



PIE Tech

POLLACHI INSTITUTE OF ENGINEERING AND TECHNOLOGY

(Approved by AICTE and Affiliated to Anna University)

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Department of Electrical and Electronics Engineering

Regulation 2021

II Year – III Semester

MA3303- probability and complex functions

UNIT-I

Probability and Random variables.

Introduction:

Random experiments:

The experiments which do not produce the same result or outcome on every trial are called random experiments.

(eg) Throw an unbiased die.

Sample Space:

The totality of the possible outcomes of a random experiment is called the sample space of the experiment and it will be denoted by S .

The number of sample points in a sample space is denoted by $n(s)$.

Example:

- 1) Tossing a coin $S = \{H, T\}$; $n(s) = 2$.
- 2) Tossing two coins simultaneously $S = \{HH, HT, TH, TT\}$; $n(s) = 4$.
- 3) Rolling a die $S = \{1, 2, 3, 4, 5, 6\}$; $n(s) = 6$.

Mutually exclusive events: [Disjoint events]

Two events A and B are said to be mutually exclusive events or disjoint events provided $A \cap B$ is the null set.

Note: If A and B are mutually exclusive, then it is not possible for both events to occur on the same trial.

Example: In the throw of a single die, the events of getting 1, 2, 3, 4, 5, 6 are mutually exclusive.

Trial and event:

The performance of a random experiment is called Trial and the outcome is called an event.

Example: Throwing of a coin is a trial and getting H or T is an event.

Definition of probability:

If there are n exhaustive, mutually exclusive and equally likely events, probability of A is defined as the ratio m/n , m is favourable to A .

Notation: $P(A) = p = \frac{m}{n} = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}}$

Note: $0 \leq p \leq 1$

Permutation:

Permutation means selection and arrangement of factors.

Notation: nPr (or) $p(n, r)$ (or) $P_{n, r}$

$$nPr = \frac{n!}{(n-r)!}$$

Permutations with repetitions:

Example: The number of permutations of the word

'RADAR' is $\frac{5!}{2!2!1!} = 30$.

Rule of sum:

If an event can occur in m ways and another event can occur in n ways, there are $m+n$ ways in which

exactly one event can occur.

Example: Consider a box containing 8 red balls and 5 green balls, then 8+5 ways to choose either red ball or a green ball.

Rule of product:

If there are m outcomes for event E_1 and n possible outcomes for event E_2 then there are mn outcomes for the composite event $E_1 E_2$.

Example: When a pair of dice is thrown once, the number of possible outcomes are $6 \times 6 = 36$. Since each dice has 6 possible outcomes.

Combination:

Combination means selection of factors.

$${}^n C_r = \frac{n!}{(n-r)! r!}$$

$$\text{Note: } {}^n C_n = {}^n C_0 = \frac{n!}{(n-n)! n!} = \frac{n!}{0! n!} = 1$$

Statistical or Empirical probability:

If in n trials an event E happens m times, then the probability P of the happening of E is given by

$$P = P(E) = \lim_{n \rightarrow \infty} \frac{m}{n}$$

$$\{T, H\} = 2$$

$$S = (2)^n$$

$$1 = (A)^n$$

$$P = \frac{(A)^n}{(2)^n} = (A)^n$$



Axioms of probability.

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

Axiom 3: For any sequence of mutually exclusive events

$E_1, E_2, \dots$ (i.e. events for which $E_i \cap E_j = \emptyset$ when $i \neq j$)

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Addition Law of probability:

If A and B are any two events, and are not disjoint, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\text{(or)} \\ P(A \cup B) = P(A) + P(B) - P(AB)$$

If $A_1, A_2, \dots, A_n$ are mutually exclusive events then $P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$

Problems: Type 1: $P(A) = \frac{n(A)}{n(S)}$

1) Find the probability that exactly one head appears in a single throw of a fair coin.

Soln:

Let A be the event of getting exactly one head in a single throw of a fair coin

$$S = \{H, T\}$$

$$n(S) = 2$$

$$n(A) = 1$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{2}$$

2) If two dice are rolled, what is the probability that the sum of the upturned faces will be equal to 7?

Soln: $n(s) = 36$

Let $A = \{\text{sum of the upturned face will equal 7}\}$

$$= \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

$$n(A) = 6$$

$$\therefore P(A) = \frac{n(A)}{n(s)} = \frac{6}{36} = \frac{1}{6}$$

3) A lot of integrated circuit chips consists of 10 good, 4 with minor defects and 2 with major defects. Two chips are randomly chosen from the lot. What is the probability that at least one chip is good?

Soln:

$$P(\text{at least one is good}) = \frac{n(A)}{n(s)} = \frac{(10C_1)(6C_1) + 10C_2}{(10C_1)(6C_1) + 10C_2} = \frac{(10)(6) + 45}{60 + 45} = \frac{105}{105} = \frac{7}{8}$$

4) From a pack of 52 cards two cards are drawn the first being replaced before the second is drawn. Find the probability that the first one is a diamond and second is a king.

