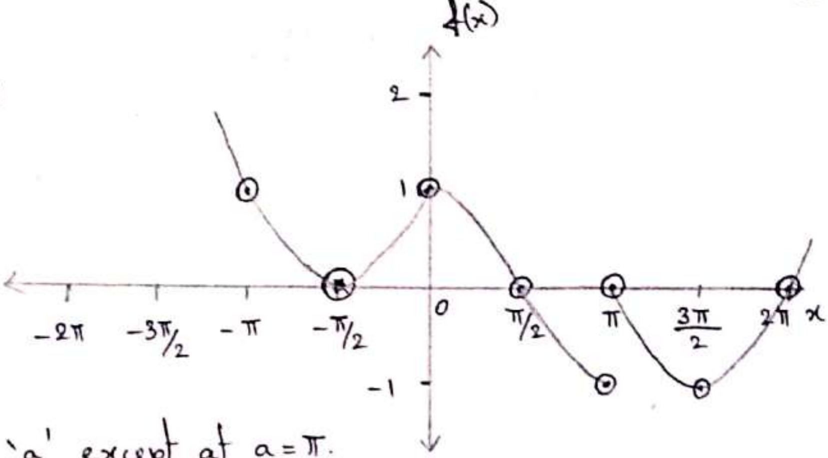
	$f(x)=1+\sin x,\ x<0$		$f(x)=\cos x,\ 0\le x\le\pi$			$f(x)=\sin x,\ x>\pi$	
x	$-\pi/2$	$-\pi$	0	$\pi/2$	π	$\frac{3\pi}{2}$	2π
$f(x)$	0	1	1	0	-1	-1	0

At $x=0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1+\sin x) = 1+\sin 0 = 1+0 = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \cos x = \cos 0 = 1$$

$\therefore \lim_{x\to 0} f(x)$ exists.



At $x = \pi$

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^-} \cos x = \cos \pi = -1$$

$$\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} \sin x = \sin \pi = 0$$

$\therefore \lim_{x \to \pi} f(x)$ doesn't exist.

Hence $\lim_{x \to a} f(x)$ exists for all 'a' except at $a = \pi$.

AU (13) Check whether $\lim_{x \to -3} \dfrac{3x+9}{|x+3|}$ exist.

Sol: $$\lim_{x \to -3^-} \frac{3x+9}{-(x+3)} = \lim_{x \to -3^-} \frac{3(x+3)}{-(x+3)} = -3$$

$$\lim_{x \to -3^+} \frac{3x+9}{x+3} = \lim_{x \to -3^+} \frac{3(x+3)}{x+3} = 3.$$ Here $\lim_{x \to -3^-} f(x) \neq \lim_{x \to -3^+} f(x)$

$\therefore \lim_{x \to -3} f(x)$ doesn't exist.

Definition: (Continuity)

A function f is continuous at 'a' if $\lim_{x \to a} f(x) = f(a)$.

(ii) If f is continuous at a, then

(i) $f(a)$ should exist

(ii) $\lim_{x \to a} f(x)$ exists both on the left & right.

(iii) $\lim_{x \to a} f(x) = f(a)$.

Eg: Polynomials, rational functions, root functions, trignometric functions, inverse trignometric functions, exponential functions, logarithmic functions.

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(14) Find the numbers that at which f is discontinuous, at which of these numbers if f is continuous from the right from the left or neither? When $f(x) = \begin{cases} x+2, & x < 0 \\ e^x, & 0 \le x \le 1 \\ 2-x, & x > 1 \end{cases}$.

Sol:

At $x=0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (x+2) = 0+2 = 2$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} e^x = e^0 = 1$

$f(0) = e^0 = 1$

$\therefore \lim\limits_{x\to 0^+} f(x) = f(0) \neq \lim\limits_{x\to 0^-} f(x)$.

Hence f is continuous on the right at $x=0$ & f is discontinuous on the left at $x=0$.

$\therefore f$ is discontinuous at $x=0$.

At $x=1$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} e^x = e^1 = e$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (2-x) = 2-1 = 1$

$f(1) = e^1 = e$

$\therefore \lim\limits_{x\to 1^-} f(x) = f(1) \neq \lim\limits_{x\to 1^+} f(x)$

Hence f is continuous on the left at $x=1$ & f is discontinuous on the right at $x=1$.

$\therefore f$ is discontinuous at $x=1$.

Thus f is continuous in $(-\infty,0)\cup(0,1)\cup(1,\infty)$.

(H.W) (1) Find the domain where the function f is continuous. Also find the numbers at which the function f is discontinuous, where

$$f(x) = \begin{cases} 1+x^2, & x \le 0 \\ 2-x, & 0 < x \le 2 \\ (x-2)^2, & x > 2 \end{cases}.$$

(AU) (15) For what value of the constant b, is the function f continuous on $(-\infty,\infty)$ if $f(x) = \begin{cases} bx^2+2x & \text{if } x<2 \\ x^3-bx & \text{if } x \ge 2 \end{cases}$.

Sol: At $x=2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (bx^2+2x) = 4b+4$$

$$f(2) = (2)^3 - b(2) = 8-2b$$

Since f is continuous, $\lim_{x\to 2^-} f(x) = f(2)$

$$\Rightarrow 4b+4 = 8-2b \Rightarrow 4b+2b = 8-4$$

$$\Rightarrow 6b = 4 \Rightarrow b = \frac{4}{6} = 2/3$$

$$\therefore \boxed{b = 2/3}$$

(AU) (16) Find the values of a & b that make f continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} \dfrac{x^3-8}{x-2}, & \text{if } x<2 \\ ax^2-bx+3, & \text{if } 2\le x<3 \\ 2x-a+b, & \text{if } x\ge 3 \end{cases}$$

$$\boxed{\frac{d(x^n)}{dx} = nx^{n-1}}$$

Sol: At $x=2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^3-8}{x-2} = \lim_{x\to 2^-} \frac{3x^2}{1} = \lim_{x\to 2^-} 3x^2 = 3(2)^2 = 12$$

$$f(2) = a(2)^2 - b(2) + 3 = 4a-2b+3$$

Since f is continuous, $\lim_{x\to 2^-} f(x) = f(2)$

$$\Rightarrow 12 = 4a-2b+3 \Rightarrow 4a-2b = 12-3 = 9 \Rightarrow 4a-2b = 9 \quad \text{—} (1)$$

At $x=3$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} ax^2-bx+3 = a(3)^2 - b(3)+3 = 9a-3b+3$$

$$f(3) = 2(3)-a+b = 6-a+b$$

Since f is continuous, $\lim_{x\to 3^-} f(x) = f(3)$

$$\Rightarrow 9a-3b+3 = 6-a+b \Rightarrow 9a+a-3b-b = 6-3$$

$$\Rightarrow 10a-4b = 3 \quad \text{—} (2)$$

$① \times 2 \Rightarrow \quad 8a - 4b = 18$

$\qquad 10a - 4b = 3 \quad — ②$

$\qquad (-) \; (+) \quad (-)$

$-2a \quad = 15 \Rightarrow a = \frac{15}{-2}$

$\therefore \boxed{a = \frac{-15}{2}}$

Substituting a value in ①, $4\left(\frac{-15}{2}\right) - 2b = 9$

$\Rightarrow -30 - 2b = 9 \Rightarrow 2b = -30 - 9 = -39 \Rightarrow \boxed{b = \frac{-39}{2}}$

Hence $a = \frac{-15}{2}$ & $b = \frac{-39}{2}$.

H.W

① If $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x < 2 \\ ax^2 - bx + 3, & 2 \leq x < 3 \\ 2x - a + b, & x \geq 3 \end{cases}$ is continuous for all real x, find the values of a & b.

Formulae:

① $\frac{d}{dx}(x^n) = nx^{n-1}$

② $\frac{d}{dx}(c) = 0$ where c is constant.

③ $\frac{d}{dx}(c\,f(x)) = c\frac{d}{dx}(f(x))$

④ Equation of tangent line is $y - y_1 = m(x - x_1)$ where $m = \frac{dy}{dx}$.

⑤ Equation of normal line is $y - y_1 = \frac{-1}{m}(x - x_1)$ where $m = \frac{dy}{dx}$.

Problems:

(17) Find the derivative of the following:-

(i) $f(x) = x^{1000}$

$f'(x) = 1000x^{1000-1} = 1000x^{999}$

(ii) $y = \frac{1}{x^2}$

$y = \frac{1}{x^2} = x^{-2}$

$y' = (-2)x^{-2-1} = (-2)x^{-3} = \frac{-2}{x^3}$

(iii) $y = \sqrt[3]{x^2}$

$y = (x^2)^{1/3} = x^{2/3}$

$y' = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

(iv) $y = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 7$

$y' = 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6$

$y' = 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$

(v) $y = ax^{2n} + bx^n + c$

$y' = a(2n)x^{2n-1} + bnx^{n-1} = 2anx^{2n-1} + bnx^{n-1}$

(vi) $y = \frac{x^2+4x+3}{\sqrt{x}}$

$y = x^{-1/2}(x^2+4x+3) = x^{-1/2}x^2 + 4x\,x^{-1/2} + 3x^{-1/2} = x^{3/2} + 4x^{1/2} + 3x^{-1/2}$

$y' = \frac{3}{2}x^{3/2-1} + \frac{1}{2} \times 4\, x^{1/2-1} + 3\left(-\frac{1}{2}\right)x^{-1/2-1}$

$= \frac{3}{2}x^{1/2} + 2x^{-1/2} - \frac{3}{2}x^{-3/2} = \frac{3}{2}\sqrt{x} + \frac{2}{\sqrt{x}} - \frac{3}{2}x^{-3/2}$

AU

(18) Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so where?

Sol: Given $y = x^4 - 2x^2 + 2$

Horizontal tangents occur where the derivative is zero.

(ie) $\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 4x = 0 \Rightarrow 4x(x^2-1) = 0$

$\Rightarrow x = 0, \; x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$\therefore x = 0, 1, -1$

x	-1	0	1
y	1	2	1

Hence the corresponding points are $(-1,1)$, $(0,2)$ & $(1,1)$.

(19) The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 7$, where s is measured in centimeters & t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

Sol:

$$\text{Velocity} = \frac{ds}{dt} = 6t^2 - 10t + 3$$

$$\text{Acceleration} = \frac{d^2s}{dt^2} = 12t - 10$$

$$\left[\frac{d^2s}{dt^2}\right]_{t=2} = 12(2) - 10 = 24 - 10 = 14$$

H.W Problem:

(1) Find the derivative of the following functions:

(i) $y = \sqrt{x}$ (ii) $y = x^{\sqrt{2}}$ (iii) $y = x^2(1-2x)$ (iv) $y = x^{2.4} + e^{2.4}$

Formulae:

(1) $\frac{d}{dx}(e^x) = e^x$ (2) $\frac{d}{dx}(e^{2x}) = e^{2x} \cdot 2 = 2e^{2x}$

Problems:

(20) Find the derivative of the following functions:

(i) $y = 3e^x + \frac{4}{\sqrt[3]{x}}$

$$y = 3e^x + \frac{4}{x^{1/3}} = 3e^x + 4x^{-1/3}$$

$$y' = 3e^x + 4\left(-\tfrac{1}{3}\right)x^{-1/3-1} = 3e^x - \frac{4}{3}x^{-4/3}$$

(ii) $y = a^x$

$$y = a^x = e^{\log a^x} = e^{x \log a} = e^{(\log a)x}$$

$$y' = e^{(\log a)x} \cdot \log a = \log a \, e^{x(\log a)} = \log a \, e^{\log a^x} = \log a \cdot a^x = a^x \log a$$

H.W (1) Find the derivative of the following functions:

(i) $y = e^x - x$

(ii) $y = 2^x$

(iii) $y = e^{-x} - 7$

Formulae:

① $\frac{d}{dx}(uv) = uv' + vu'$ ② $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

(21) Find $f'(x)$ & $f''(x)$ of $f(x) = x^4 e^x$.

Sol: Given $f(x) = x^4 e^x$

$u = x^4$, $v = e^x$

$u' = 4x^3$, $v' = e^x$

$d(uv) = uv' + vu'$

$f'(x) = x^4(e^x) + e^x(4x^3)$

$= x^4 e^x + 4e^x x^3 = e^x(x^4 + 4x^3)$

$u = e^x$, $v = x^4 + 4x^3$

$u' = e^x$, $v' = 4x^3 + 12x^2$

$f''(x) = e^x(4x^3 + 12x^2) + (x^4 + 4x^3)e^x$

$= e^x(4x^3 + 12x^2 + x^4 + 4x^3)$

$= e^x(x^4 + 8x^3 + 12x^2)$

(22) If $f(x) = \frac{x^2}{1+2x}$, then find $f'(x)$ & $f''(x)$.

Sol: Given $f(x) = \frac{x^2}{1+2x}$

$d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

$u = x^2$, $v = 1 + 2x$

$u' = 2x$, $v' = 2$

$f'(x) = \frac{(1+2x)(2x) - x^2(2)}{(1+2x)^2}$

$= \frac{2x + 4x^2 - 2x^2}{(1+2x)^2} = \frac{2x^2 + 2x}{(1+2x)^2}$

$u = 2x^2 + 2x$, $v = (1+2x)^2$

$u' = 4x + 2$, $v' = 2(1+2x) \cdot 2$

$v' = 4(1+2x)$

$f''(x) = \frac{(1+2x)^2(4x+2) - (2x^2+2x)4(1+2x)}{(1+2x)^4}$

$= \frac{(1+2x)\left[(1+2x)(4x+2) - 4(2x^2+2x)\right]}{(1+2x)^4}$

$= \frac{4x + 2 + 8x^2 + 4x - 8x^2 - 8x}{(1+2x)^3} = \frac{2}{(1+2x)^3}$

(AU) (23) If $f(x) = xe^x$ then find the expression for $f''(x)$.

Sol: Given $f(x) = xe^x$

$u = x$, $v = e^x$

$u' = 1$, $v' = e^x$

$d(uv) = uv' + vu'$

$f'(x) = xe^x + e^x(1) = xe^x + e^x$

$f''(x) = xe^x + e^x(1) + e^x = xe^x + 2e^x = e^x(x+2)$

(24) Find $\frac{dy}{dx}$ if $y = x^2 e^{2x}(x^2+1)^4$.

Sol: Given $y = x^2 e^{2x}(x^2+1)^4$.

$d(uv) = uv' + vu'$

$u = x^2 e^{2x}$ $\qquad\qquad$ $v = (x^2+1)^4$

$u' = x^2(e^{2x}.2) + e^{2x}(2x)$ $\qquad\qquad$ $v' = 4(x^2+1)^3(2x)$

$$\frac{dy}{dx} = x^2 e^{2x}\left(4(x^2+1)^3(2x)\right) + (x^2+1)^4\left(2x^2e^{2x} + 2xe^{2x}\right)$$

$$= (x^2+1)^3\left[8x^3e^{2x} + (x^2+1)\,2xe^{2x}(x+1)\right]$$

$$= (x^2+1)^3\,2xe^{2x}\left[4x^2 + (x^2+1)(x+1)\right]$$

$$= (x^2+1)^3\,2xe^{2x}\left[4x^2 + x^3 + x^2 + x + 1\right] = (x^2+1)^3\,2xe^{2x}(x^3+5x^2+x+1)$$

$$= 2xe^{2x}(x^2+1)^3(x^3+5x^2+x+1)$$

AU (25) If $f(x) = \frac{1-x}{2+x}$ then find the equation for $f'(x)$ using the concept of derivatives.

Sol: Given $f(x) = \frac{1-x}{2+x}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}}{h}$$

$$= \lim_{h\to 0}\frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x)(2+x+h)}$$

$$= \lim_{h\to 0}\frac{2+x-2x-x^2-2h-xh-(2+x+h-2x-x^2-xh)}{h(2+x)(2+x+h)}$$

$$= \lim_{h\to 0}\frac{\cancel{2}+\cancel{x}-\cancel{2x}-\cancel{x^2}-2h-\cancel{xh}-\cancel{2}-\cancel{x}-h+\cancel{2x}+\cancel{x^2}+\cancel{xh}}{h(2+x)(2+x+h)}$$

$$= \lim_{h\to 0}\frac{-3h}{h(2+x)(2+x+h)} = \lim_{h\to 0}\frac{-3}{(2+x)(2+x+h)}$$

$$= \frac{-3}{(2+x)(2+x)} = \frac{-3}{(2+x)^2}$$

Q: Differentiate the following functions

(i) $f(x) = (x^3+2x)e^x$ (ii) $f(x) = \sqrt{x}\,(a+bx)$

(iii) $f(x) = \dfrac{x^2+x-2}{x^3+6}$ (iv) $f(x) = \dfrac{e^x}{x}$

Formulae:

① $\dfrac{d}{dx}(\sin x) = \cos x$

② $\dfrac{d}{dx}(\cos x) = -\sin x$

③ $\dfrac{d}{dx}(\tan x) = \sec^2 x$

④ $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

⑤ $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

⑥ $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

⑦ $\dfrac{d}{dx}(\sin mx) = m\cos mx$

⑧ $\dfrac{d}{dx}(\cos mx) = -m\sin mx$

⑨ $\operatorname{cosec} x = \dfrac{1}{\sin x}$

⑩ $\sec x = \dfrac{1}{\cos x}$

⑪ $\cot x = \dfrac{1}{\tan x}$

⑫ $\tan x = \dfrac{\sin x}{\cos x}$

⑬ $\sin^2 x + \cos^2 x = 1$

⑭ $1+\tan^2 x = \sec^2 x$

⑮ $1+\cot^2 x = \operatorname{cosec}^2 x$

Problems:

(26) Find the derivative of the following:

(i) $y = \operatorname{cosec} x + e^x \cot x$

$$\boxed{\begin{array}{ll} u = e^x & v = \cot x \\ u' = e^x & v' = -\operatorname{cosec}^2 x \\ d(uv) = uv' + vu' & \end{array}}$$

$$y' = -\operatorname{cosec} x \cot x + \left[-e^x \operatorname{cosec}^2 x + e^x \cot x\right]$$

$$= -\operatorname{cosec} x \cot x + e^x\left(-\operatorname{cosec}^2 x + \cot x\right)$$

(ii) $y = \dfrac{\sec x}{1+\tan x}$

$$\boxed{\begin{array}{ll} u = \sec x, & v = 1+\tan x \\ u' = \sec x \tan x, & v' = \sec^2 x \\ d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} & \end{array}}$$

$$y' = \frac{(1+\tan x)\sec x \tan x - \sec x \sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\sec x\left[\tan x + \tan^2 x - \sec^2 x\right]}{(1+\tan x)^2} = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2} \quad (\because 1+\tan^2 x = \sec^2 x)$$

(27) Find the 25^{th} derivative of $\cos x$.

Sol: Given $f(x) = \cos x$.

$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x$$
$$f^{(4)}(x) = \cos x$$
$$\vdots$$
$$f^{(24)}(x) = \cos x$$
$$f^{(25)}(x) = -\sin x$$

H.W Problem:

(1) Find the derivative of the following:

(i) $y = \dfrac{\cos x}{1-\sin x}$ (ii) $y = \sin x \tan x$ (iii) $f(x) = xe^{x}\operatorname{cosec} x$

(2) Find $\dfrac{d^{99}}{dx^{99}}(\sin x)$.

Formulae:

(1) $\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1-x^2}}$

(2) $\dfrac{d}{dx}(\cos^{-1}x) = \dfrac{-1}{\sqrt{1-x^2}}$

(3) $\dfrac{d}{dx}(\tan^{-1}x) = \dfrac{1}{1+x^2}$

(4) $\dfrac{d}{dx}(\cot^{-1}x) = \dfrac{-1}{1+x^2}$

(5) $\dfrac{d}{dx}(\operatorname{cosec}^{-1}x) = \dfrac{-1}{x\sqrt{x^2-1}}$

(6) $\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{x\sqrt{x^2-1}}$

(7) $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$

(8) $\dfrac{d}{dx}(\sinh x) = \cosh x$

(9) $\dfrac{d}{dx}(\cosh x) = \sinh x$

(10) $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

(11) $\dfrac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

(12) $\dfrac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$

(13) $\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

(14) $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1+x^2}}$

(15) $\dfrac{d}{dx}(\cosh^{-1}x) = \dfrac{1}{\sqrt{x^2-1}}$

(16) $\dfrac{d}{dx}(\tanh^{-1}x) = \dfrac{1}{1-x^2}$

(17) $\dfrac{d}{dx}(\coth^{-1}x) = \dfrac{1}{1-x^2}$

(18) $\dfrac{d}{dx}(\operatorname{cosech}^{-1}x) = \dfrac{-1}{x\sqrt{x^2+1}}$

(19) $\frac{d}{dx}(\text{sech}^{-1}x) = \frac{-1}{x\sqrt{1-x^2}}$

(20) $\sinh x = \frac{e^x - e^{-x}}{2}$

(21) $\cosh x = \frac{e^x + e^{-x}}{2}$

(22) $\text{cosech}\, x = \frac{1}{\sinh x}$

(23) $\text{sech}\, x = \frac{1}{\cosh x}$

(24) $\tanh x = \frac{\sinh x}{\cosh x}$

(25) $\coth x = \frac{1}{\tanh x}$

(26) $\sinh(-x) = -\sinh x$

(27) $\cosh(-x) = \cosh x$

(28) $\cosh^2 x - \sinh^2 x = 1$

(29) $1 - \tanh^2 x = \text{sech}^2 x$

(28) Find y' if $y = \sqrt{\cos\sqrt{x}}$.

Sol: Given $y = \sqrt{\cos\sqrt{x}} = (\cos\sqrt{x})^{1/2}$

$$y' = \frac{1}{2}(\cos\sqrt{x})^{1/2 - 1}(-\sin\sqrt{x})\left(\frac{1}{2}x^{1/2 - 1}\right)$$

$$= \frac{1}{2}(\cos\sqrt{x})^{-1/2}(-\sin\sqrt{x})\left(\frac{1}{2}x^{-1/2}\right) = \frac{-\sin\sqrt{x}}{4\sqrt{\cos\sqrt{x}}\,\sqrt{x}}$$

(H.W) (1) Find y' if (i) $y = \sin^5 x$ (ii) $y = \cos(x^2)$ (iii) $y = e^{\sqrt{x}}$ (iv) $y = \sin(\sin(\sin x))$

(AU) (29) Find y'' if $x^4 + y^4 = 16$.

Sol: Given $x^4 + y^4 = 16$ — (1)

Differentiating (1), with respect to x, we get

$4x^3 + 4y^3 \cdot y' = 0 \Rightarrow x^3 + y^3 \cdot y' = 0$ — (2) $\Rightarrow y^3 y' = -x^3 \Rightarrow y' = \frac{-x^3}{y^3}$ — (3)

$$\boxed{\begin{array}{ll} u = y^3, & v = y' \\ u' = 3y^2 \cdot y' & v' = y'' \\ d(uv) = uv' + vu' & \end{array}}$$

Differentiating (2) with respect to x, we get

$$3x^2 + y^3 \cdot y'' + y' \cdot 3y^2 \cdot y' = 0$$

$$3x^2 + y^3 \cdot y'' + 3y^2\left(\frac{-x^3}{y^3}\right)^2 = 0$$

$$3x^2 + y^3 \cdot y'' + 3y^2\left(\frac{x^6}{y^6}\right) = 0 \Rightarrow 3x^2 + y^3 \cdot y'' + \frac{3x^6}{y^4} = 0$$

$$\Rightarrow y^3 y'' = -3x^2 - \frac{3x^6}{y^4} = -3x^2\left(1 + \frac{x^4}{y^4}\right) = -3x^2\left(\frac{y^4 + x^4}{y^4}\right) = -3x^2\left(\frac{16}{y^4}\right)$$

($\because$ by (1))

$$y'' = \frac{-48x^2}{y^7}$$

(AU)
30) Find y' for $\cos(xy) = 1 + \sin y$.

$u = x \quad , v = y$
$u' = 1 \quad , v' = y'$
$d(uv) = uv' + vu'$

Sol: Given $\cos(xy) = 1 + \sin y$. —— ①

Diff. ① w.r.t. x, we get

$-\sin(xy) \cdot (xy' + y \cdot 1) = \cos y \cdot y'$

$-xy' \sin(xy) - y\sin(xy) = \cos y \cdot y'$

$-y\sin(xy) = y'\cos y + xy'\sin(xy) = y'(\cos y + x\sin(xy))$

$$\therefore y' = \frac{-y\sin(xy)}{\cos y + x\sin(xy)}$$

(AU)
31) Find the derivative of $f(x) = \cos^{-1}\left(\dfrac{b + a\cos x}{a + b\cos x}\right)$.

$\dfrac{d}{dx}(\cos^{-1}x) = \dfrac{-1}{\sqrt{1-x^2}}$

Sol: Given $f(x) = \cos^{-1}\left(\dfrac{b + a\cos x}{a + b\cos x}\right)$

$$f'(x) = \frac{-1}{\sqrt{1 - \left(\dfrac{b + a\cos x}{a + b\cos x}\right)^2}} \left[\frac{(a + b\cos x)(-a\sin x) - (b + a\cos x)(-b\sin x)}{(a + b\cos x)^2}\right]$$

$u = b + a\cos x \quad , v = a + b\cos x$
$u' = -a\sin x \quad , v' = -b\sin x$
$d\left(\dfrac{u}{v}\right) = \dfrac{vu' - uv'}{v^2}$

$$f'(x) = \frac{-(a + b\cos x)}{\sqrt{(a + b\cos x)^2 - (b + a\cos x)^2}} \left[\frac{-a^2\sin x - \cancel{ab\sin x\cos x} + b^2\sin x + \cancel{ab\sin x\cos x}}{(a + b\cos x)^2}\right]$$

$$= \frac{-1}{\sqrt{a^2 + b^2\cos^2 x + \cancel{2ab\cos x} - b^2 - a^2\cos^2 x - \cancel{2ab\cos x}}} \left(\frac{\sin x (b^2 - a^2)}{a + b\cos x}\right)$$

$$= \frac{(a^2 - b^2)\sin x}{(a + b\cos x)\sqrt{(a^2 - b^2) - \cos^2 x(a^2 - b^2)}} = \frac{(a^2 - b^2)\sin x}{(a + b\cos x)\sqrt{(a^2 - b^2)(1 - \cos^2 x)}}$$

$$= \frac{(a^2-b^2)\sin x}{(a+b\cos x)\sqrt{(a^2-b^2)\sin^2 x}} \quad (\because \sin^2 x + \cos^2 x = 1)$$

$$= \frac{(a^2-b^2)\sin x}{(a+b\cos x)\sin x\sqrt{a^2-b^2}} = \frac{\sqrt{a^2-b^2}}{a+b\cos x}$$

AU

(32) Find the derivative of $f(x) = \tanh^{-1}\left[\tan\frac{x}{2}\right]$.

Sol: Given $f(x) = \tanh^{-1}\left[\tan\frac{x}{2}\right]$.

$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$$f'(x) = \frac{1}{1-\left(\tan\frac{x}{2}\right)^2}\left(\sec^2\frac{x}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{1-\tan^2\frac{x}{2}}\left(\sec^2\frac{x}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{1-\frac{\sin^2 x/2}{\cos^2 x/2}}\left(\sec^2 x/2\right)(1/2)$$

$$= \frac{\cos^2 x/2}{\cos^2 x/2 - \sin^2 x/2}\left(\frac{1}{2}\sec^2 x/2\right) = \frac{\cos^2 x/2}{\cos^2 x/2 - \sin^2 x/2}\left(\frac{1}{2\cos^2 x/2}\right)$$

$$= \frac{1}{2\left(\cos^2 x/2 - \sin^2 x/2\right)}$$

AU

(33) Find the tangent line to the equation $x^3+y^3=6xy$ at the point $(3,3)$ & at what point the tangent line horizontal in the first quadrant.

Sol: Given $x^3+y^3=6xy$ — ①

$$\boxed{\begin{array}{ll} u=x, & v=y \\ u'=1 & v'=y' \\ d(uv)=uv'+vu' & \end{array}}$$

Diff. ① w.r.t. x, we get

$$3x^2+3y^2\cdot y' = 6(xy'+y\cdot 1)$$

$$\Rightarrow 3x^2+3y^2y' = 6xy'+6y \Rightarrow 3y^2y'-6xy' = 6y-3x^2$$

$$\Rightarrow y'(3y^2-6x) = 6y-3x^2 \Rightarrow y' = \frac{6y-3x^2}{3y^2-6x}$$

$$(y')_{(3,3)} = \frac{6(3)-3(3)^2}{3(3)^2-6(3)} = \frac{18-27}{27-18} = \frac{-9}{9} = -1 = m \text{ (Slope)}$$

$$\boxed{\begin{array}{l} m=-1 \\ x_1=3 \\ y_1=3 \end{array}}$$

Equation of tangent line is $y-y_1 = m(x-x_1)$

$$y-3 = -1(x-3) \Rightarrow y-3 = -x+3$$

$$\Rightarrow x+y = 3+3 = 6 \Rightarrow x+y=6$$

The tangent line is horizontal if $y'=0$

(ie) $y' = \dfrac{6y-3x^2}{3y^2-6x} = \dfrac{2y-x^2}{y^2-2x} = 0$

$\Rightarrow 2y - x^2 = 0 \Rightarrow 2y = x^2 \Rightarrow y = \dfrac{x^2}{2}$ — (2)

Substituting (2) in (1),

$x^3 + \left(\dfrac{x^2}{2}\right)^3 = 6x\left(\dfrac{x^2}{2}\right) \Rightarrow x^3 + \dfrac{x^6}{8} = \dfrac{6x^3}{2} = 3x^3$

$\Rightarrow \dfrac{x^6}{8} = 3x^3 - x^3 = 2x^3 \Rightarrow \dfrac{x^3}{8} = 2 \rightarrow x^3 = 16 = 2^4$

$\Rightarrow \boxed{x = 2^{4/3}}$ — (3)

Subst. (3) in (2), $y = \dfrac{(2^{4/3})^2}{2} = \dfrac{2^{8/3}}{2} = 2^{8/3} \cdot 2^{-1} = 2^{8/3 - 1} = 2^{5/3}$

Hence the tangent line is horizontal at $(2^{4/3}, 2^{5/3})$.

(34) Find y' if $y = (\sin x)^{(\sin x)^{(\sin x)^{\cdot^{\cdot^{\infty}}}}}$

Sol: Given $y = (\sin x)^{(\sin x)^{(\sin x)^{\cdot^{\cdot^{\infty}}}}}$

$\Rightarrow y = (\sin x)^y$ — (*)

$\log y = \log(\sin x)^y = y\log(\sin x)$ — (1)

Diffl. (1) w.r.t. x, we get

$\dfrac{1}{y} y' = y \dfrac{1}{\sin x} \cos x + \log(\sin x) \cdot y'$

$\dfrac{1}{y} y' - \log(\sin x) \cdot y' = \dfrac{y}{\sin x} \cos x = y \cot x$

$y'\left(\dfrac{1}{y} - \log(\sin x)\right) = y\cot x \Rightarrow y'\left(\dfrac{1 - y\log(\sin x)}{y}\right) = y\cot x$

$\Rightarrow y' = \dfrac{y^2 \cot x}{1 - y\log(\sin x)} = \dfrac{y^2\cot x}{1 - \log(\sin x)^y} = \dfrac{y^2 \cot x}{1 - \log y}$ ($\because$ by (*))

(35) Find an equation of the normal line to the curve $y = \sqrt[4]{x}$ at the point $(1,1)$.

Sol: Given $y = \sqrt[4]{x}$, $(1,1)$.

$y = x^{1/4}$

$\frac{dy}{dx} = m = \frac{1}{4} x^{1/4 - 1} = \frac{1}{4} x^{-3/4}$

$(m)_{(1,1)} = \frac{1}{4}(1)^{-3/4} = \frac{1}{4} \times 1 = \frac{1}{4}$

Equation of the normal line is

$\boxed{m = \frac{1}{4} \quad x_1 = 1 \quad y_1 = 1}$

$y - y_1 = -\frac{1}{m}(x - x_1)$

$y - 1 = \frac{-1}{1/4}(x-1) \Rightarrow y - 1 = -4(x-1)$

$\Rightarrow y - 1 = -4x + 4 \Rightarrow 4x + y = 4 + 1 = 5 \Rightarrow 4x + y = 5$

H.W

1. If $x^3 + y^3 = 16$ find the value of $\frac{d^2y}{dx^2}$ at $(2, 2)$
2. Find $\frac{dy}{dx}$ if $y = x \sin^{-1} x + \sqrt{1 - x^2}$.
3. If $e^y \cos x = 1 + \sin(xy)$, then find $\frac{dy}{dx}$.
4. Find an equation of the tangent line to the curve $y \sin(2x) = x \cos(2y)$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$.

AU

(36) Find the critical points of $y = 5x^3 - 6x$.

Sol: Given $y = 5x^3 - 6x$.

Critical points: $y' = 0$ (i.e.) $\frac{dy}{dx} = 0$

$y' = 15x^2 - 6 = 0 \Rightarrow 15x^2 = 6 \Rightarrow x^2 = \frac{6}{15} = \frac{2}{5}$

$\therefore x = \pm\sqrt{\frac{2}{5}}$

Definition: (Critical number)

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

---

(37) Find the critical points of $f(x) = x^{3/5}(4 - x)$.

Sol: Given $f(x) = x^{3/5}(4-x) = 4x^{3/5} - x\,x^{3/5} = 4x^{3/5} - x^{8/5}$

Critical points: $f'(x) = 0$

$$f'(x) = 4\left(\frac{3}{5}\right)x^{\frac{3}{5}-1} - \frac{8}{5}x^{\frac{8}{5}-1} = 0$$

$$\Rightarrow \frac{12}{5}x^{-2/5} - \frac{8}{5}x^{3/5} = 0$$

$$\Rightarrow \frac{12}{5}x^{-2/5} = \frac{8}{5}x^{3/5} \Rightarrow \frac{12}{5} \times \frac{5}{8} = \frac{x^{3/5}}{x^{-2/5}} = x^{3/5}x^{2/5}$$

$$\Rightarrow \boxed{\frac{3}{2} = x}$$

$f'(x)$ doesn't exist when $x=0$.

Hence the critical points are 0 & $3/2$.

First derivative test:

Suppose that c is a critical number of a continuous function f.

(i) If f' changes from $+$ to $-$ at c, then f has a local maximum at c.

(ii) If f' changes from $-$ to $+$ at c, then f has a local minimum at c.

(iii) If f' does not change sign at c, then f has no local maximum or minimum at c.

Second derivative test:

Suppose f'' is continuous near c.

(i) If $f'(c)=0$ & $f''(c)>0$, then f has a local minimum at c.

(ii) If $f'(c)=0$ & $f''(c)<0$, then f has a local maximum at c.

AU 38) If $f(x) = 2x^3+3x^2-36x$, find the intervals on which it is increasing or decreasing, the local maximum & local minimum values of $f(x)$. Also find the intervals of concavity & the inflection points.

Sol: Given $f(x) = 2x^3+3x^2-36x$

$\Rightarrow f'(x) = 6x^2+6x-36$

$\times$	$+$
-6	1
$+3$	-2
$x+3$	$x-2$

Critical points: $f'(x)=0$

$f'(x) = 6x^2+6x-36=0 \Rightarrow x^2+x-6=0$

$\Rightarrow (x+3)(x-2)=0$

$\Rightarrow x = -3, 2$

Critical points are -3 & 2.

Interval	Sign of f'	Behaviour of f
$-\infty < x < -3$	$+$	increasing
$-3 < x < 2$	$-$	decreasing
$2 < x < \infty$	$+$	increasing

} local maximum

} local minimum

At $x=-3$, we get local maximum & at $x=2$, we get local minimum values.

$f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) = 81$

$f(2) = 2(2)^3 + 3(2)^2 - 36(2) = -44$

Hence the local maximum value is 81 & the local minimum value is -44.

$f''(x) = 12x + 6$

$f''(x) = 0 \Rightarrow 12x + 6 = 0 \Rightarrow 12x = -6 \Rightarrow x = \frac{-6}{12} = \frac{-1}{2}$

$\therefore \boxed{x = \frac{-1}{2}}$

$-\infty \quad -1/2 \quad \infty$

Interval	Sign of f''	Behaviour of f
$-\infty < x < -1/2$	$-$	Concave down
$-1/2 < x < \infty$	$+$	Concave up

Inflection points:

$f(-1/2) = 2(-1/2)^3 + 3\left(\frac{-1}{2}\right)^2 - 36\left(\frac{-1}{2}\right) = \frac{37}{2}$

Hence the inflection point is $\left(\frac{-1}{2}, \frac{37}{2}\right)$.

AU

(39) For the function $f(x) = 2 + 2x^2 - x^4$, find the intervals of increase or decrease, local maximum & minimum values, the intervals of concavity & the inflection points.

Sol: Given $f(x) = 2 + 2x^2 - x^4$

$f'(x) = 4x - 4x^3$

Critical points: $f'(x) = 0$

$f'(x) = 0 \Rightarrow 4x - 4x^3 = 0 \Rightarrow 4x(1 - x^2) = 0 \Rightarrow x = 0, \ 1 - x^2 = 0$

$\Rightarrow x = 0 \; , \; x^2 = 1 \Rightarrow x = \sqrt{1} = \pm 1$

Hence the critical points are $-1, 0$ & 1.

Interval	Sign of f'	Behaviour of f
$-\infty < x < -1$	$+$	increasing
$-1 < x < 0$	$-$	decreasing
$0 < x < 1$	$+$	increasing
$1 < x < \infty$	$-$	decreasing

local maximum

local minimum

local maximum

At $x = \pm 1$, we get local maximum value.

$$f(1) = 2 + 2(1)^2 - (1)^4 = 2 + 2 - 1 = 3$$

$\therefore$ Local maximum value is 3.

At $x = 0$, we get local minimum value.

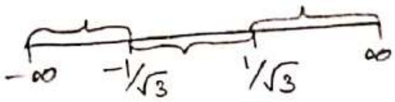
$$f(0) = 2 + 2(0)^2 - (0)^4 = 2$$

$\therefore$ Local minimum value is 2.

$$f''(x) = 4 - 12x^2$$

$$f''(x) = 0 \Rightarrow 4 - 12x^2 = 0 \Rightarrow 12x^2 = 4 \Rightarrow x^2 = \frac{4}{12} = 1/3 \Rightarrow x = \pm\sqrt{1/3}$$

$$\therefore \boxed{x = \pm\sqrt{\frac{1}{3}}} \Rightarrow \boxed{x = \pm\frac{1}{\sqrt{3}}}$$



Interval	Sign of f''	Behaviour of f
$-\infty < x < \frac{-1}{\sqrt{3}}$	$-$	Concave down
$\frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	$+$	Concave up
$\frac{1}{\sqrt{3}} < x < \infty$	$-$	Concave down

$$\boxed{\begin{array}{l} \frac{-1}{\sqrt{3}} = -0.6 \\ \frac{1}{\sqrt{3}} = 0.6 \end{array}}$$

<u>Inflection points:</u>

$$f\left(\frac{1}{\sqrt{3}}\right) = 2 + 2\left(\frac{1}{\sqrt{3}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^4 = 2 + \frac{2}{3} - \frac{1}{9} = \frac{23}{9}$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = 2 + 2\left(\frac{-1}{\sqrt{3}}\right)^2 - \left(\frac{-1}{\sqrt{3}}\right)^4 = 2 + \frac{2}{3} - \frac{1}{9} = \frac{23}{9}$$

Hence the inflection points are $\left(\frac{-1}{\sqrt{3}}, \frac{23}{9}\right)$ & $\left(\frac{1}{\sqrt{3}}, \frac{23}{9}\right)$.

40 (40) Find the local maximum & minimum values of $f(x)=\sqrt{x}-\sqrt[4]{x}$ using both the first & second derivative tests.

Sol: Given $f(x)=\sqrt{x}-\sqrt[4]{x} = x^{1/2}-x^{1/4}$

$$f'(x)=\frac{1}{2}x^{1/2-1}-\frac{1}{4}x^{1/4-1}=\frac{1}{2}x^{-1/2}-\frac{1}{4}x^{-3/4}$$

Critical points:

$$f'(x)=0 \Rightarrow \frac{1}{2}x^{-1/2}-\frac{1}{4}x^{-3/4}=0 \Rightarrow \frac{1}{2}x^{-1/2}=\frac{1}{4}x^{-3/4}$$

$$\Rightarrow \frac{4}{2}=\frac{x^{-3/4}}{x^{-1/2}} \Rightarrow 2=x^{-3/4}\cdot x^{1/2}=x^{-3/4+1/2}=x^{\frac{-3+2}{4}}=x^{-1/4}$$

$$\Rightarrow 2=x^{-1/4} \Rightarrow \frac{2}{x^{-1/4}}=1 \Rightarrow 2x^{1/4}=1 \Rightarrow x^{1/4}=1/2$$

$$\Rightarrow x=\left(1/2\right)^4=\frac{1}{16}$$

At $x=0$, $f'(x)$ doesn't exist.

Hence the critical points are 0 & $\frac{1}{16}$.

$\boxed{\frac{1}{16}=0.06}$

First derivative test:

Interval	Sign of f'	Behaviour of f
$-\infty<x<0$	(not defined)	(not defined)
$0<x<1/16$	$-$	decreasing
$1/16<x<\infty$	$+$	increasing

} local minimum

At $x=1/16$, we get local minimum value.

$$f\left(\frac{1}{16}\right)=\sqrt{1/16}-\sqrt[4]{1/16}=\frac{1}{4}-\frac{1}{2}=-1/4$$

Hence the local minimum value is $-1/4$.

Second derivative test:

$$f''(x)=\frac{1}{2}\left(\frac{-1}{2}\right)x^{-1/2-1}-1/4\left(-3/4\right)x^{-3/4-1}=\frac{-1}{4}x^{-3/2}+\frac{3}{16}x^{-7/4}$$

$$\therefore f''\left(\frac{1}{16}\right)=-16+24=8>0 \Rightarrow \text{local minimum at } x=1/16.$$

$$\therefore f(1/16)=-1/4$$

Hence the local minimum value is $-1/4$.

H.W

① For the function $f(x) = x^3 - 3x^2 + 1$, find the intervals of increase or decrease, local maximum & minimum values, the intervals of concavity & the inflection points.

② For the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$, find the intervals of increase or decrease, local maximum & minimum values, the intervals of concavity & the inflection points.

Functions Of Several Variables

1. Evaluate: $\lim\limits_{\substack{x\to\infty \\ y\to 2}} \dfrac{xy+5}{x^2+2y^2}$.

Sol: $\lim\limits_{\substack{x\to\infty \\ y\to 2}} \dfrac{xy+5}{x^2+2y^2} = \lim\limits_{x\to\infty}\left[\lim\limits_{y\to 2} \dfrac{xy+5}{x^2+2y^2}\right]$

$$= \lim_{x\to\infty}\left[\frac{x(2)+5}{x^2+2(2)^2}\right] = \lim_{x\to\infty}\left[\frac{2x+5}{x^2+8}\right]$$

$$= \lim_{x\to\infty}\left[\frac{x(2+5/x)}{x^2(1+8/x^2)}\right] = \lim_{x\to\infty}\left[\frac{2+5/x}{x(1+8/x^2)}\right]$$

$$= \frac{2+5/\infty}{\infty(1+8/\infty)} = \frac{2+0}{\infty(1+0)} = \frac{2}{\infty} = 0$$

H.W

1. Evaluate: $\lim\limits_{\substack{x\to 1 \\ y\to 2}} \dfrac{3x^2y}{x^2+y^2+5}$.

2. If $f(x,y) = \log\sqrt{x^2+y^2}$, show that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

Sol: Given $f(x,y) = \log\sqrt{x^2+y^2} = \log(x^2+y^2)^{1/2} = \frac{1}{2}\log(x^2+y^2)$.

$$\Rightarrow f(x,y) = \frac{1}{2}\log(x^2+y^2)$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\,\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}$$

w.r.t. x

$$\boxed{\begin{array}{ll} u = x & v = x^2+y^2 \\ u' = 1 & v' = 2x \\ \multicolumn{2}{l}{d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}} \end{array}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \text{—(1)}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\,\frac{1}{x^2+y^2}\cdot 2y = \frac{y}{x^2+y^2}$$

w.r.t. y

$$\boxed{\begin{array}{ll} u = y & v = x^2+y^2 \\ u' = 1 & v' = 2y \end{array}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)\cdot 1 - y(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{—(2)}$$

$$(1)+(2) \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$$

(3) If $x = r\cos\theta$, $y = r\sin\theta$, find (i) $\frac{\partial x}{\partial r}$ (ii) $\frac{\partial y}{\partial \theta}$ (iii) $\frac{\partial r}{\partial x}$ (iv) $\frac{\partial \theta}{\partial y}$ (AU)

Sol: Given $x = r\cos\theta$, $y = r\sin\theta$

(i) $\frac{\partial x}{\partial r} = \cos\theta$

(ii) $\frac{\partial y}{\partial \theta} = r\cos\theta$

We know that $r = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow r = (x^2+y^2)^{1/2}$ & $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

(iii) $\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{1/2-1} \cdot 2x = x(x^2+y^2)^{-1/2} = \frac{x}{(x^2+y^2)^{1/2}} = \frac{x}{\sqrt{x^2+y^2}}$

(iv) $\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{1}{\frac{x^2+y^2}{x^2}} \cdot \frac{1}{x}$ $\boxed{\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}}$

$= \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$

(AU) (4) Find $\frac{du}{dt}$ in terms of t, if $u = x^3+y^3$ where $x = at^2$, $y = 2at$.

Sol: Given $u = x^3+y^3$, $x = at^2$, $y = 2at$

$\therefore u = (at^2)^3 + (2at)^3 = a^3t^6 + 8a^3t^3$

$\frac{du}{dt} = a^3 6t^5 + 8a^3 3t^2 = 6a^3t^5 + 24a^3t^2 = 6a^3(t^5+4t^2) = 6a^3t^2(t^3+4)$

Euler's theorem on homogeneous function:

If u is a homogeneous function of degree n in x & y, then

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$

(H.W) (1) If $u = (x-y)(y-z)(z-x)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

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(5) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Sol: Given $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$\Rightarrow \tan u = \frac{x^3+y^3}{x-y} = f(x,y)$

$$f(tx,ty) = \frac{(tx)^3+(ty)^3}{tx-ty} = \frac{t^3x^3+t^3y^3}{tx-ty} = \frac{t^3(x^3+y^3)}{t(x-y)} = t^2\left(\frac{x^3+y^3}{x-y}\right) = t^2 f(x,y)$$

$\therefore f$ is a homogeneous function of degree 2 in x & y.

$\therefore$ By Euler's theorem, we get $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$ —①

Here $f = \tan u$

$$\frac{\partial f}{\partial x} = \sec^2 u \frac{\partial u}{\partial x}$$

$$\frac{\partial f}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$$

$$\therefore ① \Rightarrow x\sec^2 u \frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u \times \frac{1}{\sec^2 u} = 2\,\frac{\sin u}{\cos u} \times \cos^2 u = 2\sin u\cos u = \sin 2u$$

$$\boxed{2\sin A\cos A = \sin 2A}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

H.w

① If $u = \sin^{-1}\left(\frac{x^3-y^3}{x+y}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u$.

② If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$

AU

⑥ Verify the Euler's theorem for the function $u = x^2+y^2+2xy$.

Sol: Given $u = x^2+y^2+2xy = u(x,y)$

Euler's theorem: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

LHS $\frac{\partial u}{\partial x} = 2x+2y$, $\frac{\partial u}{\partial y} = 2y+2x$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x(2x+2y) + y(2y+2x) = 2x^2+2xy+2y^2+2xy$$
$$= 2x^2+2y^2+4xy$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2+2y^2+4xy \quad —①$$

RHS $u(tx,ty) = (tx)^2+(ty)^2+2(tx)(ty)$
$$= t^2x^2+t^2y^2+2t^2xy = t^2(x^2+y^2+2xy) = t^2u(x,y)$$

$\therefore u$ is a homogeneous function of degree 2 in x & y.

$\therefore n = 2$

$nu = 2(x^2+y^2+2xy) = 2x^2+2y^2+4xy$ — (2)

From (1) & (2), $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$

Definition:

A function $f(x,y)$ is said to be a homogeneous function of degree n in x & y, if $f(tx,ty) = t^n f(x,y)$ for any positive t.

(7) If $u = \sin^{-1}\left(\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$, then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} + 3\tan u = 0$.

Sol: Given $u = \sin^{-1}\left(\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right) \Rightarrow \sin u = \dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}} = f(x,y,z)$

$$f(tx,ty,tz) = \frac{tx+2ty+3tz}{\sqrt{(tx)^8+(ty)^8+(tz)^8}} = \frac{t(x+2y+3z)}{t^4\sqrt{x^8+y^8+z^8}} = t^{-3}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$$

$$= t^{-3}f(x,y,z)$$

$\therefore f$ is a homogeneous function of degree (-3) in x, y & z.

$\therefore$ By Euler's theorem, we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf = -3f \quad (\because n=-3) \text{ — (1)}$$

Here $f = \sin u$

$\dfrac{\partial f}{\partial x} = \cos u\dfrac{\partial u}{\partial x}$, $\dfrac{\partial f}{\partial y} = \cos u\dfrac{\partial u}{\partial y}$, $\dfrac{\partial f}{\partial z} = \cos u\dfrac{\partial u}{\partial z}$ — (2)

Subst. (2) in (1),

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -3\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{-3\sin u}{\cos u} = -3\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0$$

(8) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Sol: Given $u(x, y, z) = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$

$$u(tx, ty, tz) = f\left(\frac{tx}{ty}, \frac{ty}{tz}, \frac{tz}{tx}\right) = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right) = t^0 u(x, y, z)$$

$\therefore u$ is a homogeneous function of degree 0 in x, y & z.