Soln: Let A be an event of drawing a diamond.

$$P(A) = \frac{n(A)}{n(s)} = \frac{13}{52} = \frac{1}{4}$$

Let B be an event of drawing a king.

$$P(B) = \frac{n(B)}{n(s)} = \frac{4}{52} = \frac{1}{13}$$

A and B are Independent events.

$$\text{Hence } P(A \cap B) = P(A) \cdot P(B) = \left(\frac{1}{4}\right) \left(\frac{1}{3}\right) = \frac{1}{12}$$

Type 2. Mutually Exclusive events.

$$P(A \cup B) = P(A) + P(B) \quad (\text{or}) \quad P(A+B) = P(A) + P(B)$$

1) One card is drawn from a pack of 52 cards. What is the probability that it is either a king or queen.

Soln:

A = {an event that the card drawn is king}

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

B = {an event that the card drawn is queen}

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

$A \cup B$ = {an event that the card to be either a king or a queen}

$$P(A \cup B) = P(A) + P(B) \quad (\because A \text{ and } B \text{ are mutually exclusive})$$

$$= \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

2) From a group of 5 first year, 4 second year and 4 third year students, 3 students are selected at random. Find the probability that they are first year (or) second year or third year students.

Soln:

Total number of students = 13.

Three students are selected at random $n(S) = {}^{13}C_3 = 286$

A → The three students are from I year

$$n(A) = {}^5C_3 = 10$$

$$P(A) = 10/286$$

B → The three students are from third year

$$n(B) = {}^4C_3 = 4$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{286}$$

Here A and B are disjoint events since both cannot occur together.

$$P(A \cup B) = P(A) + P(B) = 10/286 + 4/286 = 14/286 = 7/143$$

Type 3: Not mutually exclusive, Independent events.

$$i) P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$ii) P(A \cap B) = P(A) \cdot P(B)$$

1) One card is drawn from a deck of 52 cards.

What is the probability of the card being either red or a king.

Soln:

Let A = {an event that the card drawn is red}

B = {an event that the card drawn is king}

$A \cup B$ = {an event that a card to be either red or a king}

$$P(A) = \frac{n(A)}{n(S)} = 26/52 = 1/2$$

$$P(B) = \frac{n(B)}{n(S)} = 4/52 = 1/13$$

There are two red coloured king cards.

$$\text{So } P(A \cap B) = 2/52 = 1/26$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad [\because A \text{ and } B \text{ are not mutually exclusive}]$$
$$= 1/2 + 1/13 - 1/26 = 7/13$$



2) If A and B are events with $P(A) = 3/8$, $P(B) = 1/2$ and $P(A \cap B) = 1/4$. Find $P(A^c \cap B^c)$.

Soln: We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 3/8 + 1/2 - 1/4 = \frac{3+4-2}{8} = 5/8$$

$$P(A^c \cap B^c) = P(A \cup B)^c$$

$$= 1 - P(A \cup B) = 1 - 5/8 = 3/8$$

3) Let events A and B be independent with $P(A) = 0.5$ and $P(B) = 0.8$. Find the probability that neither of the events nor B occurs.

Soln: Since A and B are independent, $P(A \cap B) = P(A)P(B)$

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [P(A) + P(B) - P(A)P(B)]$$

$$= 1 - [0.5 + 0.8 - (0.5)(0.8)] = 0.1$$

Conditional probability

Marginal probability:

A probability of only one event that takes place is called a marginal probability.

Joint probability:

The probability of occurrence of both events A and B together, denoted by $P(A \cap B)$ is known as joint probability of A and B.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional probability:

The conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0$$

The conditional probability of B given A is

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \text{ if } P(A) \neq 0.$$

Note:

$$i) \frac{P(A|B)}{P(B|A)} = \frac{P(A \cap B) / P(B)}{P(A \cap B) / P(A)} = \frac{P(A)}{P(B)} \quad [\because P(A \cap B) = P(B \cap A)]$$

$$ii) P(A|A) = P(A \cap A) / P(A) = P(A) / P(A) = 1$$

$$iii) \text{ If A and B are independent events then } P(A \cap B) = P(A)P(B)$$

$\therefore$ The conditional probability is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

$$P(B|A) = P(B)$$

i) A bag contains 5 red and 3 green balls and a second bag 4 red and 5 green balls. One of the bags is selected at random and a draw of 2 balls is made from it. What is the probability that one of them is red and the other is green.

Soln:

Let A_1 and A_2 denote the events of selecting the first bag and second bag respectively.

$P(A_1) = \frac{1}{2} = P(A_2)$ and A_1 and A_2 are mutually exclusive events.

$$\therefore S = A_1 \cup A_2$$

Let B denote the event of selecting one red and one green ball.

$$\text{Then } P(B/A_1) = \frac{{}^5C_1 {}^3C_1}{{}^8C_2} = \frac{15}{36} \times 2 = \frac{15}{18}$$

$$P(B/A_2) = \frac{{}^4C_1 {}^5C_1}{{}^9C_2} = \frac{20}{36} = \frac{5}{9}$$

∴ The required probability = $P(A_1)P(B/A_1) + P(A_2)P(B/A_2)$

$$= \left(\frac{1}{2}\right)\left(\frac{15}{18}\right) + \left(\frac{1}{2}\right)\left(\frac{5}{9}\right) = \frac{15}{54} + \frac{5}{18} = \frac{275}{504}$$

2) A bag contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

Soln: Let A = One of the tubes drawn is good.

B = the other tube is good.

$$P(A \cap B) = P[\text{Both the tubes drawn are good}] = \frac{{}^6C_2}{{}^{10}C_2} = \frac{1}{3}$$

Knowing that one tube is good, the conditional probability that the other tube is also good is required.

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{6/10} = \left(\frac{1}{3}\right)\left(\frac{10}{6}\right) = \frac{5}{9}$$

Cumulative distribution (or) Distribution function of X

The cumulative distribution function $F(x)$ of a discrete random variable X with Probability distribution $p(x)$ is given by $F(x) = p(X \leq x) = \sum_{t \leq x} p(t)$.

Expected Value of a Discrete Random Variable X .

Let X be a discrete random variable assuming values $x_1, x_2, \dots, x_n$ with corresponding probabilities $p_1, p_2, \dots, p_n$.

Then $E(X) = \sum_i x_i p(x_i)$ is called the Expected Value of X .

$$E[g(x)] = \sum_{x_i} g(x_i) p(x_i).$$

Variance of a random variable X .

$$\text{Var}(X) = E[X - E(X)]^2 \quad \text{or} \quad \text{Var}(X) = E[X^2] - [E(X)]^2$$

The quantity $\sqrt{\text{Var}(X)}$ is called the standard deviation of X .

Note:

$$1) E[ax+b] = aE(X) + b.$$

$$2) \text{Var}(ax+b) = a^2 \text{Var}(X).$$

Problems:

1) The number of hardware failures of a computer system in a week of operations has the following pmf:

Number of failures 0 1 2 3 4 5 6

Probability .18 .28 .25 .18 .06 .04 .01

Find the mean of the number of failures in a week.

Solution:

$X = x_i$	0	1	2	3	4	5	6	Total
$p(x_i)$	0.18	0.28	0.25	0.18	0.06	0.04	0.01	
	0	0.28	0.50	0.54	0.24	0.20	0.06	1.82

$$\text{Mean} = E[X] = \sum x_i p(x_i) = 1.82.$$

2. The monthly demand for Allwyn watches is known to have the following distribution.

Demand	1	2	3	4	5	6	7	8
Probability	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Determine the expected demand for watches. Also compute the Variance.

Solution:

Let X denote the demand.

$X = x_i$	$p(x_i)$	$x_i p(x_i)$	x_i^2	$x_i^2 p(x_i)$
1	0.08	0.08	1	0.08
2	0.12	0.24	4	0.48
3	0.19	0.57	9	1.71
4	0.24	0.96	16	3.84
5	0.16	0.80	25	4.00
6	0.10	0.60	36	3.60
7	0.07	0.49	49	3.43
8	0.04	0.32	64	2.56

$$\begin{aligned} \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= 19.70 - (1.82)^2 \\ &= 19.70 - 3.3164 \\ &= 16.3836 \end{aligned}$$

$$E[X] = \sum x_i p(x_i) = 1.82$$

$$E[X^2] = \sum x_i^2 p(x_i) = 19.70$$

3. A random variable X has the following probability distribution.

$X = x_i$	-2	-1	0	1	2	3
$p(X = x_i)$	0.1	K	0.2	$2K$	0.3	$3K$

Find (1) The value of K (2) Evaluate $p(X < 2)$ and $p(-2 < X < 2)$
(3) Find the cumulative distribution of X and evaluate m

Solution:

(1) We know that $\sum_i p(x_i) = 1$ Here $\sum_{i=-2}^3 p(x_i) = 1$ $\frac{100}{90} = \frac{10}{9}$

$$\therefore 0.1 + K + 0.2 + 2K + 0.3 + 3K = 1$$

$$6K + 0.6 = 1 \Rightarrow 6K = 1 - 0.6 \Rightarrow 6K = 0.4$$

$$\Rightarrow K = 0.4/6 = 1/15 = 0.06$$

(2) i) $P[X < 2]$

$$= P[X = -2] + P[X = -1] + P[X = 0] + P[X = 1]$$

$$= 0.1 + 1/15 + 0.2 + 2/15 = 0.3 + 3/15 = 1/2 = 0.3 + 0.5 = 0.8$$

ii) $P[-2 < X < 2] =$

$$= P[X = -1] + P[X = 0] + P[X = 1] = 1 - [P(X = 2) + P(X = 3)]$$

$$= 1/15 + 0.2 + 2/15 = 0.2 + 3/15 = 2/5 = 1 - [0.3 + 3/15] = 1 - 0.5 = 0.5$$

(3)

x_i	$P(x_i)$	Cumulative distribution $F(x_i)$	$x_i P(x_i)$
-2	$P(-2) = 0.1$	$F(-2) = P(-2) = 0.1$	-0.2
-1	$P(-1) = 1/15$	$F(-1) = F(-2) + P(-1) = 0.1 + 1/15 = 0.17$	-1/15
0	$P(0) = 0.2$	$F(0) = F(-1) + P(0) = 0.17 + 0.2 = 0.37$	0
1	$P(1) = 2/15$	$F(1) = F(0) + P(1) = 0.37 + 2/15 = 0.50$	2/15
2	$P(2) = 0.3$	$F(2) = F(1) + P(2) = 0.50 + 0.3 = 0.80$	0.6
3	$P(3) = 3/15$	$F(3) = F(2) + P(3) = 0.80 + 3/15 = 1.00$	9/15