$\therefore$ By Euler's theorem, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 0.u = 0 \quad (\because n = 0)$$

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(9) If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then prove that (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4\cos^3 u}$.

Sol: Given $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}} \Rightarrow \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = f(x, y)$

$$f(tx, ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t(x+y)}{\sqrt{t}(\sqrt{x}+\sqrt{y})} = \sqrt{t}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) = t^{1/2} f(x, y)$$

$\therefore f$ is a homogeneous function of degree $1/2$ in x & y.

$\therefore$ By Euler's theorem, we get

(i) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf = \frac{1}{2}f$ — (1)

Here $f = \sin u$

$\frac{\partial f}{\partial x} = \cos u \frac{\partial u}{\partial x}$, $\frac{\partial f}{\partial y} = \cos u \frac{\partial u}{\partial y}$ — (2)

Subst. (2) in (1), $x\cos u \frac{\partial u}{\partial x} + y\cos u \frac{\partial u}{\partial y} = \frac{1}{2}\sin u$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{1}{2}\tan u \quad \text{— (3)}$$

(ii) Diff. (3) partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}.1 + y\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\left(\frac{1}{2}\sec^2 u - 1\right) \quad \text{— (4)}$$

Diff. (3) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\cdot 1=\frac{1}{2}\sec^2 u\,\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}-\frac{\partial u}{\partial y}=\frac{\partial u}{\partial y}\left(\frac{1}{2}\sec^2 u-1\right) \quad \text{--- (5)}$$

$(④ \times x)+(⑤\times y) \Rightarrow$

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=x\frac{\partial u}{\partial x}\left(\frac{1}{2}\sec^2 u-1\right)+y\frac{\partial u}{\partial y}\left(\frac{1}{2}\sec^2 u-1\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=\left(\frac{1}{2}\sec^2 u-1\right)\frac{1}{2}\tan u \quad (\because \text{by } (3))$$

$$=\left(\frac{1}{2\cos^2 u}-1\right)\frac{1}{2}\tan u=\left(\frac{1-2\cos^2 u}{2\cos^2 u}\right)\frac{1}{2}\frac{\sin u}{\cos u}$$

$$=-\left(\frac{2\cos^2 u-1}{2\cos^2 u}\right)\frac{1}{2}\frac{\sin u}{\cos u}$$

$$=-\frac{\sin u\cos 2u}{4\cos^3 u}$$

$$\boxed{\begin{aligned}\cos^2 u&=\frac{1+\cos 2u}{2}\\ 2\cos^2 u&=1+\cos 2u\\ 2\cos^2 u-1&=\cos 2u\end{aligned}}$$

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(10) If $u=(x-y)f\left(\frac{y}{x}\right)$, then find $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$.

<u>Sol:</u> Given $u=(x-y)f\left(\frac{y}{x}\right)$

$$u(tx,ty)=(tx-ty)f\left(\frac{ty}{tx}\right)=t(x-y)f\left(\frac{y}{x}\right)=t\,u(x,y)$$

$\therefore u$ is a homogeneous function of degree 1 in x & y.

$\therefore$ By Euler's theorem, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u=1(1-1)u=1(0)u=0 \quad (\because n=1)$$

H.W

(1) If $u=\frac{x^2+y^2}{\sqrt{x+y}}$, then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{3}{2}u$.

(2) If $u=\cos^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$, then prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$.

(3) If $u=\sin^{-1}\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$, then (i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$.

Jacobians:

(AU) (11) If $x=r\cos\theta$ & $y=r\sin\theta$, then find $\frac{\partial(x,y)}{\partial(r,\theta)}$.

Sol: Given $x=r\cos\theta$, $y=r\sin\theta$

$\frac{\partial x}{\partial r}=\cos\theta$ $\qquad$ $\frac{\partial y}{\partial r}=\sin\theta$

$\frac{\partial x}{\partial \theta}=-r\sin\theta$ $\qquad$ $\frac{\partial y}{\partial \theta}=r\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=\cos\theta(r\cos\theta)+r\sin\theta(\sin\theta)$$

$$=r(\cos^2\theta+\sin^2\theta)=r \quad (\because \cos^2\theta+\sin^2\theta=1)$$

(AU) (12) If $x=uv$ & $y=\frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$.

Sol: Given $x=uv$, $y=\frac{u}{v}=uv^{-1}$

$\frac{\partial x}{\partial u}=v$ $\qquad$ $\frac{\partial y}{\partial u}=\frac{1}{v}$

$\frac{\partial x}{\partial v}=u$ $\qquad$ $\frac{\partial y}{\partial v}=u(-1)v^{-1-1}=-uv^{-2}=\frac{-u}{v^2}$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}v & u\\ \frac{1}{v} & \frac{-u}{v^2}\end{vmatrix}=v\left(\frac{-u}{v^2}\right)-u\left(\frac{1}{v}\right)$$

$$=\frac{-u}{v}-\frac{u}{v}=\frac{-2u}{v}$$

(HW) (AU) (1) If $x=u^2-v^2$, $y=2uv$ find the Jacobian of x,y with respect to u & v.
[Hint: $\frac{\partial(x,y)}{\partial(u,v)}$]

(AU) (13) State the properties of Jacobians.

Sol: (1) If u & v are the functions of x & y, then

$$\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(u,v)}=1.$$

(2) If u,v are functions of x,y & x,y are functions of r,s then $\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(r,s)}=\frac{\partial(u,v)}{\partial(r,s)}$.

(3) If u,v,w are functionally dependent functions of three independent variables x,y,z then $\frac{\partial(u,v,w)}{\partial(x,y,z)}=0$.

(14) If $u = 2xy$, $v = x^2 - y^2$ & $x = r\cos\theta$, $y = r\sin\theta$. Evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Sol: Given $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial u}{\partial x} = 2y \qquad \frac{\partial v}{\partial x} = 2x \qquad \frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial u}{\partial y} = 2x \qquad \frac{\partial v}{\partial y} = -2y \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \cdot \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-4y^2 - 4x^2)(r\cos^2\theta + r\sin^2\theta)$$

$$= -4(x^2+y^2)r(\cos^2\theta + \sin^2\theta) = -4(x^2+y^2)r \quad (\because \cos^2\theta + \sin^2\theta = 1)$$

$$= -4r^3 \quad (\because x^2+y^2 = r^2)$$

(15) Show that the functions $u = x+y-z$, $v = x-y+z$, $w = x^2+y^2+z^2-2yz$ are dependent. Find the relation between them.

Sol: Given $u = x+y-z$, $v = x-y+z$, $w = x^2+y^2+z^2-2yz$.

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 1 \qquad \frac{\partial w}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial y} = -1 \qquad \frac{\partial w}{\partial y} = 2y - 2z$$

$$\frac{\partial u}{\partial z} = -1 \qquad \frac{\partial v}{\partial z} = 1 \qquad \frac{\partial w}{\partial z} = 2z - 2y$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix}$$

$$= 1\left(-1(2z-2y) - 1(2y-2z)\right) - 1(2z-2y-2x) - 1(2y-2z+2x)$$

$$= \cancel{-2z} + \cancel{2y} - \cancel{2y} + \cancel{2z} - \cancel{2z} + \cancel{2y} + \cancel{2x} - \cancel{2y} + \cancel{2z} - \cancel{2x}$$

$$= 0$$

$\therefore u, v$ & w are functionally dependent.

$u+v = x+y-z+x-y+z = 2x \Rightarrow u+v = 2x$ — ①

$u-v = x+y-z-(x-y+z) = x+y-z-x+y-z = 2y-2z$

$\Rightarrow u-v = 2y-2z$ — ②

$\therefore (u+v)^2 + (u-v)^2 = (2x)^2 + (2y-2z)^2 = 4x^2 + 4(y-z)^2$

$= 4x^2 + 4(y^2+z^2-2yz) = 4(x^2+y^2+z^2-2yz) = 4w$ ($\because$ Given)

$\Rightarrow u^2+v^2+2uv+u^2+v^2-2uv = 4w$

$\Rightarrow 2u^2+2v^2 = 4w \Rightarrow u^2+v^2 = 2w$

H.W ① Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3, if

$y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$.

② Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ of the transformation $x = r\sin\theta\cos\phi$,

$y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

③ Prove $u = x+y+z$, $v = xy+yz+zx$, $w = x^2+y^2+z^2$ are functionally dependent. Find the relationship between them.

AU (1b) For the given function $z = \tan^{-1}\left(\frac{x}{y}\right) - (xy)$, verify whether the statement $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$, is correct or not.

$$\boxed{d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \qquad \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}}$$

Sol: Given $z = \tan^{-1}\left(\frac{x}{y}\right) - (xy)$

LHS $\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\, x\left(\frac{-1}{y^2}\right) - x = \frac{y^2}{y^2+x^2} \cdot \frac{-x}{y^2} - x = \frac{-x}{x^2+y^2} - x$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x^2+y^2)(-1) - (-x)2x}{(x^2+y^2)^2} - 1 = \frac{-x^2-y^2+2x^2}{(x^2+y^2)^2} - 1 = \frac{x^2-y^2}{(x^2+y^2)^2} - 1 \quad \text{— ①}$$

RHS $\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} - y = \frac{y^2}{y^2+x^2} \cdot \frac{1}{y} - y = \frac{y}{x^2+y^2} - y$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{(x^2+y^2)\cdot 1 - y(2y)}{(x^2+y^2)^2} - 1 = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} - 1 = \frac{x^2-y^2}{(x^2+y^2)^2} - 1 \quad \text{— ②}$$

From ① & ②, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

AU (17) If $u=(x^2+y^2+z^2)^{-1/2}$ then find the value of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$.

Sol: Given $u=(x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial u}{\partial x}=\frac{-1}{2}(x^2+y^2+z^2)^{-1/2-1}\cdot 2x = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2}=-\left[x\left(-3/2\right)(x^2+y^2+z^2)^{-3/2-1}(2x)+(x^2+y^2+z^2)^{-3/2}\cdot 1\right]$$

$$=3x^2(x^2+y^2+z^2)^{-5/2}-(x^2+y^2+z^2)^{-3/2} \quad \text{—(1)}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2}=3y^2(x^2+y^2+z^2)^{-5/2}-(x^2+y^2+z^2)^{-3/2} \quad \text{—(2)}$$

$$\frac{\partial^2 u}{\partial z^2}=3z^2(x^2+y^2+z^2)^{-5/2}-(x^2+y^2+z^2)^{-3/2} \quad \text{—(3)}$$

(1)+(2)+(3) $\Rightarrow$ $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2)-3(x^2+y^2+z^2)^{-3/2}$

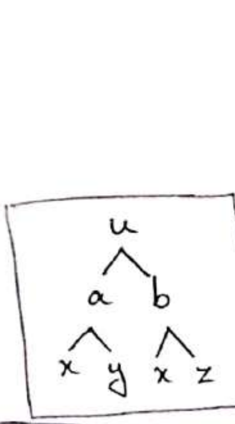
$$=3(x^2+y^2+z^2)^{-\frac{5}{2}+1}-3(x^2+y^2+z^2)^{-3/2}$$

$$=3(x^2+y^2+z^2)^{-3/2}-3(x^2+y^2+z^2)^{-3/2}=0$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

AU (18) If $u=f\left(\frac{y-x}{xy},\frac{z-x}{xz}\right)$, find $x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}$.

Sol: Given $u=f\left(\frac{y-x}{xy},\frac{z-x}{xz}\right)$



Let $a=\frac{y-x}{xy}$, $b=\frac{z-x}{xz}$

$u=f(a,b)$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial x}+\frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial x}$$

$$=\frac{\partial u}{\partial a}\cdot\frac{-1}{x^2}+\frac{\partial u}{\partial b}\cdot\frac{-1}{x^2}$$

$$\frac{\partial u}{\partial x}=-\frac{1}{x^2}\frac{\partial u}{\partial a}-\frac{1}{x^2}\frac{\partial u}{\partial b}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial y}=\frac{\partial u}{\partial a}\cdot\frac{1}{x}\left[\frac{y(1)-(y-x)\cdot 1}{y^2}\right]$$

$$\frac{\partial a}{\partial x}=\frac{1}{y}\left[\frac{x(-1)-(y-x)\cdot 1}{x^2}\right]$$

$$=\frac{1}{y}\left[\frac{-x-y+x}{x^2}\right]=\frac{1}{y}\left(\frac{-y}{x^2}\right)$$

$$\frac{\partial a}{\partial x}=\frac{-1}{x^2}$$

$$\frac{\partial b}{\partial x}=\frac{1}{z}\left[\frac{x(-1)-(z-x)\cdot 1}{x^2}\right]$$

$$=\frac{1}{z}\left[\frac{-x-z+x}{x^2}\right]=\frac{-1}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\cdot\frac{1}{x}\left[\frac{y-y+x}{y^2}\right] = \frac{1}{y^2}\frac{\partial u}{\partial a}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial z} = \frac{\partial u}{\partial b}\cdot\frac{1}{x}\left[\frac{z(1)-(z-x)\cdot 1}{z^2}\right]$$

$$= \frac{\partial u}{\partial b}\cdot\frac{1}{x}\left[\frac{z-z+x}{z^2}\right] = \frac{1}{z^2}\frac{\partial u}{\partial b}$$

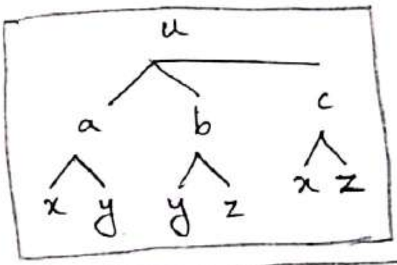
$$\therefore x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial a} - \frac{\partial u}{\partial b} + \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} = 0$$

AU (19) If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$, then find $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$.

Sol: Given $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$

Let $a = 2x-3y$, $b = 3y-4z$, $c = 4z-2x$

$u = f(a,b,c)$



$$\boxed{\begin{array}{l} \frac{\partial a}{\partial x} = 2,\ \frac{\partial c}{\partial x} = -2 \\ \frac{\partial a}{\partial y} = -3,\ \frac{\partial b}{\partial y} = 3 \\ \frac{\partial b}{\partial z} = -4,\ \frac{\partial c}{\partial z} = 4 \end{array}}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial x} = 2\frac{\partial u}{\partial a} - 2\frac{\partial u}{\partial c}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial y} = -3\frac{\partial u}{\partial a} + 3\frac{\partial u}{\partial b}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial z} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial z} = -4\frac{\partial u}{\partial b} + 4\frac{\partial u}{\partial c}$$

$$\therefore \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a} - \frac{\partial u}{\partial c} - \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} - \frac{\partial u}{\partial b} + \frac{\partial u}{\partial c} = 0$$

AU (20) Find $\frac{dy}{dx}$, if $x^y + y^x = c$, where c is a constant.

$$\boxed{\frac{d}{dx}(a^x) = a^x \log a}$$

Sol: Given $x^y + y^x = c$

Let $f(x,y) = x^y + y^x - c = 0$

$$\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y \qquad \frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

$$\therefore \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\left(\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right)$$

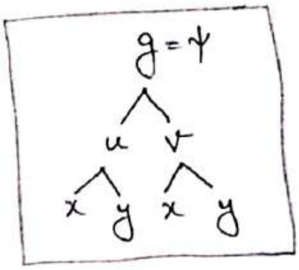
H.W

(1) If $u = f(y-z,\ z-x,\ x-y)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

(2) Find $\frac{dy}{dx}$ when $y\sin x = x\cos y$.

(21) If $g(x,y)=\psi(u,v)$ where $u=x^2-y^2$ & $v=2xy$, then prove that

$$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}=4(x^2+y^2)\left[\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}\right].$$



Sol: Given $g(x,y)=\psi(u,v)$, $u=x^2-y^2$, $v=2xy$.

$$\frac{\partial u}{\partial x}=2x \qquad \frac{\partial v}{\partial x}=2y$$

$$\frac{\partial u}{\partial y}=-2y \qquad \frac{\partial v}{\partial y}=2x$$

$$\frac{\partial g}{\partial x}=\frac{\partial\psi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\psi}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial g}{\partial x}=2x\frac{\partial\psi}{\partial u}+2y\frac{\partial\psi}{\partial v}$$

$$\frac{\partial}{\partial x}=2x\frac{\partial}{\partial u}+2y\frac{\partial}{\partial v}$$

$$\frac{\partial g}{\partial y}=\frac{\partial\psi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\psi}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial g}{\partial y}=-2y\frac{\partial\psi}{\partial u}+2x\frac{\partial\psi}{\partial v}$$

$$\frac{\partial}{\partial y}=-2y\frac{\partial}{\partial u}+2x\frac{\partial}{\partial v}$$

$$\frac{\partial^2 g}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right)=\left(2x\frac{\partial}{\partial u}+2y\frac{\partial}{\partial v}\right)\left(2x\frac{\partial\psi}{\partial u}+2y\frac{\partial\psi}{\partial v}\right)$$

$$\frac{\partial^2 g}{\partial x^2}=4x^2\frac{\partial^2\psi}{\partial u^2}+4xy\frac{\partial^2\psi}{\partial u\partial v}+4xy\frac{\partial^2\psi}{\partial v\partial u}+4y^2\frac{\partial^2\psi}{\partial v^2} \quad \text{—(1)}$$

$$\frac{\partial^2 g}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right)=\left(-2y\frac{\partial}{\partial u}+2x\frac{\partial}{\partial v}\right)\left(-2y\frac{\partial\psi}{\partial u}+2x\frac{\partial\psi}{\partial v}\right)$$

$$\frac{\partial^2 g}{\partial y^2}=4y^2\frac{\partial^2\psi}{\partial u^2}-4xy\frac{\partial^2\psi}{\partial u\partial v}-4xy\frac{\partial^2\psi}{\partial v\partial u}+4x^2\frac{\partial^2\psi}{\partial v^2} \quad \text{—(2)}$$

$$(1)+(2) \Rightarrow \frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}=4x^2\frac{\partial^2\psi}{\partial u^2}+4y^2\frac{\partial^2\psi}{\partial v^2}+4y^2\frac{\partial^2\psi}{\partial u^2}+4x^2\frac{\partial^2\psi}{\partial v^2}$$

$$=\frac{\partial^2\psi}{\partial u^2}(4x^2+4y^2)+\frac{\partial^2\psi}{\partial v^2}(4x^2+4y^2)$$

$$=(4x^2+4y^2)\left(\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}\right)$$

$$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}=4(x^2+y^2)\left(\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}\right)$$

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(22) If $u=\log(x^3+y^3+z^3-3xyz)$, show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\frac{-9}{(x+y+z)^2}$.

Sol: Given $u=\log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x}=\frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)=\frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz},$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{3(y^2 - xz)}{x^3+y^3+z^3-3xyz} \quad \& \quad \frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2-yz+y^2-xz+z^2-xy)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-xz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-xz)} = \frac{3}{x+y+z}$$

$$\Rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z} = 3(x+y+z)^{-1}$$

$$\frac{\partial}{\partial x}\left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right) = 3(-1)(x+y+z)^{-1-1}\cdot 1 = \frac{-3}{(x+y+z)^2} \quad \text{—} (1)$$

Similarly,

$$\frac{\partial}{\partial y}\left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right) = \frac{-3}{(x+y+z)^2} \quad \text{—} (2)$$

$$\frac{\partial}{\partial z}\left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right) = \frac{-3}{(x+y+z)^2} \quad \text{—} (3)$$

$$(1)+(2)+(3) \Rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right) = \frac{-9}{(x+y+z)^2}$$

$$\Rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

H.W

(1) If $z = f(x,y)$ where $x = r\cos\theta$ & $y = r\sin\theta$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

(2) If z is a function of x & y & u & v are other two variables, such that $u = lx+my$, $v = ly-mx$. Show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$

Taylor's series:

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$
$$+ \cdots$$

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(23) Expand $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x+2)$ & $(y-1)$ using Taylor's series upto third degree terms.

Sol:

Function	$x=-2,\ y=1$ $\boxed{(-2,1)}$
$f(x,y) = x^2y^2 + 2x^2y + 3xy^2$	$f = (-2)^2(1)^2 + 2(-2)^2(1) + 3(-2)(1)^2 = 4+8-6 = 6$
$f_x = 2xy^2 + 4xy + 3y^2$ $f_y = 2x^2y + 2x^2 + 6xy$	$f_x = 2(-2)(1)^2 + 4(-2)(1) + 3(1)^2 = -4-8+3 = -9$ $f_y = 2(-2)^2(1) + 2(-2)^2 + 6(-2)(1) = 8+8-12 = 4$
$f_{xx} = 2y^2 + 4y$ $f_{xy} = 4xy + 4x + 6y$ $f_{yy} = 2x^2 + 6x$	$f_{xx} = 2(1)^2 + 4(1) = 2+4 = 6$ $f_{xy} = 4(-2)(1) + 4(-2) + 6(1) = -8-8+6 = -10$ $f_{yy} = 2(-2)^2 + 6(-2) = 8-12 = -4$
$f_{xxx} = 0$ $f_{xxy} = 4y+4$ $f_{xyy} = 4x+6$ $f_{yyy} = 0$	$f_{xxx} = 0$ $f_{xxy} = 4(1)+4 = 8$ $f_{xyy} = 4(-2)+6 = -8+6 = -2$ $f_{yyy} = 0$

By Taylor's theorem,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$
$$+ \cdots$$

Here $a=-2,\ b=1$; $h = x-a = x-(-2) = x+2$, $k = y-b = y-1$

$$f(x,y) = 6 + \frac{1}{1!}\left[(x+2)(-9) + (y-1)(4)\right]$$
$$+ \frac{1}{2!}\left[(x+2)^2(6) + 2(x+2)(y-1)(-10) + (y-1)^2(-4)\right]$$
$$+ \frac{1}{3!}\left[(x+2)^3(0) + 3(x+2)^2(y-1)(8) + 3(x+2)(y-1)^2(-2) + (y-1)^3(0)\right]$$
$$+ \cdots$$
$$= 6 - 9(x+2) + 4(y-1) + \frac{1}{2}\left[6(x+2)^2 - 20(x+2)(y-1) - 4(y-1)^2\right]$$
$$+ \frac{1}{6}\left[24(x+2)^2(y-1) - 6(x+2)(y-1)^2\right] + \cdots$$

H.W (1) Obtain the Taylor's series expansion of $x^3+y^3+xy^2$ in terms of powers of $(x-1)$ & $(y-2)$ up to third degree terms.

H.W (2) Find Taylor's series expansion of function of $f(x) = \sqrt{1+x+y^2}$ in powers of $(x-1)$ & y up to second degree terms.

H.W (24) Obtain the Taylor's series expansion of $e^x \sin y$ in terms of powers of x & y upto third degree terms.

Sol:

Function	$(0,0)$ $[x=0, y=0]$
$f(x,y) = e^x \sin y$	$f = e^0 \sin 0 = (1)(0) = 0$
$f_x = e^x \sin y$ $f_y = e^x \cos y$	$f_x = e^0 \sin 0 = (1)(0) = 0$ $f_y = e^0 \cos 0 = (1)(1) = 1$
$f_{xx} = e^x \sin y$ $f_{xy} = e^x \cos y$ $f_{yy} = -e^x \sin y$	$f_{xx} = e^0 \sin 0 = (1)(0) = 0$ $f_{xy} = e^0 \cos 0 = (1)(1) = 1$ $f_{yy} = -e^0 \sin 0 = -(1)(0) = 0$
$f_{xxx} = e^x \sin y$ $f_{xxy} = e^x \cos y$ $f_{xyy} = -e^x \sin y$ $f_{yyy} = -e^x \cos y$	$f_{xxx} = e^0 \sin 0 = (1)(0) = 0$ $f_{xxy} = e^0 \cos 0 = (1)(1) = 1$ $f_{xyy} = -e^0 \sin 0 = -(1)(0) = 0$ $f_{yyy} = -e^0 \cos 0 = -(1)(1) = -1$

Here $a=0, b=0, h = x-a = x-0 = x$, $k = y-b = y-0 = y$

By Taylor's theorem,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$
$$+ \cdots$$

$$f(x,y) = 0 + \frac{1}{1!}\left[x(0) + y(1)\right] + \frac{1}{2!}\left[x^2(0) + 2xy(1) + y^2(0)\right]$$
$$+ \frac{1}{3!}\left[x^3(0) + 3x^2y(1) + 3xy^2(0) + y^3(-1)\right] + \cdots$$
$$= y + \frac{1}{2}(2xy) + \frac{1}{6}(3x^2y - y^3) + \cdots$$
$$= y + xy + \frac{1}{6}(3x^2y - y^3) + \cdots$$

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(25) Expand the function $\sin xy$ in powers of $x-1$ & $y-\frac{\pi}{2}$ upto second degree terms, using Taylor's series.

Sol:

Function	$x=1,\ y=\pi/2$
$f(x,y) = \sin xy$	$f = \sin(1)(\pi/2) = \sin \pi/2 = 1$
$f_x = \cos xy \cdot y$ $f_y = \cos xy \cdot x$	$f_x = \cos(1)(\pi/2) \cdot \pi/2 = \cos \pi/2 \cdot \pi/2 = 0 \cdot \pi/2 = 0$ $f_y = \cos(1)(\pi/2) \cdot 1 = \cos \pi/2 = 0$
$f_{xx} = y(-\sin xy) \cdot y$ $= -y^2 \sin xy$ $f_{xy} = \cos xy \cdot 1 + y(-\sin xy) \cdot x$ $= \cos xy - xy \sin xy$ $f_{yy} = x(-\sin xy) \cdot x$ $= -x^2 \sin xy$	$f_{xx} = -(\pi/2)^2 \sin(1)(\pi/2) = -\frac{\pi^2}{4}\sin \pi/2 = -\frac{\pi^2}{4}$ $f_{xy} = \cos(1)(\pi/2) - (1)(\pi/2)\sin(1)(\pi/2)$ $= \cos \pi/2 - \pi/2 \sin \pi/2 = 0 - \frac{\pi}{2}(1) = -\pi/2$ $f_{yy} = -(1)^2 \sin(1)(\pi/2) = -\sin(\pi/2) = -1$

By Taylor's theorem, $f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$

$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \cdots$$

$$\boxed{\begin{aligned} h &= x-a = x-1 \\ k &= y-b = y-\pi/2 \end{aligned}}$$

$$f(x,y) = 1 + \frac{1}{1!}\left[(x-1)(0) + (y-\pi/2)(0)\right]$$
$$+\frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right) + 2(x-1)(y-\pi/2)(-\pi/2) + (y-\pi/2)^2(-1)\right] + \cdots$$
$$= 1 + \frac{1}{2}\left[-\frac{\pi^2}{4}(x-1)^2 - \pi(x-1)(y-\pi/2) - (y-\pi/2)^2\right] + \cdots$$

(26) Expand $e^x \log(1+y)$ in powers of x & y upto the third degree terms, using Taylor's series.

$\boxed{e^0 = 1,\ \log 1 = 0}$

Sol:

Function	$x=0,\ y=0$
$f(x,y) = e^x \log(1+y)$	$f = e^0 \log(1+0) = e^0 \log 1 = (1)(0) = 0$
$f_x = e^x \log(1+y)$ $f_y = e^x \cdot \frac{1}{1+y} \cdot 1 = e^x (1+y)^{-1}$	$f_x = e^0 \log(1+0) = e^0 \log 1 = (1)(0) = 0$ $f_y = e^0 (1+0)^{-1} = e^0 (1)^{-1} = (1)(1) = 1$
$f_{xx} = e^x \log(1+y)$ $f_{xy} = e^x (1+y)^{-1}$ $f_{yy} = e^x(-1)(1+y)^{-2} = -e^x(1+y)^{-2}$	$f_{xx} = e^0 \log(1+0) = e^0 \log 1 = (1)(0) = 0$ $f_{xy} = e^0(1+0)^{-1} = (1)(1) = 1$ $f_{yy} = -e^0(1+0)^{-2} = -(1)(1)^{-2} = -1$
$f_{xxx} = e^x \log(1+y)$ $f_{xxy} = e^x(1+y)^{-1}$ $f_{xyy} = -e^x(1+y)^{-2}$ $f_{yyy} = -e^x(-2)(1+y)^{-3} = 2e^x(1+y)^{-3}$	$f_{xxx} = e^0 \log(1+0) = (1)(0) = 0$ $f_{xxy} = e^0(1+0)^{-1} = (1)(1) = 1$ $f_{xyy} = -e^0(1+0)^{-2} = -(1)(1) = -1$ $f_{yyy} = 2e^0(1+0)^{-3} = 2(1)(1) = 2$

By Taylor's theorem,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$
$$+ \cdots$$

Here $a=0, b=0$

$h = x-a = x-0 = x \quad , \quad k = y-b = y-0 = y$

$$f(x,y) = 0 + \frac{1}{1!}\left[x(0) + y(1)\right] + \frac{1}{2!}\left[x^2(0) + 2xy(1) + y^2(-1)\right]$$
$$+ \frac{1}{3!}\left[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)\right] + \cdots$$
$$= y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}(3x^2y - 3xy^2 + 2y^3) + \cdots$$

H.W

1. Expand $e^x \cos y$ about $(0, \pi/2)$ upto the third term using Taylor's series.
2. Obtain terms upto the third degree in the Taylor's series expansion of $e^x \sin y$ around the point $(1, \pi/2)$.
3. Expand $f(x,y) = e^{xy}$ in Taylor series at $(1,1)$ upto second degree.

Maxima & minima for functions of two variables:

Definitions:

Extremum value:

$f(a,b)$ is said to be an extremum value of $f(x,y)$ if it is either a maximum or a minimum.

Notations: $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$, $\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$, $\frac{\partial^2 f}{\partial y^2} = f_{yy}$

Sufficient conditions:

If $f_x(a,b) = 0$, $f_y(a,b) = 0$ & $f_{xx}(a,b) = A$, $f_{xy}(a,b) = B$, $f_{yy}(a,b) = C$,

then

(i) $f(a,b)$ is maximum value if $AC - B^2 > 0$ & $A < 0$ (or $B < 0$)

(ii) $f(a,b)$ is minimum value if $AC - B^2 > 0$ & $A > 0$ (or $B > 0$)

(iii) $f(a,b)$ is not an extremum (saddle) if $AC - B^2 < 0$ &

(iv) If $AC - B^2 = 0$, then the test is inconclusive.

Stationary value:

A function $f(x,y)$ is said to be stationary at (a,b) or $f(a,b)$ is said to be a stationary value of $f(x,y)$ if $f_x(a,b) = 0$ & $f_y(a,b) = 0$.

<u>Note</u>: Every extremum value is a stationary value but a stationary value need not be an extremum value.

(AU) (27) Examine $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ for extreme values.

<u>Sol</u>: Given $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

×	+
24	−10
−6	−4
$x-6$	$x-4$

$f_x = 3x^2 + 3y^2 - 30x + 72$

$f_y = 6xy - 30y$

<u>Stationary points</u>:

$f_x = 0$

$3x^2 + 3y^2 - 30x + 72 = 0$ —— ①

<u>$y=0$</u> in ①

$3x^2 - 30x + 72 = 0 \Rightarrow x^2 - 10x + 24 = 0$

$\Rightarrow (x-6)(x-4) = 0$

$\Rightarrow x = 4, 6$ $\quad\therefore$ The points are $(4,0)$ & $(6,0)$

$f_y = 0$

$6xy - 30y = 0 \Rightarrow 6y(x-5) = 0$

$\Rightarrow y = 0,\ x = 5$

<u>$x=5$</u> in ①

$75 + 3y^2 - 150 + 72 = 0 \Rightarrow 3y^2 - 3 = 0 \Rightarrow 3y^2 = 3 \Rightarrow y^2 = 1 \Rightarrow y = \pm\sqrt{1} = \pm 1$

$\therefore$ The points are $(5,1)$ & $(5,-1)$.

Hence the stationary points are $(4,0), (6,0), (5,1)$ & $(5,-1)$.

$A = f_{xx} = 6x - 30$; $B = f_{xy} = 6y$; $C = f_{yy} = 6x - 30$

	$(4,0)$	$(6,0)$	$(5,1)$	$(5,-1)$
$A = 6x-30$	$-6 < 0$	$6 > 0$	0	0
$B = 6y$	0	0	6	-6
$C = 6x-30$	-6	6	0	0
$AC - B^2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 < 0$
Conclusion	Maximum value	Minimum value	Saddle point	Saddle point

$f(4,0) = (4)^3 + 3(4)(0)^2 - 15(4)^2 - 15(0)^2 + 72(4) = 112$

$f(6,0) = (6)^3 + 3(6)(0)^2 - 15(6)^2 - 15(0)^2 + 72(6) = 108$

Hence the maximum value is 112 & the minimum value is 108.

(A) 28) Find the maxima & minima of $f(x,y) = x^4+y^4-2x^2+4xy-2y^2$.

Sol: Given $f(x,y) = x^4+y^4-2x^2+4xy-2y^2$.

$f_x = 4x^3-4x+4y$ $\quad A = f_{xx} = 12x^2-4$ $\quad C = f_{yy} = 12y^2-4$

$f_y = 4y^3+4x-4y$ $\quad B = f_{xy} = 4$

Stationary points:

$f_x = 0$

$\Rightarrow 4x^3-4x+4y=0$

$\Rightarrow x^3-x+y=0$ —(1)

$f_y = 0$

$\Rightarrow 4y^3+4x-4y=0$

$\Rightarrow y^3+x-y=0$ —(2)

(1)+(2) $\Rightarrow x^3-x+y+y^3+x-y=0 \Rightarrow x^3+y^3=0 \Rightarrow x^3=-y^3 \Rightarrow x=-y$

$\Rightarrow \boxed{y=-x}$ —(3)

Substituting (3) in (1),

$x^3-x-x=0 \Rightarrow x^3-2x=0 \Rightarrow x(x^2-2)=0$

$\Rightarrow x=0, \quad x^2-2=0$

$\Rightarrow x=0, \quad x^2=2$

$\Rightarrow x=0, \quad x=\pm\sqrt{2}$ —(4)

Substituting (4) in (3),

$x=0 \Rightarrow y=0$; $x=\sqrt{2} \Rightarrow y=-\sqrt{2}$; $x=-\sqrt{2} \Rightarrow y=\sqrt{2}$

Hence the stationary points are $(0,0), (\sqrt{2},-\sqrt{2})$ & $(-\sqrt{2},\sqrt{2})$.

	$(0,0)$	$(\sqrt{2},-\sqrt{2})$	$(-\sqrt{2},\sqrt{2})$
$A=12x^2-4$	-4	$20>0$	$20>0$
$B=4$	4	4	4
$C=12y^2-4$	-4	20	20
$AC-B^2$	0	$384>0$	$384>0$
Conclusion	Inconclusive	Minimum value	Minimum value.

$f(\sqrt{2},-\sqrt{2}) = (\sqrt{2})^4+(-\sqrt{2})^4-2(\sqrt{2})^2+4(\sqrt{2})(-\sqrt{2})-2(-\sqrt{2})^2 = 4+4-4-8-4 = -8$

$f(-\sqrt{2},\sqrt{2}) = (-\sqrt{2})^4+(\sqrt{2})^4-2(-\sqrt{2})^2+4(-\sqrt{2})(\sqrt{2})-2(\sqrt{2})^2 = 4+4-4-8-4 = -8$

Hence the minimum value is -8.

29) Find the extreme values of $f(x,y) = x^3y^2(1-x-y)$.

<u>Sol:</u> Given $f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$f_y = 2x^3y - 2x^4y - 3x^3y^2$

$A = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$

$B = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$

$C = f_{yy} = 2x^3 - 2x^4 - 6x^3y$

<u>Stationary points:</u>

$f_x = 0$

$\Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$\Rightarrow x^2y^2(3-4x-3y) = 0$

$\Rightarrow x=0,\ y=0,\ 3-4x-3y=0$

$\Rightarrow x=0,\ y=0,\ 4x+3y=3$ — (1)

$f_y = 0$

$\Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0$

$\Rightarrow x^3y(2-2x-3y) = 0$

$\Rightarrow x=0,\ y=0,\ 2-2x-3y=0$

$\Rightarrow x=0,\ y=0,\ 2x+3y=2$ — (2)

(1) − (2) $\Rightarrow 4x+3y-2x-3y = 3-2 \Rightarrow 2x = 1 \Rightarrow \boxed{x = 1/2}$

Substituting $x = 1/2$ in (2),

$2(1/2) + 3y = 2 \Rightarrow 1+3y = 2 \Rightarrow 3y = 2-1 \Rightarrow 3y = 1 \Rightarrow \boxed{y = 1/3}$

Hence the stationary points are $(0,0)$ & $(1/2, 1/3)$.

	$(0,0)$	$(1/2, 1/3)$
$A = 6xy^2 - 12x^2y^2 - 6xy^3$	0	$-\frac{1}{9} < 0$
$B = 6x^2y - 8x^3y - 9x^2y^2$	0	$-\frac{1}{12}$
$C = 2x^3 - 2x^4 - 6x^3y$	0	$-\frac{1}{8}$
$AC - B^2$	0	$\frac{1}{144} > 0$
Conclusion	Inconclusive	Maximum value

$$f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{8} \times \frac{1}{9} \times \left(\frac{1}{6}\right) = \frac{1}{432}$$

Hence the maximum value is $\frac{1}{432}$.

(30) Discuss the maxima & minima of the function $f(x,y)=x^3+y^3-3axy$.

Sol. Given $f(x,y)=x^3+y^3-3axy$

$f_x=3x^2-3ay$ $\quad A=f_{xx}=6x$ $\quad C=f_{yy}=6y$

$f_y=3y^2-3ax$ $\quad B=f_{xy}=-3a$

Stationary points:

$f_x=0$

$\Rightarrow 3x^2-3ay=0$

$\Rightarrow x^2-ay=0 \Rightarrow x^2=ay$ — (1)

$f_y=0$

$\Rightarrow 3y^2-3ax=0 \Rightarrow y^2-ax=0$

$\Rightarrow y^2=ax$ — (2)

(1) $\Rightarrow y=\frac{x^2}{a}$ — (3)

Substituting (3) in (2), $\left(\frac{x^2}{a}\right)^2=ax \Rightarrow \frac{x^4}{a^2}=ax \Rightarrow \frac{x^4}{x}=a^3 \Rightarrow x^3=a^3$

$\Rightarrow \boxed{x=a}$ — (4)

Substituting (4) in (3), $y=\frac{a^2}{a}=a \Rightarrow \boxed{y=a}$

Hence the stationary point is (a,a).

	(a,a)
$A=6x$	$6a$
$B=-3a$	$-3a$
$C=6y$	$6a$
$AC-B^2$	$36a^2-9a^2=27a^2>0$
Conclusion	

If $a>0$, then $A>0 \Rightarrow$ Minimum value at (a,a).

If $a<0$, then $A<0 \Rightarrow$ Maximum value at (a,a).

$f(a,a)=a^3+a^3-3a(a)(a)=2a^3-3a^3=-a^3$

Hence the maximum or minimum value at (a,a) is $-a^3$.

AU (1) Find the maximum or minimum values of $f(x,y)=3x^2-y^2+x^3$.

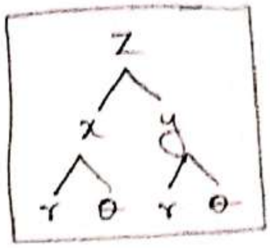
AU (2) Find the maximum or minimum values of $f(x,y)=x^2+y^2+6x+12$.

(3) Examine $x^3y^2(12-x-y)$ for extreme values.

(4) Find the maxima & minima of $xy(a-x-y)$.

(31) If $z=f(x,y)$ where $x=r\cos\theta$ & $y=r\sin\theta$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$



Sol: Given $z=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$

$$\frac{\partial x}{\partial r}=\cos\theta=\frac{x}{r} \qquad \frac{\partial y}{\partial r}=\sin\theta=\frac{y}{r}$$

$$\frac{\partial x}{\partial \theta}=-r\sin\theta=-y \qquad \frac{\partial y}{\partial \theta}=r\cos\theta=x$$

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}=\frac{x}{r}\frac{\partial z}{\partial x}+\frac{y}{r}\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}=-y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}$$

$$\left(\frac{\partial z}{\partial r}\right)^2=\left(\frac{x}{r}\frac{\partial z}{\partial x}+\frac{y}{r}\frac{\partial z}{\partial y}\right)^2=\frac{1}{r^2}\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)^2$$

$$=\frac{1}{r^2}\left(x^2\left(\frac{\partial z}{\partial x}\right)^2+y^2\left(\frac{\partial z}{\partial y}\right)^2+2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right)$$

$$\left(\frac{\partial z}{\partial \theta}\right)^2=\left(-y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}\right)^2=\left(x\frac{\partial z}{\partial y}-y\frac{\partial z}{\partial x}\right)^2$$

$$=x^2\left(\frac{\partial z}{\partial y}\right)^2+y^2\left(\frac{\partial z}{\partial x}\right)^2-2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$\therefore\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\frac{1}{r^2}\left(x^2\left(\frac{\partial z}{\partial x}\right)^2+y^2\left(\frac{\partial z}{\partial y}\right)^2+2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right)$$

$$+\frac{1}{r^2}\left(x^2\left(\frac{\partial z}{\partial y}\right)^2+y^2\left(\frac{\partial z}{\partial x}\right)^2-2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right)$$

$$=\frac{1}{r^2}\left(x^2\left(\frac{\partial z}{\partial x}\right)^2+y^2\left(\frac{\partial z}{\partial y}\right)^2+x^2\left(\frac{\partial z}{\partial y}\right)^2+y^2\left(\frac{\partial z}{\partial x}\right)^2\right)$$

$$=\frac{1}{r^2}\left[\left(\frac{\partial z}{\partial x}\right)^2(x^2+y^2)+\left(\frac{\partial z}{\partial y}\right)^2(x^2+y^2)\right]$$

$$=\frac{1}{r^2}\left[(x^2+y^2)\left(\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right)\right]$$

$$=\frac{1}{r^2}\times r^2\times\left(\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right)$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

Hence $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$.

Lagrange's method of undetermined multipliers:

AU (32) A thin closed rectangular box is to have one edge equal to twice the other & constant volume $72\,m^3$. Find the least surface area of the box.

Sol: Let $x, y, 2y$ be the length, breadth & height of the box respectively.

Surface area $= 2xy + 2(y)(2y) + 2(x)(2y) = 2xy + 4y^2 + 4xy = 6xy + 4y^2$

Volume $= (x)(y)(2y) = 2xy^2 = 72 \Rightarrow xy^2 = \frac{72}{2} = 36 \Rightarrow xy^2 = 36$ — (*)

$F = (6xy + 4y^2) + \lambda(xy^2 - 36) = 6xy + 4y^2 + \lambda xy^2 - 36\lambda$

$F_x = 6y + \lambda y^2$; $F_y = 6x + 8y + 2\lambda xy$

$F_x = 0$

$\Rightarrow 6y + \lambda y^2 = 0 \Rightarrow 6y = -\lambda y^2$

$\Rightarrow 6 = -\lambda y \Rightarrow \frac{6}{y} = -\lambda$ — (1)

$F_y = 0$

$\Rightarrow 6x + 8y + 2\lambda xy = 0 \Rightarrow 6x + 8y = -2\lambda xy$

$\Rightarrow 3x + 4y = -\lambda xy \Rightarrow \frac{3x + 4y}{xy} = -\lambda$

$\Rightarrow \frac{3}{y} + \frac{4}{x} = -\lambda$ — (2)

From (1) & (2), $\frac{6}{y} = \frac{3}{y} + \frac{4}{x} \Rightarrow \frac{6}{y} - \frac{3}{y} = \frac{4}{x} \Rightarrow \frac{3}{y} = \frac{4}{x}$

$\Rightarrow 3x = 4y \Rightarrow \boxed{y = \frac{3}{4}x}$ — (3)

Substituting (3) in (*),

$x\left(\frac{3}{4}x\right)^2 = 36 \Rightarrow x\,\frac{9}{16}x^2 = 36 \Rightarrow \frac{9}{16}x^3 = 36 \Rightarrow x^3 = \frac{36 \times 16}{9} = 64 = 4^3$

$\Rightarrow \boxed{x = 4}$

$\therefore y = \frac{3}{4}(4) = 3 \Rightarrow \boxed{y = 3}$

$\therefore$ Least surface area $= 6xy + 4y^2 = 6(4)(3) + 4(3)^2 = 108.$

AU (33) Find the dimensions of the rectangular box without a top of maximum capacity, whose surface area is 108 sq.cm.

Sol: Let x, y, z be the length, breadth & height of the box.

Surface area $= xy + 2yz + 2zx = 108$ — (1)

Volume $= xyz$

$F = xyz + \lambda(xy + 2yz + 2zx - 108) = xyz + \lambda xy + 2\lambda yz + 2\lambda zx - 108\lambda$

$F_x = yz + \lambda y + 2\lambda z$; $F_y = xz + \lambda x + 2\lambda z$; $F_z = xy + 2\lambda y + 2\lambda x$

$F_x = 0$

$\Rightarrow yz + \lambda(y+2z) = 0$

$\Rightarrow yz = -\lambda(y+2z)$

$\Rightarrow \dfrac{yz}{y+2z} = -\lambda$

$\Rightarrow \dfrac{y+2z}{yz} = \dfrac{-1}{\lambda}$

$\Rightarrow \dfrac{1}{z} + \dfrac{2}{y} = \dfrac{-1}{\lambda}$ — (2)

$F_y = 0$

$\Rightarrow xz + \lambda(x+2z) = 0$

$\Rightarrow xz = -\lambda(x+2z)$

$\Rightarrow \dfrac{xz}{x+2z} = -\lambda$

$\Rightarrow \dfrac{x+2z}{xz} = \dfrac{-1}{\lambda}$

$\Rightarrow \dfrac{1}{z} + \dfrac{2}{x} = -\dfrac{1}{\lambda}$ — (3)

$F_z = 0$

$\Rightarrow xy + \lambda(2y+2x) = 0$

$\Rightarrow xy = -\lambda(2y+2x)$

$\Rightarrow \dfrac{xy}{2y+2x} = -\lambda$

$\Rightarrow \dfrac{2y+2x}{xy} = \dfrac{-1}{\lambda}$

$\Rightarrow \dfrac{2}{x} + \dfrac{2}{y} = -\dfrac{1}{\lambda}$ — (4)

From (2) & (3),

$\dfrac{1}{z} + \dfrac{2}{y} = \dfrac{1}{z} + \dfrac{2}{x}$

$\Rightarrow \dfrac{2}{y} = \dfrac{2}{x} \Rightarrow 2x = 2y$

$\Rightarrow \boxed{x = y}$ — (5)

From (3) & (4),

$\dfrac{1}{z} + \dfrac{2}{x} = \dfrac{2}{x} + \dfrac{2}{y}$

$\Rightarrow \dfrac{1}{z} = \dfrac{2}{y}$

$\Rightarrow \boxed{y = 2z}$ — (6)

From (5) & (6), $x = y = 2z$

$\therefore$ (1) $\Rightarrow xy + 2yz + 2zx = 108 \Rightarrow (2z)(2z) + 2(2z)z + 2z(2z) = 108$

$\Rightarrow 4z^2 + 4z^2 + 4z^2 = 108 \Rightarrow 12z^2 = 108 \Rightarrow z^2 = 9 \Rightarrow \boxed{z = 3}$

$\therefore x = 6,\ y = 6,\ z = 3$

Maximum volume $= xyz = (6)(6)(3) = 108$.

AU

(34) Find the shortest & the longest distances from the point $(1,2,-1)$ to the sphere $x^2+y^2+z^2 = 24$.

Here $x_1 = 1,\ y_1 = 2,\ z_1 = -1$

$d = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}$

Sol: Given $x^2+y^2+z^2 = 24$ & $(1,2,-1)$

$d = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$

$\Rightarrow d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2$

$F = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2+y^2+z^2-24)$

$F_x = 2(x-1) + 2x\lambda$; $F_y = 2(y-2) + 2y\lambda$; $F_z = 2(z+1) + 2z\lambda$

$F_x = 0$

$2(x-1)+2x\lambda = 0$

$x-1+x\lambda = 0$

$x+x\lambda = 1$

$x(1+\lambda) = 1$

$x = \dfrac{1}{1+\lambda}$ — ①

$F_y = 0$

$2(y-2)+2y\lambda = 0$

$y-2+y\lambda = 0$

$y+y\lambda = 2$

$y(1+\lambda) = 2$

$\dfrac{y}{2} = \dfrac{1}{1+\lambda}$ — ②

$F_z = 0$

$2(z+1)+2z\lambda = 0$

$z+1+z\lambda = 0$

$z+z\lambda = -1$

$z(1+\lambda) = -1$

$-z = \dfrac{1}{1+\lambda}$ — ③

From ①, ② & ③,

$$x = \frac{y}{2} = -z \Rightarrow x = -z, \ \frac{y}{2} = -z \Rightarrow x = -z, \ y = -2z$$

$$\therefore x^2+y^2+z^2 = 24 \Rightarrow (-z)^2+(-2z)^2+z^2 = 24$$

$$\Rightarrow z^2+4z^2+z^2 = 24 \Rightarrow 6z^2 = 24 \Rightarrow z^2 = 4 \rightarrow z = \pm\sqrt{4} = \pm 2$$

$z = 2 \Rightarrow x = -2, \ y = -2(2) = -4$

$z = -2 \Rightarrow x = 2, \ y = -2(-2) = 4$

Hence the stationary points are $(-2,-4,2)$ & $(2,4,-2)$.

$$d = \sqrt{(-2-1)^2+(-4-2)^2+(2+1)^2} = \sqrt{9+36+9} = \sqrt{54} = 3\sqrt{6}$$

$$d = \sqrt{(2-1)^2+(4-2)^2+(-2+1)^2} = \sqrt{1+4+1} = \sqrt{6}$$

Hence the shortest & longest distances are $\sqrt{6}$ & $3\sqrt{6}$ respectively.