(4) Mean $= E[X] = \sum x_i P(x_i) = 16/15$

(4) A discrete random variable X has the following probability distribution.

x	0	1	2	3	4	5	6	7	8
$P(X=x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Find the value of a . $a = 1/8$

ii) Find $P(X < 3)$, $P(0 < X < 3)$, $P(X > 3)$

iii) Find the distribution function of X .

(5) If $\text{Var}(X) = 4$, then find $\text{Var}(3X+8)$, where X is a r.v.

Solution: We know that, $\text{Var}(aX+b) = a^2 \text{Var}(X)$

$$\text{Var}(3X+8) = 3^2 \text{Var}(X) = 3^2 [4] = 36$$

(6) If the probability mass function of a r.v X is given by

$$P(X=r) = Kr^3, r=1, 2, 3, 4 \text{ find}$$

(i) The value of K

We know that $\sum p(x_i) = 1 \Rightarrow 100K = 1 \Rightarrow K = \frac{1}{100}$

(ii) $P(\frac{1}{2} < X < \frac{5}{2} | X > 1)$ (iii) Mean & Variance of X (iv) Distribution function of X .

Given: $P(X=r) = Kr^3, r=1, 2, 3, 4$

x_i	$P(x_i)$	$x_i P(x_i)$	x_i^2	$x_i^2 P(x_i)$	$F(X)$
1	$K = \frac{1}{100}$	$\frac{1}{100}$	1	$\frac{1}{100}$	$F[1] = P(1) = \frac{1}{100}$
2	$8K = \frac{8}{100}$	$\frac{16}{100}$	4	$\frac{32}{100}$	$F[2] = F[1] + P(2)$ $= \frac{1}{100} + \frac{8}{100}$ $= \frac{9}{100}$
3	$27K = \frac{27}{100}$	$\frac{81}{100}$	9	$\frac{243}{100}$	$F[3] = F[2] + P(3)$ $= \frac{9}{100} + \frac{27}{100} = \frac{36}{100}$
4	$64K = \frac{64}{100}$	$\frac{256}{100}$	16	$\frac{1024}{100}$	$F[4] = F[3] + P(4)$ $= \frac{36}{100} + \frac{64}{100} = 1$

$$E[X] = \sum x_i P(x_i) = \frac{354}{100} = 3.54$$

$$E[X^2] = \sum x_i^2 P(x_i) = (1300) \left(\frac{1}{100} \right) = 13$$

$$\text{Var}(X) = E[X^2] - [E[X]]^2 = 13 - \left(\frac{354}{100} \right)^2 = 13 - (3.54)^2 = 0.4684$$

$$P\left[\frac{1}{2} < X < \frac{5}{2} | X > 1\right] = \frac{P\left[\left(\frac{1}{2} < X < \frac{5}{2}\right) \cap X > 1\right]}{P(X > 1)} = \frac{P[X=2]}{P(X > 1)}$$

$$= \frac{P[X=2]}{P[X=2] + P[X=3] + P[X=4]} = \frac{\frac{8}{100}}{\frac{99}{100}} = \frac{8}{99}$$

Continuous Random variables

Definition:

A random variable X is said to be continuous if it takes all possible values between certain limits say from real number 'a' to real number 'b'.

Probability density function:

For a continuous random variable X , a probability density function is a function such that

$$1) f(x) \geq 0 \quad (2) \int_{-\infty}^{\infty} f(x) dx = 1 \quad (3) P(a \leq X \leq b) = \int_a^b f(x) dx.$$

Cumulative distribution function

The cumulative distribution function of a continuous random variable X is $F(x) = P(X \leq x) = \int_{-\infty}^x f(x) dx$ for $-\infty < x < \infty$.

Note: $\frac{d}{dx} \int_{-\infty}^x f(t) dt = f(x)$

$$\text{ie) } f(x) = \frac{d}{dx} F(x)$$

The mean or expected value of a continuous random variable

The mean or expected value of X , denoted by μ or $E[X]$ is $\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$.

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x) dx.$$

The variance of a continuous random variable X .

(A) The variance of X denoted as $V(X)$ or σ^2 is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

P $= E[X^2] - (E[X])^2$

(C) The standard deviation of X is $\sigma = \sqrt{\text{var}(X)}$

Problems:

1) If $f(x) = \begin{cases} kxe^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$ is the p.d.f of a random

Variable X . Find K .