AU

35 Find the minimum distance from the point $(1,2,0)$ to the cone $z^2 = x^2+y^2$.

Sol: Given $z^2 = x^2+y^2$ & $(1,2,0)$

$$d = \sqrt{(x-1)^2+(y-2)^2+(z-0)^2}$$

$$d^2 = (x-1)^2+(y-2)^2+z^2$$

$$F = (x-1)^2+(y-2)^2+z^2+\lambda(z^2-x^2-y^2)$$

$F_x = 2(x-1)-2x\lambda$; $F_y = 2(y-2)-2y\lambda$; $F_z = 2z+2z\lambda$

$F_x = 0$

$2(x-1)-2x\lambda = 0$

$x-1-x\lambda = 0$

$x-1 = x\lambda$

$\dfrac{x-1}{x} = \lambda$

$1-\dfrac{1}{x} = \lambda$ — ①

$F_y = 0$

$2(y-2)-2y\lambda = 0$

$y-2-y\lambda = 0$

$y-2 = y\lambda$

$\dfrac{y-2}{y} = \lambda$

$1-\dfrac{2}{y} = \lambda$ — ②

$F_z = 0$

$2z+2z\lambda = 0$

$z+z\lambda = 0$

$z = -z\lambda$

$\dfrac{z}{-z} = \lambda$

$\boxed{\lambda = -1}$ — ③

Substituting ③ in ① & ②,

$$1-\frac{1}{x}=-1 \Rightarrow 1+1=\frac{1}{x} \Rightarrow 2=\frac{1}{x} \Rightarrow \boxed{x=1/2}$$

$$1-\frac{2}{y}=-1 \Rightarrow 1+1=\frac{2}{y} \Rightarrow 2=\frac{2}{y} \Rightarrow y=\frac{2}{2} \Rightarrow \boxed{y=1}$$

$$\therefore z^2=x^2+y^2 \Rightarrow z^2=(1/2)^2+1^2=\frac{1}{4}+1=\frac{5}{4} \Rightarrow z=\pm\sqrt{\frac{5}{4}}=\pm\frac{\sqrt{5}}{2}$$

Hence the stationary points are $\left(1/2, 1, \frac{\sqrt{5}}{2}\right)$ & $\left(\frac{1}{2}, 1, -\frac{\sqrt{5}}{2}\right)$.

$$d=\sqrt{(1/2-1)^2+(1-2)^2+\left(\frac{\sqrt{5}}{2}\right)^2}=\sqrt{1/4+1+\frac{5}{4}}=\sqrt{\frac{3}{2}+1}=\sqrt{5/2}$$

$$d=\sqrt{(1/2-1)^2+(1-2)^2+\left(-\frac{\sqrt{5}}{2}\right)^2}=\sqrt{5/2}$$

Hence the minimum distance is $\sqrt{5/2}$.

(36) Find the maximum volume of the largest rectangular parallelopiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol: Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ — ①

Volume of parallelopiped $=(2x)(2y)(2z)=8xyz$

$$F=8xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

$$F=8xyz+\lambda\frac{x^2}{a^2}+\lambda\frac{y^2}{b^2}+\lambda\frac{z^2}{c^2}-\lambda$$

$F_x=8yz+\frac{2x\lambda}{a^2}$; $F_y=8xz+\frac{2\lambda y}{b^2}$; $F_z=8xy+\frac{2\lambda z}{c^2}$

$F_x=0$	$F_y=0$	$F_z=0$
$8yz+\frac{2x\lambda}{a^2}=0$	$8xz+\frac{2\lambda y}{b^2}=0$	$8xy+2\frac{\lambda z}{c^2}=0$
$8yz=-\frac{2x\lambda}{a^2}$	$8xz=-\frac{2\lambda y}{b^2}$	$8xy=-\frac{2\lambda z}{c^2}$
$4yz=\frac{-x\lambda}{a^2}$	$4xz=-\frac{\lambda y}{b^2}$	$4xy=-\frac{\lambda z}{c^2}$
$4xyz=-\frac{x^2\lambda}{a^2}$	$4xyz=\frac{-\lambda y^2}{b^2}$	$4xyz=\frac{-\lambda z^2}{c^2}$
$\frac{4xyz}{-\lambda}=\frac{x^2}{a^2}$ — ②	$\frac{4xyz}{-\lambda}=\frac{y^2}{b^2}$ — ③	$\frac{4xyz}{-\lambda}=\frac{z^2}{c^2}$ — ④

From ②, ③ & ④, $\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}$ — ⑤

Substituting ⑤ in ①,

$$\frac{x^2}{a^2}+\frac{x^2}{a^2}+\frac{x^2}{a^2}=1 \Rightarrow \frac{3x^2}{a^2}=1 \Rightarrow x^2=\frac{a^2}{3} \Rightarrow \boxed{x=\frac{a}{\sqrt{3}}}$$

Similarly, $\boxed{y=\frac{b}{\sqrt{3}}}$ & $\boxed{z=\frac{c}{\sqrt{3}}}$

Hence the stationary point is $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$.

Maximum volume $= 8xyz = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}$.

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(37) Find the maximum value of $x^m y^n z^p$, when $x+y+z=a$.

Sol: Given $x+y+z=a$ —(*)

$$F = x^m y^n z^p + \lambda(x+y+z-a) = x^m y^n z^p + \lambda x + \lambda y + \lambda z - \lambda a$$

$F_x = m x^{m-1} y^n z^p + \lambda$; $F_y = n x^m y^{n-1} z^p + \lambda$; $F_z = p x^m y^n z^{p-1} + \lambda$

$F_x = 0$

$m x^{m-1} y^n z^p + \lambda = 0$

$m x^{m-1} y^n z^p = -\lambda$

$\frac{m x^m y^n z^p}{x} = -\lambda$ —①

$F_y = 0$

$n x^m y^{n-1} z^p + \lambda = 0$

$n x^m y^{n-1} z^p = -\lambda$

$\frac{n x^m y^n z^p}{y} = -\lambda$ —②

$F_z = 0$

$p x^m y^n z^{p-1} + \lambda = 0$

$p x^m y^n z^{p-1} = -\lambda$

$\frac{p x^m y^n z^p}{z} = -\lambda$ —③

From ①, ② & ③,

$$\frac{m x^m y^n z^p}{x} = \frac{n x^m y^n z^p}{y} = \frac{p x^m y^n z^p}{z}$$

Dividing by $x^m y^n z^p$, we get

$$\frac{m}{x} = \frac{n}{y} = \frac{p}{z}$$

$\frac{m}{x} = \frac{p}{z}$

$\Rightarrow \boxed{x = \frac{mz}{p}}$ —④

$\frac{n}{y} = \frac{p}{z}$

$\Rightarrow \boxed{y = \frac{nz}{p}}$ —⑤

∴ (*) becomes,

$$\frac{mz}{p} + \frac{nz}{p} + z = a \Rightarrow z\left(\frac{m}{p}+\frac{n}{p}+1\right) = a$$

$$\Rightarrow z\left(\frac{m+n+p}{p}\right) = a \Rightarrow z = \frac{ap}{m+n+p} \quad \text{—⑥}$$

Substituting (6) in (4) & (5),

$$x = \frac{map}{p(m+n+p)} = \frac{ma}{m+n+p}$$

$$y = \frac{nap}{p(m+n+p)} = \frac{an}{m+n+p}$$

Hence the stationary point is $\left(\frac{am}{m+n+p}, \frac{an}{m+n+p}, \frac{ap}{m+n+p}\right)$.

Maximum value of $x^m y^n z^p = \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$

$$= \frac{a^m m^m}{(m+n+p)^m} \cdot \frac{a^n n^n}{(m+n+p)^n} \cdot \frac{a^p p^p}{(m+n+p)^p}$$

$$= \frac{a^{m+n+p}\, m^m n^n p^p}{(m+n+p)^{m+n+p}}.$$

H.W

(1) Find the minimum values of x^2yz^3 subject to the condition $2x+y+3z=a$.

(2) Find the maximum value of $400xyz^2$ subject to the condition $x^2+y^2+z^2=1$.

(3) Find the dimensions of the rectangular box without top of maximum capacity with surface area 432 square metres.

(4) A rectangular box open at the top, is to have a volume of 32 cc. Find the dimensions of the box, that requires the least material for its construction.

INTEGRAL CALCULUS

(AU) Fundamental theorem of calculus:

Suppose f is continuous on $[a,b]$.

(i) If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

(ii) $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any anti-derivative of f, that is $F' = f$.

(AU) 1) Find the derivative of $G(x) = \int_x^1 \cos\sqrt{t}\,dt$.

Sol: Given $G(x) = \int_x^1 \cos\sqrt{t}\,dt = -\int_1^x \cos\sqrt{t}\,dt$

Here $f(t) = \cos\sqrt{t}$ is continuous.

$\therefore G'(x) = -\cos\sqrt{x}$

(AU) 2) Evaluate $\int_0^3 (x^3 - 6x)\,dx$ by using Riemann sum with n sub intervals.

Sol: Take n sub intervals, we have $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$

$x_0 = 0,\ x_1 = \frac{3}{n},\ x_2 = \frac{6}{n},\ x_3 = \frac{9}{n},\ \ldots,\ x_i = \frac{3i}{n}$. Here $f(x) = x^3 - 6x$

$$\therefore \int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{27i^3}{n^3} - \frac{18i}{n}\right)$$

$$= \lim_{n\to\infty} \frac{81}{n^4} \sum_{i=1}^{n} i^3 - \lim_{n\to\infty} \frac{54}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{81}{n^4} \left[\frac{n(n+1)}{2}\right]^2 - \lim_{n\to\infty} \frac{54}{n^2} \left[\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{81}{n^4} \left[\frac{n^2(1+1/n)}{2}\right]^2 - \lim_{n\to\infty} \frac{54}{n^2} \left[\frac{n^2(1+1/n)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{81}{n^4} \times \frac{n^4(1+1/n)^2}{4} - \lim_{n\to\infty} \frac{54}{n^2} \times \frac{n^2(1+1/n)}{2}$$

$$= \lim_{n\to\infty} \frac{81}{4}(1+1/n)^2 - \lim_{n\to\infty} 27(1+1/n) = \frac{81}{4} - 27 = -\frac{27}{4}$$

Note:

① $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ ② $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ ③ $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

(H.W) ① Evaluate $\int_0^3 (x^2-2x)\,dx$ by using Riemann sum with n sub intervals.

(AU) ③ What is wrong with the equation $\int_{-1}^{2} \frac{4}{x^3}\,dx = \left[\frac{-2}{x^2}\right]_{-1}^{2} = \frac{3}{2}$?

Sol: Here $f(x) = \frac{4}{x^3}$ is not continuous in the interval $[-1,2]$.

Since $f(x) = \frac{4}{x^3}$ is discontinuous at $x=0$.

$\therefore \int_{-1}^{2} \frac{4}{x^3}\,dx$ doesn't exist.

Formulae:

① $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$ ② $\int \frac{1}{x}\,dx = \log x + c$ ③ $\int e^x\,dx = e^x + c$

④ $\int e^{2x} = \frac{e^{2x}}{2} + c$ ⑤ $\int dx = x + c$ ⑥ $\int a\,dx = ax + c$, where a is a constant.

④ Evaluate the following:

(i) $\int \left(\frac{6}{x^2} + \sqrt{x} + x^{3/2} + \frac{5}{x} + 1\right) dx$

Sol: $\int \left(\frac{6}{x^2} + \sqrt{x} + x^{3/2} + \frac{5}{x} + 1\right) dx = \int \left(6x^{-2} + x^{1/2} + x^{3/2} + \frac{5}{x} + 1\right) dx$

$$= 6\frac{x^{-2+1}}{-2+1} + \frac{x^{1/2+1}}{1/2+1} + \frac{x^{3/2+1}}{3/2+1} + 5\log x + x + c$$

$$= \frac{6x^{-1}}{-1} + \frac{x^{3/2}}{3/2} + \frac{x^{5/2}}{5/2} + 5\log x + x + c$$

$$= -\frac{6}{x} + \frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2} + 5\log x + x + c$$

(ii) $\int \frac{x^2+3x-5}{\sqrt{x}}\,dx$

Sol: $\int \frac{x^2+3x-5}{\sqrt{x}}\,dx = \int x^{-1/2}(x^2+3x-5)\,dx$

$$= \int \left(x^{-1/2}x^2 + 3x^{-1/2}x - 5x^{-1/2}\right)dx$$

$$= \int \left(x^{-1/2+2} + 3x^{-1/2+1} - 5x^{-1/2}\right)dx = \int \left(x^{3/2} + 3x^{1/2} - 5x^{-1/2}\right)dx$$

$$= \frac{x^{3/2+1}}{3/2+1} + 3\,\frac{x^{1/2+1}}{1/2+1} - 5\frac{x^{-1/2+1}}{-1/2+1} + c$$

$$= \frac{x^{5/2}}{5/2} + 3\,\frac{x^{3/2}}{3/2} - 5\,\frac{x^{1/2}}{1/2} + c = \frac{2}{5}x^{5/2} + 2x^{3/2} - 10\sqrt{x} + c$$

(iii) $\int (e^{2x} + 3x - 7)dx$

Sol: $\int (e^{2x} + 3x - 7)dx = \frac{e^{2x}}{2} + 3\frac{x^2}{2} - 7x + c$

(iv) $\int (e^{\log x} + 2)dx$

Sol: $\int (e^{\log x} + 2)dx = \int (x+2)dx = \frac{x^2}{2} + 2x + c$

(v) $\int x^2(1-x)^2 dx$

Sol: $\int x^2(1-x)^2 dx = \int x^2(1+x^2-2x)\,dx$

$$= \int (x^2 + x^4 - 2x^3)\,dx$$

$$= \frac{x^3}{3} + \frac{x^5}{5} - \frac{2x^4}{4} + c$$

$$= \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^4}{2} + c$$

H.W (1) Evaluate the following:

(i) $\int \left(x^4 - \frac{1}{2}x^3 + \frac{1}{4}x - 2\right)dx$ (ii) $\int \frac{x^3 - 2\sqrt{x}}{x}\,dx$

(iii) $\int \left(x^{2/5} - x^{-3/5}\right)^2 dx$ (iv) $\int (e^x + x^2 + 8)\,dx$

AU (5) If f is continuous & $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

Sol: Take $2x = t$

$2dx = dt \Rightarrow dx = \frac{dt}{2}$

When $x = 0 \Rightarrow t = 0$

$x = 2 \Rightarrow t = 2(2) = 4$

$$\therefore \int_0^2 f(2x)\,dx = \int_0^4 f(t)\frac{dt}{2} = \frac{1}{2}\int_0^4 f(t)\,dt = \frac{1}{2}(10) = 5$$

$$\left(\because \int_0^4 f(x)\,dx = \int_0^4 f(t)\,dt = 10\right)$$

Formulae:

1. $\int \sin x\, dx = -\cos x + c$
2. $\int \cos x\, dx = \sin x + c$
3. $\int \sec^2 x\, dx = \tan x + c$
4. $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
5. $\int \sec x \tan x\, dx = \sec x + c$
6. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$
7. $\int \cosh x\, dx = \sinh x + c$
8. $\int \sinh x\, dx = \cosh x + c$
9. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$
10. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
11. $\int \frac{1}{\sqrt{x^2-1}} dx = \log\left(x+\sqrt{x^2-1}\right) + c$
12. $\int \frac{1}{\sqrt{x^2+1}} dx = \log\left(x+\sqrt{x^2+1}\right) + c$
13. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$
14. $\int \sin 2x\, dx = -\frac{\cos 2x}{2} + c$

AU (6) Evaluate $\int \frac{\tan x}{\sec x + \cos x} dx$.

Sol: $\int \frac{\tan x}{\sec x+\cos x} dx = \int \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\cos x} dx$

$$= \int \frac{\frac{\sin x}{\cos x}}{\frac{1+\cos^2 x}{\cos x}} dx = \int \frac{\sin x}{\cos x} \times \frac{\cos x}{1+\cos^2 x} dx$$

$$\boxed{\begin{aligned} \text{Put } \cos x &= t \\ -\sin x\, dx &= dt \\ \sin x\, dx &= -dt \end{aligned}}$$

$$= \int \frac{\sin x}{1+\cos^2 x} dx$$

$$= \int \frac{-dt}{1+t^2} = -\int \frac{dt}{1+t^2}$$

$$= -\tan^{-1} t + c = -\tan^{-1}(\cos x) + c$$

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(7) Evaluate the following:

(i) $\int \frac{1}{1+\sin x} dx$

Sol: $\int \frac{1}{1+\sin x} dx = \int \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx$

$$= \int \frac{1-\sin x}{1-\sin^2 x} dx = \int \frac{1-\sin x}{\cos^2 x} dx$$

$$= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx = \int \left(\sec^2 x - \tan x \sec x\right) dx$$

$$= \tan x - \sec x + c$$

(ii) $\int \frac{\cos^2 x}{1-\sin x}\,dx$

Sol: $\int \frac{\cos^2 x}{1-\sin x}\,dx = \int \frac{1-\sin^2 x}{1-\sin x}\,dx$

$= \int \frac{(1+\sin x)(1-\sin x)}{1-\sin x}\,dx = \int (1+\sin x)\,dx$

$= x - \cos x + c$

(iii) $\int (\tan x - 2\cot x)^2\,dx$

Sol: $\int (\tan x - 2\cot x)^2\,dx = \int (\tan^2 x + 4\cot^2 x - 4\tan x \cot x)\,dx$

$= \int \left(\sec^2 x - 1 + 4(\operatorname{cosec}^2 x - 1) - 4\tan x \frac{1}{\tan x}\right)dx$

$= \int (\sec^2 x - 1 + 4\operatorname{cosec}^2 x - 4 - 4)\,dx$

$= \int (\sec^2 x + 4\operatorname{cosec}^2 x - 9)\,dx$

$= \tan x + 4(-\cot x) - 9x + c$

$= \tan x - 4\cot x - 9x + c$

H.W (1) Evaluate the following:

(i) $\int \frac{\sin^2 x}{1+\cos x}\,dx$ (ii) $\int \frac{1}{1-\cos x}\,dx$ (iii) $\int \left(\frac{3}{\sqrt{1-x^2}} + e^x + 8\right)dx$

(8) Evaluate the following:

(i) $\int_1^4 (x^2+2x-5)\,dx$

Sol: $\int_1^4 (x^2+2x-5)\,dx = \left[\frac{x^3}{3} + \frac{2x^2}{2} - 5x\right]_1^4 = \left[\frac{x^3}{3} + x^2 - 5x\right]_1^4$

$= \left[\frac{64}{3} + 16 - 20 - \left(\frac{1}{3} + 1 - 5\right)\right]$

$= \frac{64}{3} + 16 - 20 - \frac{1}{3} - 1 + 5 = 21$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

(ii) $\int_{-1}^{1} (2-|x|)\,dx$

Sol: $\int_{-1}^{1} (2-|x|)\,dx$

Here $f(x) = 2-|x|$ is an even function.

$f(x) = 2-|x|$

$f(-x) = 2-|-x| = 2-|x| = f(x)$

$$\int_{-1}^{1}(2-|x|)\,dx = 2\int_{0}^{1}(2-x)\,dx = 2\left[2x-\frac{x^2}{2}\right]_0^1$$

$$= 2\left[2-\frac{1}{2}\right] = 2\times\frac{3}{2} = 3$$

(AU) (9) Evaluate $\int_0^{\pi/2}\frac{1}{1+\tan x}\,dx$. $\left((\text{or}) \int_0^{\pi/2}\frac{\cos x}{\sin x+\cos x}\,dx\right)$

Sol: Let $I = \int_0^{\pi/2}\frac{\cos x}{\sin x+\cos x}\,dx$ — (1) $= \int_0^{\pi/2}\frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$

$\left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$$\therefore I = \int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}\,dx \quad \text{— (2)}$$

$$(1)+(2) \Rightarrow 2I = \int_0^{\pi/2}\frac{\cos x}{\sin x+\cos x}\,dx + \int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}\,dx$$

$$= \int_0^{\pi/2}\frac{\sin x+\cos x}{\sin x+\cos x}\,dx = \int_0^{\pi/2}dx = (x)_0^{\pi/2} = \frac{\pi}{2}-0 = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{2\times 2} = \frac{\pi}{4}$$

$$\therefore \int_0^{\pi/2}\frac{1}{1+\tan x}\,dx = \frac{\pi}{4}$$

(H.W) (1) Evaluate: (i) $\int_0^1 (4+3x^2)\,dx$ (ii) $\int_2^1\left(1+\frac{z}{2}\right)dz$

(2) Evaluate: (i) $\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}\,dx$ (ii) $\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

---

(10) Evaluate $\int_0^{\pi/2}\log(\tan x)\,dx$.

Sol: Let $I = \int_0^{\pi/2}\log(\tan x)\,dx$ — (1)

$$I = \int_0^{\pi/2}\log\left(\tan\left(\frac{\pi}{2}-x\right)\right)dx = \int_0^{\pi/2}\log(\cot x)\,dx \quad \text{— (2)}$$

$$(1)+(2) \Rightarrow 2I = \int_0^{\pi/2}\log(\tan x)\,dx + \int_0^{\pi/2}\log(\cot x)\,dx$$

$$= \int_0^{\pi/2}\left(\log(\tan x)+\log(\cot x)\right)dx = \int_0^{\pi/2}\log(\tan x\cdot\cot x)\,dx$$

$$= \int_0^{\pi/2} \log 1 \, dx = \int_0^{\pi/2} 0 . dx = 0$$

$$\therefore I = 0 \Rightarrow \int_0^{\pi/2} \log(\tan x) dx = 0$$

Substitution rule:

(11) Evaluate: $\int (x+1)\sqrt{2x+x^2}\, dx$

Sol: Put $u = 2x + x^2$

$$du = (2+2x)dx = 2(1+x)dx \Rightarrow (x+1)dx = \frac{du}{2}$$

$$\therefore \int (x+1)\sqrt{2x+x^2}\, dx = \int \sqrt{u}\, \frac{du}{2} = \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{2}\int u^{1/2} du$$

$$= \frac{1}{2}\left[\frac{u^{1/2+1}}{1/2+1}\right] + c = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right] + c$$

$$= \frac{1}{2} \times \frac{2}{3} u^{3/2} + c = \frac{1}{3}(2x+x^2)^{3/2} + c$$

(12) Evaluate: $\int \frac{x^2}{\sqrt{x+5}} dx$

$$\boxed{u^2 = x+5 \Rightarrow u = \sqrt{x+5} = (x+5)^{1/2}}$$

Sol: Put $u^2 = x+5 \Rightarrow x = u^2 - 5$

$$2u\,du = dx$$

$$\therefore \int \frac{x^2}{\sqrt{x+5}} dx = \int \frac{(u^2-5)^2}{\sqrt{u^2}} 2u\,du = \int \frac{(u^2-5)^2}{u} 2u\,du = 2\int (u^2-5)^2 du$$

$$= 2\int (u^4 + 25 - 10u^2) du = 2\left(\frac{u^5}{5} + 25u - 10\frac{u^3}{3}\right) + c$$

$$= 2\left(\frac{(x+5)^{5/2}}{5} + 25\sqrt{x+5} - \frac{10(x+5)^{3/2}}{3}\right) + c$$

(13) Evaluate: $\int_1^e \frac{\log x}{x} dx$

Sol: Put $u = \log x$

$$du = \frac{1}{x} dx$$

when $x = 1 \Rightarrow u = \log 1 = 0$

$x = e \Rightarrow u = \log e = 1$

$$\therefore \int_1^e \frac{\log x}{x} dx = \int_0^1 u\,du = \left(\frac{u^2}{2}\right)_0^1 = \frac{1}{2}$$

(14) Evaluate: $\int \frac{\sec^2(\log x)}{x} dx$

Sol: Put $u = \log x \Rightarrow du = \frac{1}{x} dx$

$$\int \frac{\sec^2(\log x)}{x}dx = \int \sec^2 u\, du = \tan u + c = \tan(\log x) + c$$

(15) <u>Evaluate</u>: $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}dx$

<u>Sol</u>: Put $u = e^{1/x}$

$$du = e^{1/x}\cdot\left(\frac{-1}{x^2}\right)dx \Rightarrow \frac{dx}{x^2} = \frac{-du}{e^{1/x}} = -\frac{du}{u}$$

When $x = 1 \Rightarrow u = e$

$x = 2 \Rightarrow u = e^{1/2} = \sqrt{e}$

$$\therefore \int_1^2 \frac{e^{1/x}}{x^2}dx = \int_e^{\sqrt{e}} u\left(-\frac{du}{u}\right) = -\int_e^{\sqrt{e}} du = -(u)_e^{\sqrt{e}} = -(\sqrt{e} - e) = e - \sqrt{e}$$

(16) <u>Evaluate</u>: $\displaystyle\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx.$

<u>Sol</u>: Put $u = \tan^{-1}x \Rightarrow x = \tan u$

$$du = \frac{1}{1+x^2}dx$$

$$1+x+x^2 = 1+\tan u + \tan^2 u = \tan u + \sec^2 u$$

$$\therefore \int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx = \int e^u\left(\tan u + \sec^2 u\right)du$$

Put $t = e^u \tan u$

$$dt = (e^u \sec^2 u + \tan u\, e^u)du = e^u(\tan u + \sec^2 u)du$$

$$\therefore \int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx = \int dt = t + c$$

$$= e^u \tan u + c = e^{\tan^{-1}x}\tan(\tan^{-1}x) + c$$

$$= x e^{\tan^{-1}x} + c$$

(H.W) (1) <u>Evaluate the following</u>:

(i) $\displaystyle\int \cos^3\theta \sin\theta\, d\theta$ (ii) $\displaystyle\int \sec^2\theta \tan^2\theta\, d\theta$

(iii) $\displaystyle\int \frac{e^x}{e^x+1}dx$ (iv) $\displaystyle\int_0^1 \frac{e^x+1}{e^x+x}dx$

(v) $\displaystyle\int \frac{(\log x)^2}{x}dx$ (vi) $\displaystyle\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}dx$

Integration by parts:

(17) Evaluate: $\int x\cos 5x\,dx$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Sol: Let $u = x$, $dv = \cos 5x\,dx$

$du = dx$, $\int dv = \int \cos 5x\,dx$

$v = \frac{\sin 5x}{5}$

$$\therefore \int x\cos 5x\,dx = x\frac{\sin 5x}{5} - \int \frac{\sin 5x}{5}\,dx$$

$$= \frac{x}{5}\sin 5x - \frac{1}{5}\left(-\frac{\cos 5x}{5}\right) + c$$

$$= \frac{x}{5}\sin 5x + \frac{1}{25}\cos 5x + c$$

(18) Evaluate: $\int x^5 e^x\,dx$

Bernoulli's formula:

$$\boxed{\int u\,dv = uv - u'v_1 + u''v_2 - \cdots}$$

Sol: $u = x^5$ $\quad dv = e^x dx$

$v = e^x$

$u' = 5x^4$ $\quad v_1 = e^x$

$u'' = 20x^3$ $\quad v_2 = e^x$

$u''' = 60x^2$ $\quad v_3 = e^x$

$u^{iv} = 120x$ $\quad v_4 = e^x$

$u^{v} = 120$ $\quad v_5 = e^x$

$$\therefore \int x^5 e^x\,dx = x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + c$$

(AU) (19) Using integration by parts, evaluate $\int \frac{(\ln x)^2}{x^2}\,dx$.

$$\boxed{\ln x = \log x}$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Sol:

$u = (\log x)^2$	$dv = \frac{dx}{x^2} = x^{-2}dx$
$du = 2\log x \cdot \frac{1}{x}\,dx$	$v = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = \frac{-1}{x}$

$$\therefore \int \frac{(\ln x)^2}{x^2}\,dx = \int \frac{(\log x)^2}{x^2}\,dx$$

$$= (\log x)^2 \cdot \frac{-1}{x} - \int \frac{-1}{x}\cdot 2\log x \cdot \frac{1}{x}\,dx$$

$$= -\frac{1}{x}(\log x)^2 + 2\int \frac{\log x}{x^2}\,dx$$

$u = \log x$	$dv = \frac{dx}{x^2} = x^{-2}dx$
$du = \frac{1}{x}\,dx$	$v = \frac{-1}{x}$

$$\int \frac{(\ln x)^2}{x^2}dx = \frac{-1}{x}(\log x)^2 + 2\left[\log x \cdot \frac{-1}{x} - \int \frac{-1}{x}\cdot\frac{1}{x}dx\right]$$

$$= \frac{-1}{x}(\log x)^2 - \frac{2}{x}\log x + 2\int \frac{dx}{x^2}$$

$$= \frac{-1}{x}(\log x)^2 - \frac{2}{x}\log x + 2\int x^{-2}dx$$

$$= \frac{-1}{x}(\log x)^2 - \frac{2}{x}\log x + 2\left(\frac{x^{-2+1}}{-2+1}\right) + c$$

$$= \frac{-1}{x}(\log x)^2 - \frac{2}{x}\log x + 2\left(\frac{x^{-1}}{-1}\right) + c$$

$$= \frac{-1}{x}(\log x)^2 - \frac{2}{x}\log x - \frac{2}{x} + c$$

(N) (20) Evaluate $\int e^{ax}\cos bx\,dx$ using integration by parts.

Sol:

$u = e^{ax}$	$dv = \cos bx\,dx$
$du = e^{ax}\cdot a\,dx$	$v = \frac{\sin bx}{b}$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Let $I = \int e^{ax}\cos bx\,dx = e^{ax}\frac{\sin bx}{b} - \int \frac{\sin bx}{b}e^{ax}a\,dx$

$$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx$$

$u = e^{ax}$	$dv = \sin bx\,dx$
$du = e^{ax}a\,dx$	$v = -\frac{\cos bx}{b}$

$$\therefore I = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[-e^{ax}\frac{\cos bx}{b} - \int -\frac{\cos bx}{b}e^{ax}a\,dx\right]$$

$$I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

$$I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I$$

$$I + \frac{a^2}{b^2}I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

$$I\left(1 + \frac{a^2}{b^2}\right) = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

$$I = \frac{b^2}{a^2+b^2}\left(\frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx\right)$$

$$\therefore I = \frac{1}{a^2+b^2}\left(be^{ax}\sin bx + ae^{ax}\cos bx\right) + c$$

(AU) (21) Evaluate $\int e^x \sin x\,dx$ by using integration by parts.

Sol:

$u = e^x$	$dv = \sin x\,dx$
$du = e^x dx$	$v = -\cos x$

$\boxed{\int u\,dv = uv - \int v\,du}$

Let $I = \int e^x \sin x\,dx = e^x(-\cos x) - \int -\cos x\, e^x dx$

$= -e^x \cos x + \int e^x \cos x\,dx$

$u = e^x$	$dv = \cos x\,dx$
$du = e^x dx$	$v = \sin x$

$\therefore I = -e^x \cos x + \left[e^x \sin x - \int \sin x\, e^x dx\right]$

$= -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$

$\therefore I = -e^x \cos x + e^x \sin x - I$

$2I = -e^x \cos x + e^x \sin x \Rightarrow I = \frac{1}{2} e^x(\sin x - \cos x) + c$

(H.W)

(1) Evaluate $\int e^{ax} \sin bx\,dx$ using integration by parts.

(2) Evaluate $\int e^x \cos x\,dx$ using integration by parts.

Reduction formula:

(22) Establish a reduction formula for $I_n = \int \sin^n x\,dx$. Hence find $\int_0^{\pi/2} \sin^n x\,dx$.

Sol: Given $I_n = \int \sin^n x\,dx$ — (1)

$= \int \sin^{n-1} x \sin x\,dx$

$\boxed{\int u\,dv = uv - \int v\,du}$

$u = \sin^{n-1} x$	$dv = \sin x\,dx$
$du = (n-1)\sin^{n-2} x \cos x\,dx$	$v = -\cos x$

$\therefore I_n = \sin^{n-1} x(-\cos x) - \int (-\cos x)(n-1)\sin^{n-2} x \cos x\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1 - \sin^2 x)\,dx \quad (\because \sin^2 x + \cos^2 x = 1)$

$= -\sin^{n-1} x \cos x + (n-1)\int (\sin^{n-2} x - \sin^n x)\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$

$\therefore I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n \quad (\because \text{by } ①)$

$I_n + (n-1)I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$

$I_n(1+n-1) = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$

$nI_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$

$$\boxed{\therefore I_n = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}I_{n-2}}$$

$$\boxed{\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx}$$

$I_0 = \int \sin^0 x\,dx = \int dx = x + c$

$I_1 = \int \sin x\,dx = -\cos x + c$

Now consider, $I_n = \int_0^{\pi/2} \sin^n x\,dx$

$$I_n = \left(\frac{-1}{n}\sin^{n-1}x\cos x\right)_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$$

$$= (0+0) + \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx = \frac{n-1}{n}I_{n-2}$$

$\therefore I_n = \frac{n-1}{n}I_{n-2}$ —②

$$I_{n-2} = \frac{n-2-1}{n-2}I_{n-2-2} = \frac{n-3}{n-2}I_{n-4}$$

$$I_{n-4} = \frac{n-4-1}{n-4}I_{n-4-2} = \frac{n-5}{n-4}I_{n-6}$$

$\vdots$

$$\therefore I_n = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2}I_0 & \text{if } n \text{ is even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}I_1 & \text{if } n \text{ is odd} \end{cases}$$

$$\boxed{\begin{aligned} &② \Rightarrow I_2 = \frac{2-1}{2}I_0 \\ &I_2 = \frac{1}{2}I_0 \\ &② \Rightarrow I_3 = \frac{3-1}{3}I_{3-2} \\ &I_3 = \frac{2}{3}I_1 \end{aligned}}$$

$$I_0 = \int_0^{\pi/2}\sin^0 x\,dx = \int_0^{\pi/2}dx = (x)_0^{\pi/2} = \frac{\pi}{2}$$

$$I_1 = \int_0^{\pi/2}\sin x\,dx = (-\cos x)_0^{\pi/2} = -\cos\pi/2 + \cos 0 = -0+1 = 1$$

$$\therefore I_n = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2}\cdot\frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}\cdot 1 & \text{if } n \text{ is odd} \end{cases}$$

(23) Establish a reduction formula for $I_n = \int \cos^n x\,dx$. Hence find $\int_0^{\pi/2} \cos^n x\,dx$.

Sol: Given $I_n = \int \cos^n x\,dx$ —(1)

$= \int \cos^{n-1}x \cos x\,dx$

$\boxed{\int u\,dv = uv - \int v\,du}$

$u = \cos^{n-1}x$	$dv = \cos x\,dx$
$du = (n-1)\cos^{(n-2)}x(-\sin x)\,dx$	$v = \sin x$

$\therefore I_n = \cos^{n-1}x \sin x - \int \sin x (n-1)\cos^{n-2}x(-\sin x)\,dx$

$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x \sin^2 x\,dx$

$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)\,dx$

$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$

$I_n = \cos^{n-1}x \sin x + (n-1)I_{n-2} - (n-1)I_n$

$I_n + (n-1)I_n = \cos^{n-1}x \sin x + (n-1)I_{n-2}$

$I_n(1+n-1) = \cos^{n-1}x \sin x + (n-1)I_{n-2}$

$\therefore nI_n = \cos^{n-1}x \sin x + (n-1)I_{n-2}$

$$\boxed{I_n = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}I_{n-2}}$$

$$\therefore \boxed{\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx}$$

$I_0 = \int \cos^0 x\,dx = \int dx = x + c$

$I_1 = \int \cos x\,dx = \sin x + c$

Now consider, $I_n = \int_0^{\pi/2} \cos^n x\,dx$

$I_n = \left(\frac{1}{n}\cos^{n-1}x \sin x\right)_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2}x\,dx$

$= 0 + \frac{n-1}{n}I_{n-2}$

$\therefore I_n = \frac{n-1}{n}I_{n-2}$

$I_{n-2} = \frac{n-2-1}{n-2}I_{n-2-2} = \frac{n-3}{n-2}I_{n-4}$

$I_{n-4} = \frac{n-4-1}{n-4}I_{n-4-2} = \frac{n-5}{n-4}I_{n-6}$

$\vdots$

$$\therefore I_n = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2}\cdot I_0 & \text{if } n \text{ is even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}I_1 & \text{if } n \text{ is odd} \end{cases}$$

$$I_0 = \int_0^{\pi/2} \cos^0 x\,dx = \int_0^{\pi/2} dx = (x)_0^{\pi/2} = \frac{\pi}{2}$$

$$I_1 = \int_0^{\pi/2} \cos x\,dx = (\sin x)_0^{\pi/2} = \sin \pi/2 - \sin 0 = 1 - 0 = 1$$

$$\therefore I_n = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1 & \text{if } n \text{ is odd} \end{cases}$$

(24) Find the value of (i) $\int \sin^3 x\,dx$ (ii) $\int \sin^4 x\,dx$ (iii) $\int_0^{\pi/2} \sin^7 x\,dx$ (iv) $\int_0^{\pi/2} \sin^8 x\,dx$ (v) $\int_0^{\pi/2} \sin^{2n} x\,dx$.

Sol: We know that $\boxed{\int \sin^n x\,dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\,dx}$

(i) $\int \sin^3 x\,dx = -\frac{1}{3} \sin^2 x \cos x + \frac{3-1}{3} \int \sin x\,dx$ (Here $n=3$)

$= -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3}(-\cos x) + c$

$\therefore \int \sin^3 x\,dx = -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + c$

(ii) $\int \sin^4 x\,dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x\,dx$ (Here $n=4$)

$= -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4}\left[-\frac{1}{2} \sin x \cos x + \frac{1}{2} \int \sin^0 x\,dx\right]$ (Here $n=2$)

$= -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + c$

$= -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \frac{\sin 2x}{2} + \frac{3}{8} x + c$ ($\because 2 \sin x \cos x = \sin 2x$)

$= -\frac{1}{4} \sin^3 x \cos x - \frac{3}{16} \sin 2x + \frac{3}{8} x + c$

(iii) $\int_0^{\pi/2} \sin^7 x\,dx$ (Here $n=7 \Rightarrow$ odd)

We know that
$$\boxed{\int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1 & \text{if } n \text{ is odd} \end{cases}}$$

$$\therefore \int_0^{\pi/2} \sin^7 x\,dx = \frac{7-1}{7} \cdot \frac{7-3}{7-2} \cdot \frac{7-5}{7-4} \cdot 1 = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{16}{35}$$

(iv) $\int_0^{\pi/2} \sin^8 x\,dx = \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{35}{256}\pi$

(Here $n = 8 \Rightarrow$ even)

(v) $\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$ (Here 2n is even)

AU

(25) Evaluate $\int_0^{\pi/2} \cos^5 x\,dx$.

Sol: We know that
$$\boxed{\int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1 & \text{if } n \text{ is odd} \end{cases}}$$

Here $n=5$. $\therefore$ n is odd.

$$\therefore \int_0^{\pi/2} \cos^5 x\,dx = \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{8}{15}$$

H.W

(1) Find the value of (i) $\int \sin^2 x\,dx$ (ii) $\int_0^{\pi/2} \sin^5 x\,dx$ (iii) $\int \cos^3 x\,dx$

(iv) $\int_0^{\pi/2} \cos^{10} x\,dx$

(26) Find the reduction formula for $\int \sec^n x\,dx$, $n \geq 2$ is an integer.

Sol: Let $I_n = \int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$

$u = \sec^{n-2} x$ $\qquad dv = \sec^2 x\,dx$

$du = (n-2)\sec^{n-3} x (\sec x \tan x)\,dx$ $\qquad v = \tan x$

$\boxed{\int u\,dv = uv - \int v\,du}$

$$\therefore I_n = \sec^{n-2} x \tan x - \int \tan x (n-2) \sec^{n-3} x \sec x \tan x\,dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x (\sec^2 x - 1)\,dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int (\sec^n x - \sec^{n-2} x)\,dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$
$$\Rightarrow I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$
$$I_n (1+n-2) = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$\boxed{I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}}$$

$I_0 = \int \sec^0 x\,dx = \int dx = x + c$

$I_1 = \int \sec x\,dx = \log(\sec x + \tan x) + c$

(27) Find the reduction formula for $\int \text{cosec}^n x\,dx$, $n \geq 2$ is an integer.

Sol: Let $I_n = \int \text{cosec}^n x\,dx = \int \text{cosec}^{n-2} x\,\text{cosec}^2 x\,dx$

$u = \text{cosec}^{n-2} x$	$dv = \text{cosec}^2 x\,dx$
$du = (n-2)\,\text{cosec}^{n-3} x\,(-\text{cosec}\,x \cot x)\,dx$	$v = -\cot x$

$$\therefore I_n = \text{cosec}^{n-2} x\,(-\cot x) - \int (-\cot x)(n-2)\,\text{cosec}^{n-3} x\,(-\text{cosec}\,x\cot x)\,dx$$

$$= -\text{cosec}^{n-2} x \cot x - (n-2)\int \text{cosec}^{n-2} x \cot^2 x\,dx$$

$$= -\text{cosec}^{n-2} x \cot x - (n-2)\int \text{cosec}^{n-2} x\,(\text{cosec}^2 x - 1)\,dx$$

$$= -\text{cosec}^{n-2} x \cot x - (n-2)\int \text{cosec}^{n} x\,dx + (n-2)\int \text{cosec}^{n-2} x\,dx$$

$$I_n = -\text{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$$

$$\therefore I_n + (n-2) I_n = -\text{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$$

$$I_n (1 + n - 2) = -\text{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$$

$$\therefore \boxed{I_n = \frac{-1}{n-1}\,\text{cosec}^{n-2} x \cot x + \frac{n-2}{n-1} I_{n-2}}$$

$$I_0 = \int \text{cosec}^0 x\,dx = \int dx = x + c$$

$$I_1 = \int \text{cosec}\,x\,dx = \log(\text{cosec}\,x - \cot x) + c$$

(28) Find the reduction formula for $\int \tan^n x\,dx$.

Sol: Let $I_n = \int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx$

$$= \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

Put $u = \tan x$
$du = \sec^2 x\,dx$

$$= \int u^{n-2}\,du - \int \tan^{n-2} x\,dx$$

$$= \frac{u^{n-2+1}}{n-2+1} - I_{n-2} = \frac{u^{n-1}}{n-1} - I_{n-2}$$

$$\therefore I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$I_0 = \int \tan^0 x\,dx = \int dx = x + c$$

$$I_1 = \int \tan x\,dx = \log(\sec x) + c$$

(29) Find the reduction formula for $\int \cot^n x\,dx$, $n \neq 1$.

Sol: Let $I_n = \int \cot^n x\,dx$

$$= \int \cot^{n-2} x \cot^2 x\,dx = \int \cot^{n-2} x\,(\operatorname{cosec}^2 x - 1)\,dx$$

$$= \int \cot^{n-2} x \operatorname{cosec}^2 x\,dx - \int \cot^{n-2} x\,dx$$

Put $u = \cot x$

$du = -\operatorname{cosec}^2 x\,dx$

$-du = \operatorname{cosec}^2 x\,dx$

$$= \int u^{n-2}(-du) - I_{n-2}$$

$$= -\left[\frac{u^{n-2+1}}{n-2+1}\right] - I_{n-2}$$

$$= -\frac{u^{n-1}}{n-1} - I_{n-2} = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

$$\therefore I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

$$I_0 = \int \cot^0 x\,dx = \int dx = x + c$$

$$I_1 = \int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \log(\sin x) + c$$

(30) Evaluate: $\int \sin^6 x \cos^3 x\,dx$.

Sol: Let $I = \int \sin^6 x \cos^3 x\,dx = \int \sin^6 x \cos^2 x \cos x\,dx = \int \sin^6 x\,(1 - \sin^2 x)\cos x\,dx$

Put $u = \sin x$

$du = \cos x\,dx$

$$\therefore I = \int u^6 (1 - u^2)\,du = \int (u^6 - u^8)\,du = \left(\frac{u^7}{7} - \frac{u^9}{9}\right) + c$$

$$= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + c$$

(31) Evaluate: $\int \sin^5 x \cos^2 x\,dx$.

Sol: Let $I = \int \sin^5 x \cos^2 x\,dx = \int \sin^4 x \cos^2 x \sin x\,dx$

$$= \int (1 - \cos^2 x)^2 \cos^2 x \sin x\,dx$$

Put $u = \cos x$

$du = -\sin x\,dx \Rightarrow -du = \sin x\,dx$

$$\therefore I = \int (1-u^2)^2 u^2 (-du) = -\int (1 + u^4 - 2u^2)\,u^2\,du = -\int (u^2 + u^6 - 2u^4)\,du$$

$$= -\left(\frac{u^3}{3} + \frac{u^7}{7} - \frac{2u^5}{5}\right) + c = -\frac{\cos^3 x}{3} - \frac{\cos^7 x}{7} + \frac{2\cos^5 x}{5} + c$$

32 Evaluate: $\int \cos^2 x \sin 2x \, dx$.

Sol: Let $I = \int \cos^2 x \sin 2x \, dx = \int \cos^2 x \, 2\sin x \cos x \, dx$

$= 2\int \cos^3 x \sin x \, dx$ $\qquad (\because \sin 2x = 2\sin x \cos x)$

Put $u = \cos x$

$du = -\sin x \, dx \Rightarrow \sin x \, dx = -du$

$\therefore I = 2\int u^3 (-du) = -2\int u^3 du = -2\left(\frac{u^4}{4}\right) + c = \frac{-1}{2}\cos^4 x + c$

AU

33 Evaluate: $\int_0^{\pi} \sin^2 x \cos^4 x \, dx$

Sol: Let $I = \int_0^{\pi} \sin^2 x \cos^4 x \, dx = \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$

$= \frac{1}{8}\int_0^{\pi} (1-\cos 2x)(1+\cos^2 2x + 2\cos 2x) \, dx$

$= \frac{1}{8}\int_0^{\pi} (1+\cos^2 2x + 2\cos 2x - \cos 2x - \cos^3 2x - 2\cos^2 2x) \, dx$

$= \frac{1}{8}\int_0^{\pi} (1-\cos^2 2x + \cos 2x - \cos^3 2x) \, dx$ —— (1)

$\int \cos^2 2x \, dx = \int \left(\frac{1+\cos 4x}{2}\right) dx = \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right)$

$\int \cos^3 2x \, dx = \int \cos^2 2x \cos 2x \, dx = \int (1-\sin^2 2x)\cos 2x \, dx$

Put $u = \sin 2x$

$du = 2\cos 2x \, dx \Rightarrow \cos 2x \, dx = \frac{du}{2}$

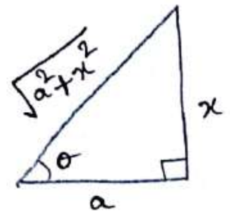
$\therefore \int \cos^3 2x \, dx = \int (1-u^2)\frac{du}{2} = \frac{1}{2}\left(u - \frac{u^3}{3}\right) = \frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right)$

$\therefore I = \frac{1}{8}\left[x - \frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{\sin 2x}{2} - \frac{1}{2}\sin 2x + \frac{1}{6}\sin^3 2x\right]_0^{\pi}$

$= \frac{1}{8}\left[\frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right]_0^{\pi}$ $\qquad \boxed{\begin{array}{l}\sin n\pi = 0 \\ \sin 0 = 0\end{array}}$

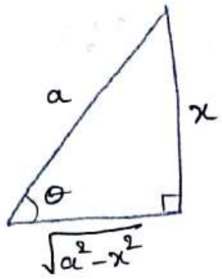
$= \frac{1}{8}\left[\frac{\pi}{2}\right] = \frac{\pi}{16}$

Trigonometric substitution:



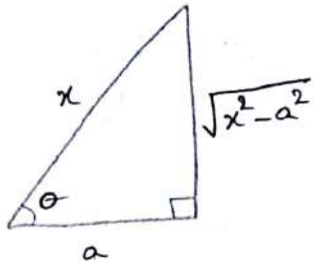
$\tan\theta = \frac{x}{a}$

$\boxed{x = a\tan\theta}$



$\sin\theta = \frac{x}{a}$

$\boxed{x = a\sin\theta}$



$\cos\theta = \frac{a}{x}$

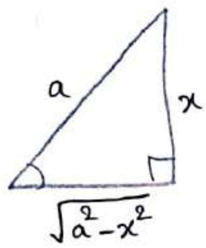
$x = \frac{a}{\cos\theta} = a\sec\theta$

$\boxed{x = a\sec\theta}$

---

(34) Evaluate $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$.

$$\boxed{\begin{aligned} x &= a\sin\theta \\ \theta &= \sin^{-1}(x/a) \\ \theta &= \sin^{-1}(x/3) \end{aligned}}$$



$\sin\theta = \frac{x}{a} = \frac{x}{3}$

$\cos\theta = \frac{\sqrt{a^2-x^2}}{a} = \frac{\sqrt{9-x^2}}{3}$

Sol: Put $x = a\sin\theta$. Here $a = 3$

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$\therefore \int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{(3\sin\theta)^2}{\sqrt{9-(3\sin\theta)^2}}\, 3\cos\theta\, d\theta$$

$$= \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\, 3\cos\theta\, d\theta = 9\int \frac{\sin^2\theta}{3\sqrt{1-\sin^2\theta}}\, 3\cos\theta\, d\theta$$

$$= 9\int \frac{\sin^2\theta}{\cos\theta}\cos\theta\, d\theta = 9\int \sin^2\theta\, d\theta = 9\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + c = \frac{9}{2}\left[\theta - \frac{2\sin\theta\cos\theta}{2}\right] + c$$

$$= \frac{9}{2}\left[\theta - \sin\theta\cos\theta\right] + c$$

$$= \frac{9}{2}\left[\sin^{-1}(x/3) - \frac{x}{3}\,\frac{\sqrt{9-x^2}}{3}\right] + c$$

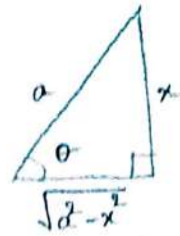
$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + c$$

AU

35) Evaluate $\int \sqrt{a^2-x^2}\,dx$ by using substitution rule.