Solution: For a p.d.f, $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\text{Here, } \int_0^{\infty} kxe^{-x} dx = 1$$

$$\Rightarrow K \left[x \left(\frac{e^{-x}}{-1} \right) - (1) \left(\frac{e^{-x}}{(-1)^2} \right) \right]_0^{\infty} = 1$$

$$\Rightarrow K \left[-xe^{-x} - e^{-x} \right]_0^{\infty} = 1$$

$$\Rightarrow K \left[(-0-0) - (-0-1) \right] = 1 \Rightarrow K = 1 \quad [\because e^{-\infty} = 0]$$

2) A continuous random variable X that can assume any value between $x=2$ and $x=5$ has a density function given by $f(x) = k(1+x)$. Find $P[X < 4]$.

Solution:

$$i) \int_2^5 f(x) dx = 1$$

$$(ii) P[X < 4] = \int_2^4 f(x) dx$$

$$\text{Here } \int_2^5 k(1+x) dx = 1$$

$$k \int_2^5 (1+x) dx = 1$$

$$k \left[x + \frac{x^2}{2} \right]_2^5 = 1$$

$$k \left[\left(5 + \frac{25}{2} \right) - \left(2 + \frac{4}{2} \right) \right] = 1$$

$$k \left[\frac{35}{2} - \frac{8}{2} \right] = 1$$

$$k \left(\frac{27}{2} \right) = 1, \quad k = \frac{2}{27}$$

3) A continuous random variable has the p.d.f of $f(x) = kx^4$; $-1 < x < 0$. Find the value of K and also $P(X > -\frac{1}{2} / X < -\frac{1}{4})$.

Solution:



Given: $f(x) = Kx^4$

(i) To find the value of K , $\int_{-\infty}^{\infty} f(x) dx = 1$

Here $\int_{-1}^0 Kx^4 dx = 1 \Rightarrow K \left[\frac{x^5}{5} \right]_{-1}^0 = 1 \Rightarrow K(0 + \frac{1}{5}) = 1 \Rightarrow K = 5$

ii) $P\left[x > -\frac{1}{2} / x < -\frac{1}{4}\right] = \frac{P\left[x > -\frac{1}{2} \cap x < -\frac{1}{4}\right]}{P\left[x < -\frac{1}{4}\right]} = \frac{P\left[-\frac{1}{2} < x < -\frac{1}{4}\right]}{P\left[x < -\frac{1}{4}\right]}$

$P\left[-\frac{1}{2} < x < -\frac{1}{4}\right] = \int_{-1/2}^{-1/4} 5x^4 dx$, $P\left(x < -\frac{1}{4}\right) = \int_{-1}^{-1/4} 5x^4 dx \rightarrow \textcircled{1}$

$= 5 \left[\frac{x^5}{5} \right]_{-1/2}^{-1/4} = \left[x^5 \right]_{-1/2}^{-1/4} = 5 \left[\frac{x^5}{5} \right]_{-1}^{-1/4} = \left[x^5 \right]_{-1}^{-1/4}$

$= \left(-\frac{1}{1024}\right) - \left(-\frac{1}{32}\right) = \frac{31}{1024}$ $= \left(-\frac{1}{1024}\right) - (-1) = \frac{1023}{1024}$

$\textcircled{1} \Rightarrow \frac{\frac{31}{1024}}{\frac{1023}{1024}} = \frac{31}{1023} = \frac{1}{33} = 0.0303$

④ The cumulative distribution function of a r. v X is

$F(x) = 1 - (1+x)e^{-x}$, $x > 0$. Find the p.d.f of X , Mean and Variance of X .

Soln: Given: $f(x) = 1 - (1+x)e^{-x}$, $x > 0$

$= 1 - e^{-x} - xe^{-x}$, $x > 0$

p.d.f $f(x) = \frac{d}{dx} [F(x)] = \frac{d}{dx} [1 - e^{-x} - xe^{-x}] = 0 + e^{-x} - [x(e^{-x}) + e^{-x}(-1)]$

$= xe^{-x}$, $x > 0$
 $E[X] = \int_0^{\infty} xf(x) dx = \int_0^{\infty} x xe^{-x} dx = \int_0^{\infty} x^2 e^{-x} dx$

$= \left[x^2 \left(\frac{e^{-x}}{-1} \right) \right] - (2x) \left[\frac{e^{-x}}{(-1)^2} \right] + (2) \left(\frac{e^{-x}}{(-1)^3} \right) \Big|_0^{\infty}$
 $= (0 - 0 + 0) - (0 - 0 - 2) = 2$ ($\because e^{-\infty} = 0, e^{-0} = 1$)

$E[X^2] = \int_0^{\infty} x^2 f(x) dx = \int_0^{\infty} x^2 (xe^{-x}) dx = \int_0^{\infty} x^3 e^{-x} dx$
 $= \left[x^3 \left(\frac{e^{-x}}{-1} \right) \right] - (3x^2) \left(\frac{e^{-x}}{(-1)^2} \right) + (6x) \left[\frac{e^{-x}}{(-1)^3} \right] - (6) \left[\frac{e^{-x}}{(-1)^4} \right] \Big|_0^{\infty} = 6$