Sol: Put $x = a\sin\theta \Rightarrow \theta = \sin^{-1}(x/a)$

$dx = a\cos\theta\,d\theta$



$\sin\theta = \dfrac{x}{a}$

$\cos\theta = \dfrac{\sqrt{a^2-x^2}}{a}$

$$\therefore \int\sqrt{a^2-x^2}\,dx = \int\sqrt{a^2-(a\sin\theta)^2}\;a\cos\theta\,d\theta$$

$$= \int\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\,d\theta$$

$$= a^2\int\sqrt{1-\sin^2\theta}\;\cos\theta\,d\theta = a^2\int\cos^2\theta\,d\theta$$

$$= a^2\int\left(\frac{1+\cos2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{2}\left[\theta+\frac{\sin2\theta}{2}\right]+c$$

$$= \frac{a^2}{2}\theta+\frac{a^2}{4}(2\sin\theta\cos\theta)+c$$

$$= \frac{a^2}{2}\theta+\frac{a^2}{2}\sin\theta\cos\theta+c$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+\frac{a^2}{2}\,\frac{x}{a}\,\frac{\sqrt{a^2-x^2}}{a}+c$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+c$$

---

36) Evaluate $\int\dfrac{x}{\sqrt{3-2x-x^2}}\,dx$.

Sol: Consider, $3-2x-x^2 = -(x^2+2x)+3$

$$= -(x^2+2x+1-1)+3$$

$$= -\left[(x+1)^2-1\right]+3$$

$$= -(x+1)^2+1+3 = 4-(x+1)^2$$

Put $u = x+1 \Rightarrow x = u-1$

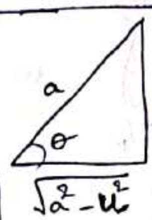
$du = dx$

$$\therefore \int\frac{x}{\sqrt{3-2x-x^2}}\,dx = \int\frac{u-1}{\sqrt{4-u^2}}\,du$$

Put $u = a\sin\theta$. Here $a=2$

$u = 2\sin\theta \Rightarrow \quad \theta = \sin^{-1}\dfrac{u}{2}$

$du = 2\cos\theta\,d\theta$



$\cos\theta = \frac{\sqrt{a^2-u^2}}{a} = \frac{\sqrt{4-u^2}}{2}$

$$\therefore \int \frac{x}{\sqrt{3-2x-x^2}}\,dx = \int \frac{u-1}{\sqrt{4-u^2}}\,du$$

$$= \int \frac{2\sin\theta - 1}{\sqrt{4-4\sin^2\theta}}\, 2\cos\theta\, d\theta$$

$$= \int \frac{2\sin\theta - 1}{2\cos\theta}\, 2\cos\theta\, d\theta = \int (2\sin\theta - 1)\,d\theta$$

$$= 2(-\cos\theta) - \theta + c$$

$$= -2\,\frac{\sqrt{4-u^2}}{2} - \sin^{-1}\frac{u}{2} + c$$

$$= -\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + c$$

$$= -\sqrt{4-x^2-1-2x} - \sin^{-1}\left(\frac{x+1}{2}\right) + c$$

$$= -\sqrt{3-x^2-2x} - \sin^{-1}\left(\frac{x+1}{2}\right) + c$$

(AO) (37) Evaluate $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$.

Sol: $$\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9\left(x^2-\frac{1}{9}\right)}} = \frac{1}{3}\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{x^2-\left(\frac{1}{3}\right)^2}} = I \text{ (Say)}$$

When $x = 2/3 \Rightarrow \frac{2}{3} = \frac{1}{3}\sec\theta$
$\Rightarrow \frac{2}{3}\times 3 = \frac{1}{\cos\theta} \Rightarrow \cos\theta = \frac{1}{2}$
$\Rightarrow \theta = \frac{\pi}{3}$

Put $x = a\sec\theta$

Here $a = 1/3$. $x = \frac{1}{3}\sec\theta$

$dx = \frac{1}{3}\sec\theta\tan\theta\, d\theta$

$$\sqrt{x^2-\left(\tfrac{1}{3}\right)^2} = \sqrt{\tfrac{1}{9}\sec^2\theta - \tfrac{1}{9}} = \frac{1}{3}\sqrt{\sec^2\theta - 1} = \frac{1}{3}\sqrt{\tan^2\theta} = \frac{1}{3}\tan\theta$$

$$\therefore I = \frac{1}{3}\int_{\pi/4}^{\pi/3} \frac{\frac{1}{3}\sec\theta\tan\theta}{\left(\frac{1}{3}\right)^5\sec^5\theta\cdot\frac{1}{3}\tan\theta}\,d\theta$$

When $x = \frac{\sqrt{2}}{3} \Rightarrow \frac{\sqrt{2}}{3} = \frac{1}{3}\sec\theta$
$\Rightarrow \frac{\sqrt{2}}{3}\times 3 = \frac{1}{\cos\theta} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$
$\Rightarrow \theta = \frac{\pi}{4}$

$$= \frac{1}{9}\times 3^6 \int_{\pi/4}^{\pi/3} \frac{1}{\sec^4\theta}\,d\theta$$

$$= 81\int_{\pi/4}^{\pi/3} \cos^4\theta\, d\theta = 81\int_{\pi/4}^{\pi/3} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{81}{4}\int_{\pi/4}^{\pi/3}(1+\cos 2\theta)^2\,d\theta$$

$$= \frac{81}{4}\int_{\pi/4}^{\pi/3}(1+\cos^2 2\theta+2\cos 2\theta)\,d\theta$$

$$= \frac{81}{4}\int_{\pi/4}^{\pi/3}\left(1+\frac{1+\cos 4\theta}{2}+2\cos 2\theta\right)d\theta$$

$$= \frac{81}{4}\int_{\pi/4}^{\pi/3}\left(1+\frac{1}{2}+\frac{\cos 4\theta}{2}+2\cos 2\theta\right)d\theta$$

$$= \frac{81}{4}\int_{\pi/4}^{\pi/3}\left(\frac{3}{2}+\frac{\cos 4\theta}{2}+2\cos 2\theta\right)d\theta$$

$$= \frac{81}{4}\left[\frac{3}{2}\theta+\frac{\sin 4\theta}{8}+\frac{2\sin 2\theta}{2}\right]_{\pi/4}^{\pi/3}$$

$$= \frac{81}{4}\left[\frac{3}{2}\left(\frac{\pi}{3}\right)+\frac{\sin 4\pi/3}{8}+\sin\frac{2\pi}{3}-\frac{3}{2}\left(\frac{\pi}{4}\right)-\frac{\sin\pi}{8}-\sin\frac{\pi}{2}\right]$$

$$= \frac{81}{4}\left[\frac{\pi}{2}-\frac{\sin\pi/3}{8}+\cos\frac{\pi}{6}-\frac{3\pi}{8}-1\right]$$

$$= \frac{81}{4}\left[\frac{\pi}{2}-\frac{\sqrt{3}}{2\times 8}+\frac{\sqrt{3}}{2}-\frac{3\pi}{8}-1\right]$$

$$= \frac{81}{4}\left[\frac{4\pi}{8}-\frac{3\pi}{8}-\frac{\sqrt{3}}{16}+\frac{\sqrt{3}}{2}-1\right] = \frac{81}{4}\left[\frac{\pi}{8}-\frac{\sqrt{3}}{16}+\frac{8\sqrt{3}}{16}-1\right]$$

$$= \frac{81}{4}\left[\frac{\pi}{8}+\frac{7\sqrt{3}}{16}-1\right] = \frac{81}{32}\left[\pi+\frac{7\sqrt{3}}{2}-8\right]$$

38. Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

Sol: Here $a=2$

Put $x = a\tan\theta \Rightarrow x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$$\therefore \int\frac{1}{x^2\sqrt{x^2+4}}\,dx = \int\frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\,2\sec^2\theta\,d\theta$$

$$= \frac{1}{2}\int \frac{1}{\tan^2\theta \, 2\sqrt{1+\tan^2\theta}} \sec^2\theta \, d\theta$$

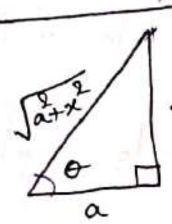
$$= \frac{1}{4}\int \frac{1}{\tan^2\theta \sec\theta} \sec^2\theta \, d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta} d\theta$$

$$= \frac{1}{4}\int \frac{1/\cos\theta}{\frac{\sin^2\theta}{\cos^2\theta}} d\theta = \frac{1}{4}\int \frac{1}{\cos\theta} \times \frac{\cos^2\theta}{\sin^2\theta} d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta} d\theta = \frac{1}{4}\int \operatorname{cosec}\theta \cot\theta \, d\theta$$

$$= -\frac{1}{4}\operatorname{cosec}\theta + c$$

$$= -\frac{1}{4}\frac{\sqrt{4+x^2}}{x} + c$$



$\sin\theta = \frac{x}{\sqrt{a^2+x^2}}$

$\sin\theta = \frac{x}{\sqrt{4+x^2}}$

$\operatorname{cosec}\theta = \frac{\sqrt{4+x^2}}{x}$

Integration of rational functions by partial fraction:

AU (39) Evaluate $\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx.$

Sol: Put $u = \cos x$

$du = -\sin x\, dx \Rightarrow \sin x\, dx = -du$

When $x = \pi/2 \Rightarrow u = \cos \pi/2 = 0$

$x = 0 \Rightarrow u = \cos 0 = 1$

$\times$	$+$
2	3
1	2
$u+1$	$u+2$

$$\therefore I = \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx = \int_1^0 \frac{u(-du)}{u^2+3u+2}$$

$$= -\int_1^0 \frac{u\,du}{(u+1)(u+2)} = \int_0^1 \frac{u\,du}{(u+1)(u+2)}$$

Consider, $\frac{u}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2} = \frac{A(u+2)+B(u+1)}{(u+1)(u+2)}$

$\therefore u = A(u+2) + B(u+1)$

Put $u = -2$

$-2 = -B \Rightarrow \boxed{B = 2}$

Put $u = -1$

$\boxed{-1 = A}$

$$\therefore \frac{u}{(u+1)(u+2)} = \frac{-1}{u+1} + \frac{2}{u+2}$$

$$\therefore I = \int_0^1 \frac{u\,du}{(u+1)(u+2)} = \int_0^1 \left(\frac{-1}{u+1} + \frac{2}{u+2}\right) du$$

$$=\left(-\log(u+1)+2\log(u+2)\right)_0^1$$

$$=-\log 2+2\log 3+\log 1-2\log 2$$

$$=-3\log 2+2\log 3=\log(2)^{-3}+\log(3)^2$$

$$=\log\frac{1}{2^3}+\log 9=\log\frac{1}{8}+\log 9$$

$$=\log\left(\frac{1}{8}\times 9\right)=\log\frac{9}{8}$$

---

(40) Evaluate $\int\frac{x^2+1}{(x-3)(x-2)^2}dx$.

Sol: Consider, $\frac{x^2+1}{(x-3)(x-2)^2}=\frac{A}{x-3}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$

$$x^2+1=A(x-2)^2+B(x-3)(x-2)+C(x-3)$$

Put $x=2$

$5=C(-1)$

$-C=5\Rightarrow \boxed{C=-5}$

Put $x=3$

$\boxed{10=A}$

Put $x=0$

$1=4A+6B-3C$

$1=40+6B+15\Rightarrow 6B=-54$

$\boxed{B=-9}$

$$\therefore\frac{x^2+1}{(x-3)(x-2)^2}=\frac{10}{x-3}-\frac{9}{x-2}-\frac{5}{(x-2)^2}$$

$$\therefore\int\frac{x^2+1}{(x-3)(x-2)^2}dx=10\int\frac{dx}{x-3}-9\int\frac{dx}{x-2}-5\int\frac{dx}{(x-2)^2}$$

$$=10\log(x-3)-9\log(x-2)-5\int(x-2)^{-2}dx$$

$$=10\log(x-3)-9\log(x-2)-5\left[\frac{(x-2)^{-2+1}}{-2+1}\right]+c$$

$$=10\log(x-3)-9\log(x-2)+5\frac{1}{x-2}+c$$

---

(41) Evaluate $\int\frac{2x^2-x+4}{x^3+4x}dx$.

Sol: Consider, $\frac{2x^2-x+4}{x^3+4x}=\frac{2x^2-x+4}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$

$2x^2-x+4 = A(x^2+4)+(Bx+c)x$

Put $x=0$

$4 = 4A \Rightarrow \boxed{A=1}$

Put $x=1$

$2-1+4 = 5A+B+C$

$5 = 5+B+C$

$\Rightarrow B+C=0$ —— (1)

Put $x=-1$

$2+1+4 = 5A-(B(-1)+c)$

$7 = 5+B-C$

$B-C=2$ —— (2)

(1)+(2) $\Rightarrow 2B=2 \Rightarrow \boxed{B=1}$

Subst. $B=1$ in (1), $1+C=0 \Rightarrow \boxed{C=-1}$

$$\therefore \int \frac{2x^2-x+4}{x^3+4x}dx = \int \frac{1}{x}dx + \int \frac{x-1}{x^2+4}dx$$

$$= \log x + \int \frac{x}{x^2+4}dx - \int \frac{dx}{x^2+4}$$

Put $x^2+4=u$
$2x\,dx = du$
$x\,dx = \frac{du}{2}$

$$= \log x + \int \frac{du/2}{u} - \frac{1}{2}\tan^{-1}\frac{x}{2}$$

$$= \log x + \frac{1}{2}\log u - \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

$$= \log x + \frac{1}{2}\log(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

(42) Evaluate $\int \frac{x^2}{x+2}dx$.

Sol:

$$\begin{array}{r|l} & x-2 \\ \hline x+2 & x^2 \\ & x^2+2x \\ & (-)\;(-) \\ \hline & -2x \\ & -2x-4 \\ & (+)\;\;(+) \\ \hline & 4 \end{array}$$

$$\frac{x^2}{x+2} = x-2+\frac{4}{x+2}$$

$$\therefore \int \frac{x^2}{x+2}dx = \int \left(x-2+\frac{4}{x+2}\right)dx = \frac{x^2}{2} - 2x + 4\log(x+2) + c$$

Working rule: $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Put $px+q = A\frac{d}{dx}(ax^2+bx+c)+B$

AU 43) Evaluate $\int \frac{2x+5}{\sqrt{x^2-2x+10}}\,dx$.

Sol: Put $2x+5 = A\frac{d}{dx}(x^2-2x+10)+B$

$$2x+5 = A(2x-2)+B \Rightarrow 2x+5 = 2Ax-2A+B$$

Equating like coefficients on both sides, we get

$2 = 2A \Rightarrow \boxed{A=1}$

$5 = -2A+B \Rightarrow 5 = -2+B \Rightarrow \boxed{B=7}$

$\therefore 2x+5 = (2x-2)+7$

$$\int \frac{2x+5}{\sqrt{x^2-2x+10}}\,dx = \int \frac{2x-2}{\sqrt{x^2-2x+10}}\,dx + \int \frac{7}{\sqrt{x^2-2x+10}}\,dx$$

Put $u = x^2-2x+10$
$du = (2x-2)\,dx$

$$= \int \frac{du}{\sqrt{u}} + 7\int \frac{dx}{\sqrt{x^2-2x+1-1+10}}$$

$$= \int u^{-1/2}\,du + 7\int \frac{dx}{\sqrt{(x-1)^2+9}}$$

Put $t = x-1$
$dt = dx$

$$\boxed{\begin{aligned}\int \frac{dx}{\sqrt{a^2-x^2}} &= \sin^{-1}\frac{x}{a}+c\\ \int \frac{dx}{\sqrt{x^2-a^2}} &= \cosh^{-1}\frac{x}{a}+c\\ \int \frac{dx}{\sqrt{a^2+x^2}} &= \sinh^{-1}\frac{x}{a}+c\end{aligned}}$$

$$= \frac{u^{-1/2+1}}{-1/2+1} + 7\int \frac{dt}{\sqrt{t^2+3^2}}$$

$$= \frac{u^{1/2}}{1/2} + 7\sinh^{-1}\frac{t}{3} + c$$

$$= 2\sqrt{x^2-2x+10} + 7\sinh^{-1}\left(\frac{x-1}{3}\right) + c$$

AU 44) Evaluate $\int \frac{x}{\sqrt{x^2+x+1}}\,dx$

Sol: Put $x = A\frac{d}{dx}(x^2+x+1)+B$

$$x = A(2x+1)+B \Rightarrow x = 2Ax+A+B$$

Equating like coefficients on both sides, we get

$1 = 2A \Rightarrow \boxed{A = 1/2}$

$0 = A+B \Rightarrow 0 = 1/2 + B \Rightarrow \boxed{B = -1/2}$

$\therefore x = 1/2(2x+1) - 1/2$

$$\therefore \int \frac{x}{\sqrt{x^2+x+1}}dx = \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$$

Put $u = x^2+x+1$
$du = (2x+1)\,dx$

$$= \frac{1}{2}\int \frac{du}{\sqrt{u}} - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+2\cdot\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+1}}$$

$$= \frac{1}{2}\int u^{-1/2}du - \frac{1}{2}\int \frac{dx}{\sqrt{(x+1/2)^2+3/4}}$$

Put $t = x+1/2$
$dt = dx$

$$= \frac{1}{2}\,\frac{u^{-1/2+1}}{-1/2+1} - \frac{1}{2}\int \frac{dt}{\sqrt{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \frac{1}{2}\,\frac{u^{1/2}}{1/2} - \frac{1}{2}\sinh^{-1}\frac{t}{\sqrt{3}/2} + c$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\sinh^{-1}\left(\frac{2}{\sqrt{3}}(x+1/2)\right) + c$$

AU (45) Evaluate $\int_3^\infty \frac{dx}{(x-2)^{3/2}}$ & determine whether it is convergent or divergent.

Sol: $$\int_3^\infty \frac{dx}{(x-2)^{3/2}} = \lim_{t\to\infty}\int_3^t \frac{dx}{(x-2)^{3/2}} = \lim_{t\to\infty}\int_3^t (x-2)^{-3/2}dx$$

$$= \lim_{t\to\infty}\left[\frac{(x-2)^{-3/2+1}}{-3/2+1}\right]_3^t = \lim_{t\to\infty}\left[\frac{(x-2)^{-1/2}}{-1/2}\right]_3^t$$

$$= \lim_{t\to\infty}\left[-2\frac{1}{\sqrt{x-2}}\right]_3^t = \lim_{t\to\infty}\left[\frac{-2}{\sqrt{t-2}}+\frac{2}{\sqrt{1}}\right]$$

$$= \frac{-2}{\infty}+2 = 0+2 = 2$$

$\therefore \int_3^\infty \frac{dx}{(x-2)^{3/2}}$ is convergent.

AU (46) Evaluate $\int_4^\infty \frac{1}{\sqrt{x}}dx$ & determine whether it is convergent or divergent.

Sol: $$\int_4^\infty \frac{1}{\sqrt{x}}dx = \lim_{t\to\infty}\int_4^t x^{-1/2}dx = \lim_{t\to\infty}\left[\frac{x^{-1/2+1}}{-1/2+1}\right]_4^t$$

$$= \lim_{t\to\infty}\left[\frac{x^{1/2}}{1/2}\right]_4^t = \lim_{t\to\infty}\left[2\sqrt{x}\right]_4^t = \lim_{t\to\infty}\left[2\sqrt{t}-2\sqrt{4}\right]$$

$$= \lim_{t\to\infty}\left[2\sqrt{t}-4\right] = \infty-4 = \infty$$

$\therefore \int_4^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent.

(AU) (47) Determine whether the given integral $\int_0^\infty e^x\,dx$ is convergent or divergent.

Sol: $\int_0^\infty e^x\,dx = \lim_{t\to\infty}\int_0^t e^x\,dx = \lim_{t\to\infty}(e^x)_0^t = \lim_{t\to\infty}(e^t - e^0)$

$= \lim_{t\to\infty}(e^t - 1) = e^\infty - 1 = \infty - 1 = \infty$

$\therefore \int_0^\infty e^x\,dx$ is divergent.

(AU) (48) For what values of p is $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

Sol: If $p \neq 1$,

$\lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t$

$= \lim_{t\to\infty}\left[\frac{t^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right]$

$= \lim_{t\to\infty}\left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right] = \lim_{t\to\infty}\left[\frac{1}{1-p}\left(t^{-p+1} - 1\right)\right]$

$= \lim_{t\to\infty}\left[\frac{1}{1-p}\left(t^{-(p-1)} - 1\right)\right] = \lim_{t\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{t^{p-1}} - 1\right)\right]$

$\boxed{= \frac{1}{1-p}\left[\frac{1}{\infty} - 1\right] = \frac{1}{1-p}\left[0 - 1\right] = \frac{-1}{1-p} = \frac{1}{p-1}}$ → Rough work

$= \lim_{t\to\infty}\left[\frac{1}{p-1}\left(1 - \frac{1}{t^{p-1}}\right)\right]$

$= \begin{cases} \frac{1}{p-1}, & p > 1, \text{ converges} \\ \infty; & p \le 1, \text{ diverges} \end{cases}$

MULTIPLE INTEGRALS

(AU) 1) Evaluate $\int_1^a \int_2^b \frac{dx\,dy}{xy}$.

Sol: $\int_{y=1}^{a} \int_{x=2}^{b} \frac{dx\,dy}{xy} = \int_1^a \int_2^b \frac{dx}{x} \frac{dy}{y} = \int_1^a (\log x)_2^b \frac{dy}{y}$

$= \int_1^a (\log b - \log 2) \frac{dy}{y}$

$= (\log b - \log 2)(\log y)_1^a = (\log b - \log 2)(\log a - \log 1)$

$= (\log b - \log 2)\log a = \log\left(\frac{b}{2}\right)\log a$

(AU) 2) Find the value of $\int_0^\infty \int_0^y \left(\frac{e^{-y}}{y}\right) dx\,dy$.

Sol: $\int_{y=0}^{\infty} \int_{x=0}^{y} \left(\frac{e^{-y}}{y}\right) dx\,dy = \int_0^\infty \left(\frac{e^{-y}}{y}\right)(x)_0^y\, dy$

$= \int_0^\infty \left(\frac{e^{-y}}{y}\right) \times y\, dy = \int_0^\infty e^{-y}\, dy$

$= \left(\frac{e^{-y}}{-1}\right)_0^\infty = -(e^{-y})_0^\infty = -(e^{-\infty} - e^{-0})$

$= -(0-1) = 1$

(AU) 3) Evaluate $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\, dx\,dy$.

Sol: $\int_{y=1}^{\ln 8} \int_{x=0}^{\ln y} e^{x+y}\, dx\,dy = \int_1^{\ln 8} \int_0^{\ln y} e^x e^y\, dx\,dy = \int_1^{\ln 8} (e^x)_0^{\ln y} e^y\, dy$

$= \int_1^{\ln 8} (e^{\ln y} - e^0) e^y\, dy = \int_1^{\ln 8} (y-1) e^y\, dy$

$= \int_1^{\ln 8} (ye^y - e^y)\, dy$

$$\boxed{\begin{array}{ll} u = y & dv = e^y\,dy \\ du = dy & v = e^y \\ \multicolumn{2}{l}{\int u\,dv = uv - \int v\,du} \end{array}}$$

$= (ye^y)_1^{\ln 8} - \int_1^{\ln 8} e^y\, dy - (e^y)_1^{\ln 8}$

$= \ln 8\, e^{\ln 8} - e - (e^y)_1^{\ln 8} - (e^{\ln 8} - e)$

$$= \ln 8 \cdot 8 - e - \left(e^{\ln 8} - e\right) - (8-e)$$

$$= 8\ln 8 - e - 8 + e - 8 + e = 8\ln 8 + e - 16$$

(AU) (4) Evaluate $\int_1^2 \int_0^{x^2} x\,dx\,dy$

Sol: $\int_1^2 \int_0^{x^2} x\,dx\,dy = \int_{x=1}^2 \int_{y=0}^{x^2} x\,dy\,dx$ (correct form)

$$= \int_1^2 x\,(y)_0^{x^2}\,dx = \int_1^2 x(x^2-0)\,dx = \int_1^2 x^3\,dx$$

$$= \left(\frac{x^4}{4}\right)_1^2 = \frac{2^4}{4} - \frac{1}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$$

(AU) (5) Evaluate $\int_0^{2a} \int_0^{x} \int_y^{x} xyz\,dz\,dy\,dx$.