$\text{Var}(X) = E[X^2] - [E(X)]^2 = 6 - (2)^2 = 6 - 4 = 2$

Moments - Moment Generating Functions & their properties

Moments [Discrete case]

Let X be a discrete R.V taking the values $x_1, x_2, \dots, x_n$ with p.m.f $p_1, p_2, \dots, p_n$ respectively then the r -th moment about origin is

$$\mu_r' = \sum_{i=1}^n x_i^r p_i \rightarrow \textcircled{1}$$

$$\mu_r' \text{ (about any point } a=A) = \sum_{i=1}^n (x_i - A)^r p_i \rightarrow \textcircled{2}$$

$$\mu_r' \text{ (about mean)} = \sum_{i=1}^n (x_i - \text{mean})^r p_i \rightarrow \textcircled{3}$$

$$\mu_1' = \sum_{i=1}^n x_i p_i = \text{Mean}(\bar{x}), \quad \mu_2' = \sum_{i=1}^n x_i^2 p_i = \text{Mean square value}$$

$$\mu_2 = \sum_{i=1}^n (x_i - \text{mean})^2 p_i = \text{Variance}$$

$$= \mu_2' - (\mu_1')^2$$

$$\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' \mu_1'^2 - 3\mu_1'^4$$

Moments [continuous case]

If X is a continuous R.V with p.d.f defined in (a, b) . Then

$$\mu_r' = \int_a^b x^r f(x) dx$$

$$\mu_r' \text{ (about a point } A) = \int_a^b (x - A)^r f(x) dx$$

$$\mu_r' \text{ (about the mean)} = \int_a^b (x - \bar{x})^r f(x) dx$$

Moment Generating Function: (M.G.F)

Moment generating function of a random variable X about the origin is defined as

$$M_X(t) = E[e^{tx}] = \sum_{\alpha} e^{t\alpha} p(\alpha), \text{ if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} e^{tx} f_X(x) dx, \text{ if } X \text{ is continuous.}$$

Note:

$$1) \mu_r' = \frac{d^r}{dt^r} [M_X(t)]_{t=0} = 1 + t^r \mu_r' + \frac{t^2}{2!} \mu_2' + \dots + \frac{t^r}{r!} \mu_r'$$

$$2) M_X(t) = E[e^{t(x-a)}] \quad \text{where } \mu_r' = r\text{-th moment about origin.}$$

(about $x=a$)

$$3) \text{Mean} = \bar{x}$$

$$M_X(t) [\text{about } x = \bar{x}] = 1 + \frac{t}{1!} \mu_1 + \frac{t^2}{2!} \mu_2 + \dots + \frac{t^r}{r!} \mu_r$$

when $\mu_r = [(x - \bar{x})^r] = r\text{-th central moment}$

Properties of Moment Generating Function.

1) Let $Y = ax + b$, where X is a R.V with m.g.f. $M_X(t)$

$$\text{Then } M_Y(t) = E[e^{tY}] = E[e^{t(ax+b)}] = E[e^{tax} e^{bt}] = e^{bt} E[e^{tax}] = e^{bt} M_X(at)$$

$$2) M_{cX}(t) = E[e^{ctX}] = E[e^{X(ct)}] = M_X(ct), \text{ where } c \text{ is a constant.}$$

3) If X and Y are two independent random variables, then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.

problems:

1) Find the Mgf of the RV X whose probability function $P(X=x) = \frac{1}{2^x}, x=1, 2, \dots$. Find its mean.

$$\text{Soln: } M_X(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \frac{(e^t)^2}{2^2} + \dots + 1$$

$$= \frac{e^t}{2} \left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \dots \right] = \frac{e^t}{2} \left[\frac{1}{1 - e^t/2} \right]$$

$$= \frac{e^t}{2} \left[\frac{2}{2 - e^t} \right] = \frac{e^t}{2 - e^t}$$

$$\begin{aligned} \text{Mean} = m_1 &= \left[\frac{d}{dt} \left[\frac{e^t}{2-e^t} \right] \right]_{t=0} \\ &= \left[\frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2} \right]_{t=0} = \frac{(2-1)(1) + (1)(1)}{(2-1)^2} \\ &= \frac{1+1}{1} = 2 \end{aligned}$$

2) For the triangular distribution $f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 2-x, & 1 \leq x < 2 \\ 0, & \text{otherwise.} \end{cases}$