Sol: $\int_{x=0}^{2a} \int_{y=0}^{x} \int_{z=y}^{x} xyz\,dz\,dy\,dx = \int_0^{2a} \int_0^{x} xy\left(\frac{z^2}{2}\right)_y^x dy\,dx$

$$= \int_0^{2a} \int_0^{x} \frac{xy}{2}(x^2-y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^{2a} \int_0^{x} x\,(yx^2 - y^3)\,dy\,dx$$

$$= \frac{1}{2}\int_0^{2a} x\left(\frac{y^2x^2}{2} - \frac{y^4}{4}\right)_0^x dx$$

$$= \frac{1}{2}\int_0^{2a} x\left(\frac{x^4}{2} - \frac{x^4}{4}\right) dx$$

$$= \frac{1}{2}\int_0^{2a} x\left(\frac{2x^4 - x^4}{4}\right) dx = \frac{1}{2}\int_0^{2a} x\left(\frac{x^4}{4}\right) dx$$

$$= \frac{1}{8}\int_0^{2a} x^5\,dx = \frac{1}{8}\left(\frac{x^6}{6}\right)_0^{2a} = \frac{1}{48}\left(x^6\right)_0^{2a}$$

$$= \frac{1}{48}\left((2a)^6 - 0\right) = \frac{64a^6}{48} = \frac{4}{3}a^6$$

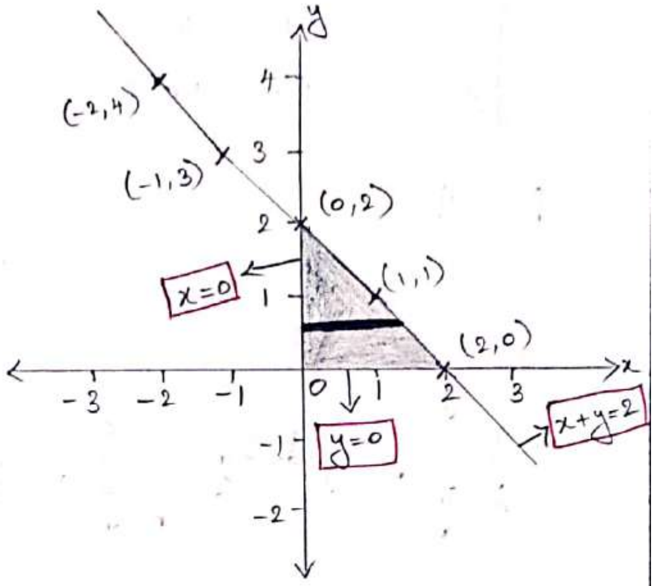
AU

6. Find the limits of integration $\iint_R f(x,y)\,dx\,dy$ where R is the triangle bounded by $x=0,\ y=0,\ x+y=2$.

Sol: Given $x=0,\ y=0,\ x+y=2$

$x+y=2 \Rightarrow y=2-x$

$x:$	-2	-1	0	1	2
$y:$	4	3	2	1	0



From the graph, we get

$x=0,\ x=2-y$ & $y=0,\ y=2$

$$\therefore \iint_R f(x,y)\,dx\,dy = \int_0^2\int_0^{2-y} f(x,y)\,dx\,dy.$$

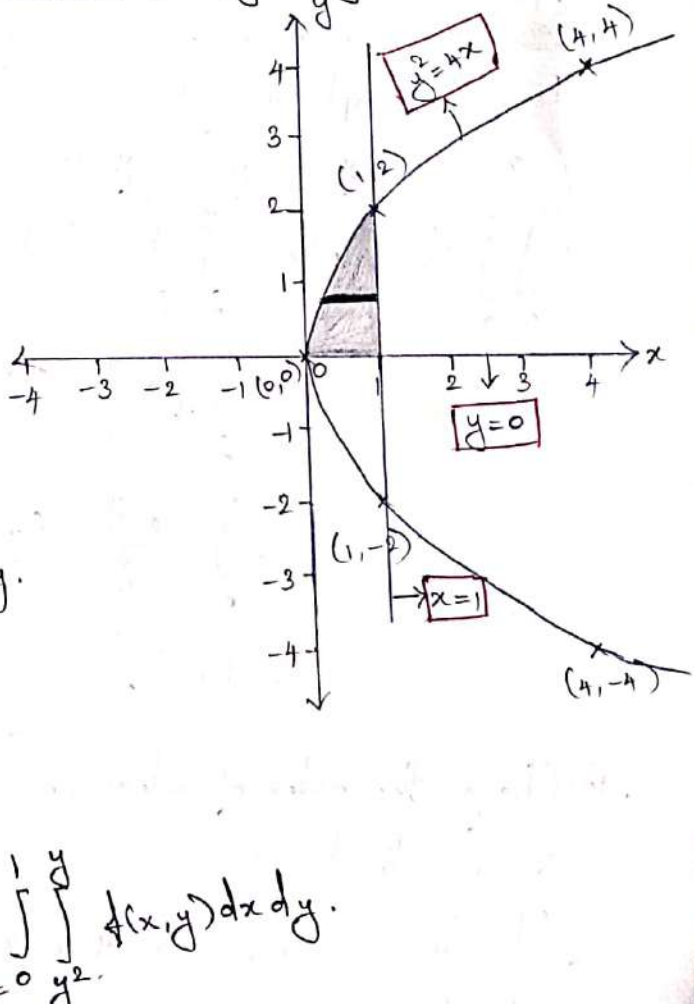
AU

7. Find the limits of integration in the double integral $\iint_R f(x,y)\,dx\,dy$ where R is in the first quadrant & bounded $x=1,\ y=0,\ y^2=4x$.

Sol: Given $x=1,\ y=0,\ y^2=4x$

$y^2=4x \Rightarrow y=\pm\sqrt{4x}=\pm 2\sqrt{x}$

$x:$	0	1	4
$y:$	0	± 2	± 4



From the graph, we get

$x=\frac{y^2}{4},\ x=1$ & $y=0,\ y=2$

$$\therefore \iint_R f(x,y)\,dx\,dy = \int_0^2\int_{y^2/4}^{1} f(x,y)\,dx\,dy.$$

Change the order of integration:

AU

8. Change the order of integration in $\int_{y=0}^{1}\int_{x=y^2}^{y} f(x,y)\,dx\,dy$.

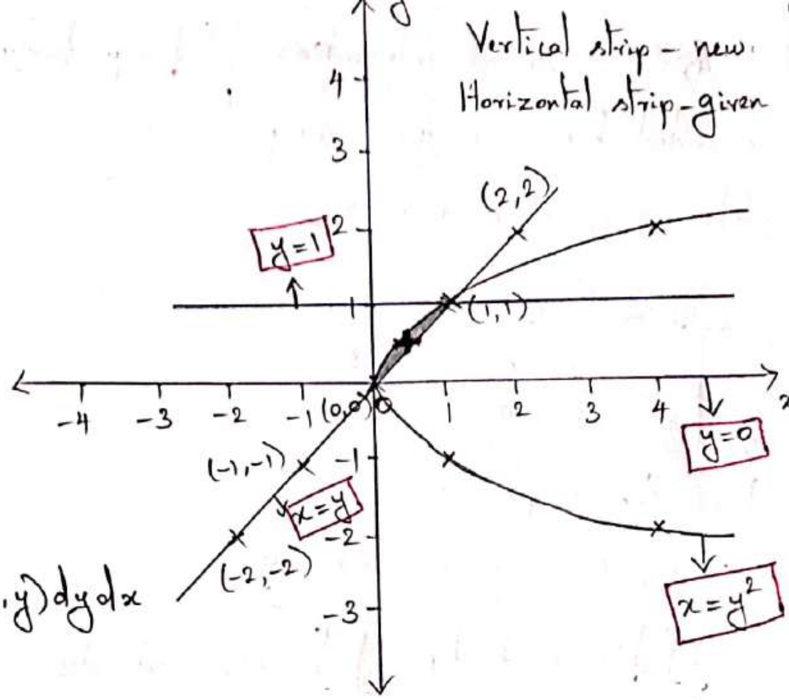
Sol: Given $y=0,\ y=1,\ x=y^2$ & $x=y$.

$x=y$

$x:$	-2	-1	0	1	2
$y:$	-2	-1	0	1	2

$x = y^2 \Rightarrow y = \pm\sqrt{x}$

x	0	1	4
y	0	± 1	± 2



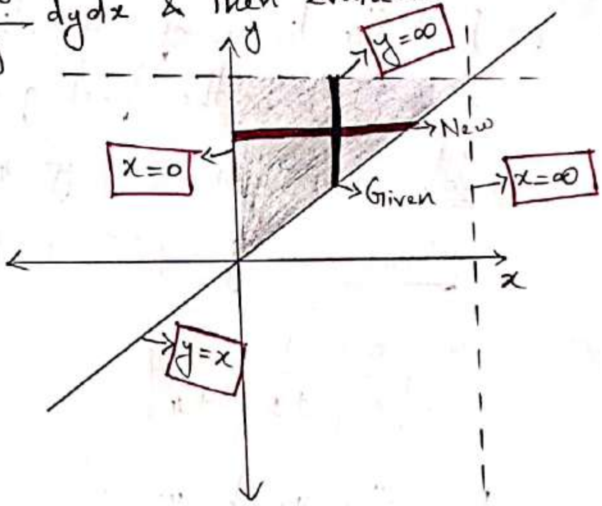
From the graph, we get

$y = \sqrt{x}$, $y = x$ &

$x = 0$, $x = 1$.

$$\therefore \int_0^1 \int_{y^2}^{y} f(x,y)\,dx\,dy = \int_0^1 \int_{\sqrt{x}}^{x} f(x,y)\,dy\,dx$$

(AU) (9) Change the order of integration in $\int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dy\,dx$ & then evaluate it.



Sol: Given $x=0$, $x=\infty$, $y=x$ & $y=\infty$.

From the graph, we get

$x=0$, $x=y$ & $y=0$, $y=\infty$.

$$\therefore \int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx = \int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_0^{\infty} \left(\frac{e^{-y}}{y}\right)(x)_0^y\,dy$$

$$= \int_0^{\infty} \left(\frac{e^{-y}}{y}\right)(y)\,dy = \int_0^{\infty} e^{-y}\,dy = \left(\frac{e^{-y}}{-1}\right)_0^{\infty}$$

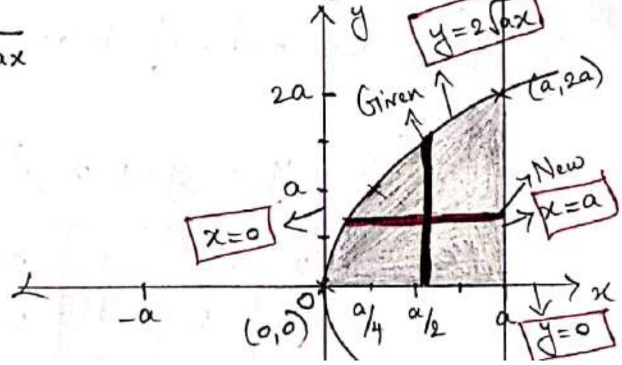
$$= -\left(e^{-y}\right)_0^{\infty} = -\left(e^{-\infty} - e^{-0}\right) = -(0-1) = 1$$

(AU) (10) Change the order of integration for the given integral $\int_{x=0}^{a}\int_{y=0}^{2\sqrt{ax}} x^2\,dy\,dx$ & evaluate it.

Sol: Given $x=0$, $x=a$, $y=0$ & $y = 2\sqrt{ax}$

$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax \Rightarrow x = \frac{y^2}{4a}$

x	0	a	$a/4$
y	0	$2a$	a



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From the graph, we get

$$x = \frac{y^2}{4a}, \quad x = a \quad \& \quad y = 0, \quad y = 2a$$

$$\therefore \int_0^a \int_0^{2\sqrt{ax}} x^2\,dy\,dx = \int_0^{2a} \int_{\frac{y^2}{4a}}^{a} x^2\,dx\,dy$$

$$= \int_0^{2a} \left(\frac{x^3}{3}\right)_{\frac{y^2}{4a}}^{a} dy = \frac{1}{3}\int_0^{2a} \left(a^3 - \left(\frac{y^2}{4a}\right)^3\right) dy$$

$$= \frac{1}{3}\int_0^{2a} \left(a^3 - \frac{y^6}{64a^3}\right) dy = \frac{1}{3}\left[a^3y - \frac{y^7}{7\times 64a^3}\right]_0^{2a}$$

$$= \frac{1}{3}\left[a^3(2a) - \frac{(2a)^7}{7\times 64a^3}\right]$$

$$= \frac{1}{3}\left[2a^4 - \frac{2\times 64a^7}{7\times 64a^3}\right] = \frac{1}{3}\left[2a^4 - \frac{2a^4}{7}\right]$$

$$= \frac{a^4}{3}\left(2 - \frac{2}{7}\right) = \frac{a^4}{3}\left(\frac{14-2}{7}\right) = \frac{a^4}{3}\left(\frac{12}{7}\right) = \frac{4}{7}a^4$$

AU (11) Change the order of integration for the given integral $\int_{x=0}^{a}\int_{y=x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx$ & evaluate it.

Sol: Given $x=0, x=a, y=\frac{x}{a}$ & $y=\sqrt{\frac{x}{a}}$

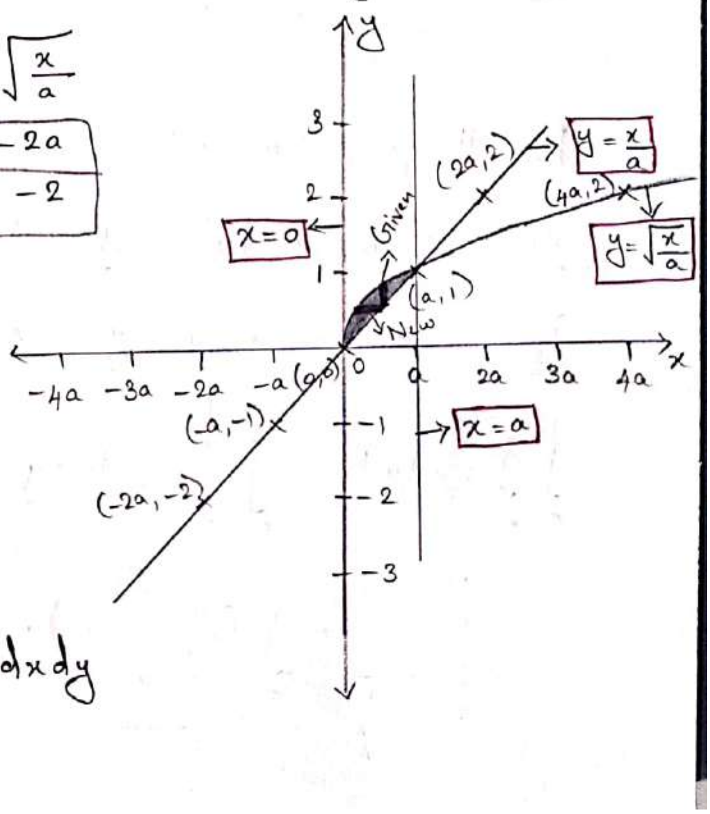
$y = \frac{x}{a}$

x	0	a	$2a$	$-a$	$-2a$
y	0	1	2	-1	-2

$y = \sqrt{\frac{x}{a}}$

x	0	a	$4a$
y	0	1	2

$\Rightarrow y^2 = \frac{x}{a}$



From the graph, we have

$$y=0, \; y=1, \; x = ay^2 \;\&\; x = ay$$

$$\therefore \int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx = \int_0^1 \int_{ay^2}^{ay} (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^3}{3} + xy^2\right)_{ay^2}^{ay} dy$$

$$= \int_0^1 \left(\frac{a^3y^3}{3} + ayy^2 - \frac{a^3y^6}{3} - ay^2y^2\right) dy$$

$$= \int_0^1 \left(\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4\right) dy$$

$$= \left(\frac{a^3y^4}{12} - \frac{ay^4}{4} - \frac{a^3y^7}{21} - \frac{ay^5}{5}\right)_0^1 = \frac{a^3}{12} - \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5}$$

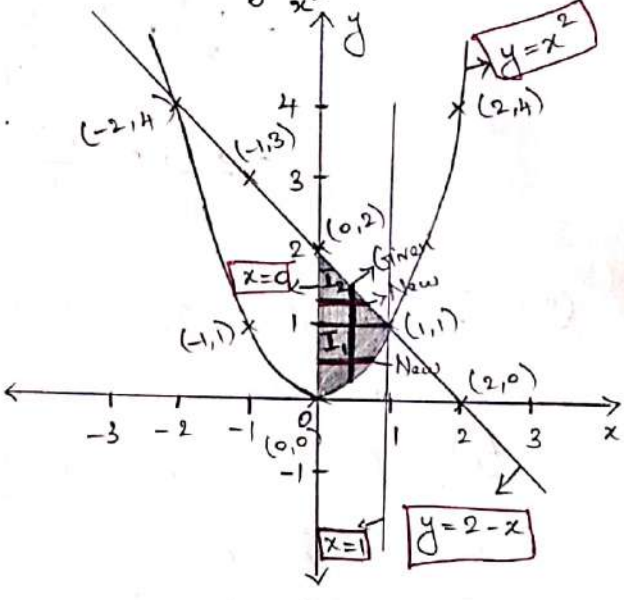
$$= a^3\left(\frac{1}{12} - \frac{1}{21}\right) - a\left(\frac{1}{4} + \frac{1}{5}\right) = \frac{a^3}{28} - \frac{9a}{20}$$

$$= \frac{a}{4}\left(\frac{a^2}{7} - \frac{9}{5}\right)$$

(12) Change the order of integration & hence evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$.

Sol:

Given $x=0$, $x=1$, $y=x^2$ & $y=2-x$.



$y=x^2$

x	-2	-1	0	1	2
y	4	1	0	1	4

$y=2-x$

x	-2	-1	0	1	2
y	4	3	2	1	0

From the graph, we get

I_1: $x=0$, $x=\sqrt{y}$, $y=0$ & $y=1$

I_2: $x=0$, $x=2-y$, $y=1$ & $y=2$

$$\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} y\,dy + \int_1^2 \left(\frac{x^2}{2}\right)_0^{2-y} y\,dy$$

$$= \int_0^1 \frac{y}{2}\, y\, dy + \int_1^2 \frac{(2-y)^2}{2}\, y\, dy$$

$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 (4+y^2-4y)\, y\, dy$$

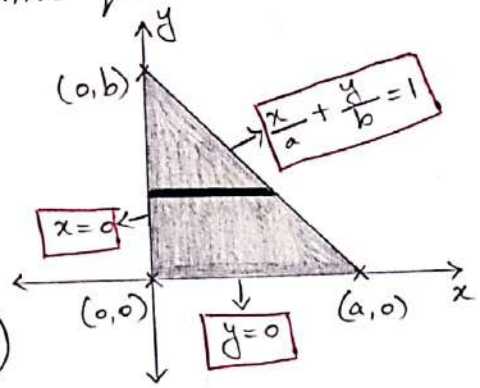
$$= \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}\int_1^2 (4y+y^3-4y^2)\, dy$$

$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left[\frac{4y^2}{2} + \frac{y^4}{4} - \frac{4y^3}{3}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{16}{2} + \frac{16}{4} - \frac{32}{3} - \left(\frac{4}{2} + \frac{1}{4} - \frac{4}{3}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[8+4-\frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right] = \frac{3}{8}$$

AU (13) Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant bounded by $\frac{x}{a}+\frac{y}{b}=1$.



Sol: Given $x=0,\ y=0,\ \frac{x}{a}+\frac{y}{b}=1$.

From the graph, we get

$x=0,\quad \frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{x}{a} = 1-\frac{y}{b} \Rightarrow x = a\left(1-\frac{y}{b}\right)$

$\Rightarrow x=0,\ x=a\left(1-\frac{y}{b}\right)$

$y=0,\ y=b$

$$\therefore \int_0^b \int_0^{a\left(1-\frac{y}{b}\right)} xy\,dx\,dy = \int_0^b \left(\frac{x^2}{2}\right)_0^{a\left(1-\frac{y}{b}\right)} y\,dy$$

$$= \frac{1}{2}\int_0^b a^2\left(1-\frac{y}{b}\right)^2 y\,dy = \frac{a^2}{2}\int_0^b \left(1+\frac{y^2}{b^2}-\frac{2y}{b}\right) y\,dy$$

$$= \frac{a^2}{2}\int_0^b \left[y + \frac{y^3}{b^2} - \frac{2y^2}{b}\right] dy$$

$$= \frac{a^2}{2}\left[\frac{y^2}{2} + \frac{y^4}{4b^2} - \frac{2y^3}{3b}\right]_0^b = \frac{a^2}{2}\left[\frac{b^2}{2} + \frac{b^4}{4b^2} - \frac{2b^3}{3b}\right]$$

$$= \frac{a^2}{2}\left[\frac{b^2}{2} + \frac{b^2}{4} - \frac{2b^2}{3}\right] = \frac{a^2b^2}{2}\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right) = \frac{a^2b^2}{24}$$

AU (14) Using double integral, find the area bounded by $y=x$ & $y=x^2$.

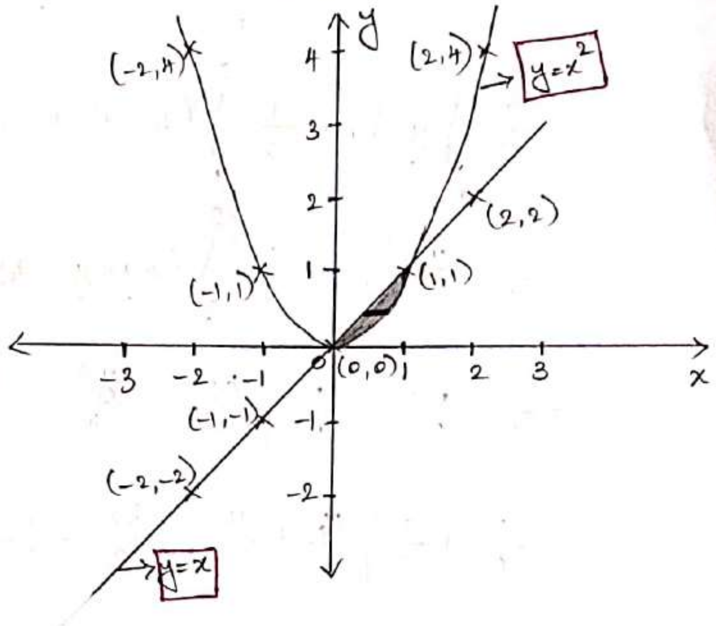
Sol: Given $y=x$ & $y=x^2$.

$y=x$

x	-2	-1	0	1	2
y	-2	-1	0	1	2

$y=x^2$

x	-2	-1	0	1	2
y	4	1	0	1	4



From the graph, we get

$y=x$, $y=x^2$, $x=0$ & $x=1$

$$\int_0^1 \int_x^{x^2} dy\,dx = \int_0^1 (y)_x^{x^2} dx$$

$$= \int_0^1 (x^2-x)\,dx = \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_0^1 = \frac{1}{3} - \frac{1}{2} = \frac{-1}{6}$$

Hence the required area is $\frac{1}{6}$.

AU (15) Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ & $y=x$.

Sol: By using Problem no. (14), we have $y=x^2$, $y=x$, $x=0$ & $x=1$.

$$\int_0^1 \int_{x^2}^{x} xy(x+y)\,dy\,dx = \int_0^1 \int_{x^2}^{x} x(xy+y^2)\,dy\,dx$$

$$= \int_0^1 x\left(x\frac{y^2}{2} + \frac{y^3}{3}\right)_{x^2}^{x} dx$$

$$= \int_0^1 x\left(\frac{x^3}{2} + \frac{x^3}{3} - \frac{x^5}{2} - \frac{x^6}{3}\right) dx$$

$$= \int_0^1 x\left(\frac{5x^3}{6} - \frac{x^5}{2} - \frac{x^6}{3}\right) dx$$

$$= \int_0^1 \left(\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right) dx$$

$$= \left(\frac{5x^5}{30} - \frac{x^7}{14} - \frac{x^8}{24}\right)_0^1 = \frac{5}{30} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{3}{56}$$