Find the mean, variance and the MGF and also find cdf of $f(x)$

Solution:

$$\text{Given: } f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 2-x, & 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Mean} = E[X] = \int_{-\infty}^{\infty} x f(x) dx \rightarrow \text{①}$$

$$= \int_0^1 (x)(x) dx + \int_1^2 (x)(2-x) dx$$

$$= \int_0^1 x^2 dx + \int_1^2 (2x - x^2) dx = \left[\frac{x^3}{3} \right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3} \right]_1^2$$

$$= \left[\frac{1}{3} - 0 \right] + \left[\left(4 - \frac{8}{3} \right) - \left(1 - \frac{1}{3} \right) \right] = \frac{1}{3} + \frac{4}{3} - \frac{2}{3} = \frac{4+1-2}{3} = 1.$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^1 x^2 x dx + \int_1^2 x^2 (2-x) dx$$

$$= \int_0^1 x^3 dx + \int_1^2 (2x^2 - x^3) dx = \left[\frac{x^4}{4} \right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_1^2$$

$$= \left(\frac{1}{4} - 0 \right) + \left(\frac{16}{3} - \frac{16}{4} \right) - \left(\frac{2}{3} - \frac{1}{4} \right) = \frac{1}{4} + \frac{16}{3} - \frac{16}{4} - \frac{2}{3} + \frac{1}{4}$$

$$= -\frac{14}{4} + \frac{14}{3} = \frac{-42+56}{12} = \frac{7}{6} \therefore \text{Var}(X) = E(X^2) - (E(X))^2 = \frac{7}{6} - 1 = \frac{1}{6}$$

The MGF of a Random Variable X is

$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_0^1 x e^{tx} dx + \int_1^2 (2-x) e^{tx} dx$$

$$= \left[\frac{x e^{tx}}{t} - \frac{e^{tx}}{t^2} \right]_0^1 + \left[(2-x) \frac{e^{tx}}{t} - (-1) \frac{e^{tx}}{t^2} \right]_1^2$$

$$= \left[\left(\frac{e^t}{t} - \frac{e^t}{t^2} \right) - \left(0 - \frac{1}{t^2} \right) \right] + \left[\left(0 - \frac{e^{2t}}{t^2} \right) - \left(\frac{e^t}{t} + \frac{e^t}{t^2} \right) \right]$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \frac{e^{2t}}{t^2} - \frac{2e^t}{t^2} + \frac{1}{t^2} = \frac{1}{t^2} [e^{2t} - 2e^t + 1] = \frac{1}{t^2} [e^t - 1]^2$$

To find the cdf of $F(x)$

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(x) dx$$

i) If $x \leq 0$, then $F(x) = 0$

ii) If $0 < x \leq 1$, then $F(x) = \int_0^x x dx = \left[\frac{x^2}{2} \right]_0^x = \frac{x^2}{2}$

iii) If $1 \leq x < 2$, then $F(x) = \int_0^1 x dx + \int_1^x (2-x) dx$

$$= \left[\frac{x^2}{2} \right]_0^1 + \left[2x - \frac{x^2}{2} \right]_1^x = \frac{1}{2} + \left(2x - \frac{x^2}{2} \right) - \left(2 - \frac{1}{2} \right)$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - 2 + \frac{1}{2} = 2x - \frac{x^2}{2} - 1$$

iv) If $x > 2$, then

$$F(x) = \int_{-\infty}^x f(x) dx$$

$$= \int_0^1 x dx + \int_1^2 (2-x) dx + \int_2^x 0 dx$$

$$= \left[\frac{x^2}{2} \right]_0^1 + \left[2x - \frac{x^2}{2} \right]_1^2 = \frac{1}{2} + (4-2) - \left(2 - \frac{1}{2} \right)$$

$$= \frac{1}{2} + 2 - 2 + \frac{1}{2} = 1$$

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3) Find the M.G.F of the random variable x having the p.d.f $f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & x > 0 \\ 0 & \text{elsewhere} \end{cases}$

Also deduce the first four moments about the origin.

Solution:

Given: $f(x) = \frac{x}{4} e^{-x/2}, x > 0$

$$\begin{aligned} M_X(t) &= E[e^{tx}] = \int_0^{\infty} e^{tx} f(x) dx = \int_0^{\infty} e^{tx} \frac{x}{4} e^{-x/2} dx \\ &= \frac{1}{4} \int_0^{\infty} x e^{-(1/2-t)x} dx = \frac{1}{4} \left[x \left[\frac{e^{-(1/2-t)x}}{-(1/2-t)} \right] - (1) \left[\frac{e^{-(1/2-t)x}}{(1/2-t)^2} \right] \right] \\ &= \frac{1}{4} \left[(0-0) - \left(0 - \frac{1}{(1/2-t)^2} \right) \right] = \frac{1}{4} \frac{1}{(1-2t)^2} = \frac{1}{(1-2t)^2} \end{aligned}$$

$$M_X(t) = (1-2t)^{-2} = 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \dots$$

$$= 1 + 4t + 12t^2 + 32t^3 + 80t^4 + \dots$$

$$= 1 + \frac{t}{1!}(4) + \frac{t^2}{2!}(24) + \frac{t^3}{3!}(192) + \frac{t^4}{4!}(1920) + \dots$$

$$\therefore \mu_1' = \text{Co-efficient of } t/1! = 4 \quad \frac{d}{dt} M_X(t) = (-2)(1-2t)^{-3}$$

$$\mu_2' = \text{Co-efficient of } t^2/2! = 24 \quad = (-2)(-3)(1-2t)^{-4}$$

$$\mu_3' = \text{Co-efficient of } t^3/3! = 192 \quad = (-2)(-3)(-4)(1-2t)^{-5}$$

$$\mu_4' = \text{Co-efficient of } t^4/4! = 1920 \quad = \frac{0+4(1+2t)}{(1-2t)^6}$$

Binomial distribution.

A random variable x is said to follow binomial distribution if it assumes only non-negative values and its p.m.f is given by $P[X=x] = p(x) = \begin{cases} nC_x p^x q^{n-x}, & x=0,1,2,\dots,n \\ 0 & \text{otherwise.} \end{cases}$ where $q=1-p$

The first four moments from the moment generating

$$\ln M_x (pe^t + q)^n$$

$$\mu_1' = \frac{d}{dt} [M_x(t)]_{t=0} = [n(pe^t + q)^{n-1} \cdot pe^t]_{t=0} = np$$

∴ Mean = np

$$\mu_2' = [n(n-1)(pe^t + q)^{n-2} \cdot p^2 e^{2t} + n(pe^t + q)^{n-1} pe^t]_{t=0}$$

$$= n(n-1)p^2 + np = n^2p^2 - np^2 + np$$

$$= n^2p^2 + np(1-p) = n^2p^2 + npq$$

$$\therefore \text{Variance } \mu_2 = \mu_2' - \mu_1'^2 = n^2p^2 + npq - n^2p^2 = npq$$

$$\mu_3' = n^3p^3 - 3n^2p^3 + 2np^3 + 3n^2p^2 - 3np^2 + np$$

$$\mu_3 = npq(q-p)$$

$$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

$$\mu_4 = 3n^2p^2q^2 + npq(1-6pq)$$

1) The mean of a Binomial distribution is 20 and S.D is 4.

Determine the parameters of the distribution.

Soln: Given: Mean = np = 20 → ①

$$S.D = \sqrt{npq} = 4$$

$$npq = 16 \rightarrow ②$$

$$\frac{②}{①} \Rightarrow \frac{npq}{np} = \frac{16}{20} = \frac{4}{5}$$

$$\therefore q = \frac{4}{5}$$

$$p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}$$

$$① \Rightarrow n\left(\frac{1}{5}\right) = 20 \Rightarrow n = 100$$

∴ The parameters of the binomial distribution are n = 100, p = 1/5

② Find the probability that in tossing a fair coin 5 times, there will appear (a) 3 heads (b) 3 tails and 2 heads

(c) atleast 1 head and (d) not more than 1 tail.

Soln: Given: $p = \frac{1}{2}$, $q = \frac{1}{2}$, $n = 5$

Let X denote the number of heads.

$$a) P[3 \text{ heads}] = P[X=3] = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = 10 \left(\frac{1}{2}\right)^5 = \frac{5}{16}$$

$$(b) P[3 \text{ tails and 2 heads}] = P[X=2] = {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = \frac{5}{16}$$

$$(c) P[\text{atleast 1 head}] = P[X \geq 1] = 1 - P[X < 1] = 1 - P[X=0] \\ = 1 - {}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 = \frac{31}{32}$$

$$(d) P[\text{not more than 1 tail}] = P[0 \text{ tail}] + P[1 \text{ tail}] \\ = P[5 \text{ head}] + P[4 \text{ head}] \\ = P[X=5] + P[X=4] \\ = {}^5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 + {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 \\ = \left(\frac{1}{2}\right)^5 + 5 \left(\frac{1}{2}\right)^5 = 6 \left(\frac{1}{2}\right)^5 = \frac{3}{16}$$

③ In a large consignment of electric bulb 10% are defective. A random sample of 20 is taken for inspection. Find the probability that (i) All are good bulbs (ii) Atmost there are 3 defective bulbs (iii) Exactly there are 3 defective bulbs.

Soln: p - probability of an electric bulb is defective = $\frac{1}{10}$.

$$q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$$

Let X denote the number of defectives.

$$(i) P(\text{All are good}) = P(\text{None of them are defective}) \\ = P[X=0] = {}^{20}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{20} = \left(\frac{9}{10}\right)^{20} = 0.1216$$

$$(ii) P(\text{Atmost three are defective}) = P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3) \\ = {}^{20}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{20} + {}^{20}C_1 \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^{19} + {}^{20}C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^{18} + {}^{20}C_3 \left(\frac{1}{10}\right)^3 \left(\frac{9}{10}\right)^{17} = 0.8666$$

$$(iii) P[\text{Exactly 3 are defective}] = P[X=3] = {}^{20}C_3 \left(\frac{1}{10}\right)^3 \left(\frac{9}{10}\right)^{17} = 0.19$$

The first four moments from the moment generating

$$\text{in } M_x(pe^t + q)^n$$

$$\mu_1' = \frac{d}{dt} [M_x(t)]_{t=0} = \left[n(pe^t + q)^{n-1} \cdot pe^t \right]_{t=0} = np$$

$$\therefore \text{Mean} = np$$

$$\mu_2' = \left[n(n-1)(pe^t + q)^{n-2} \cdot p^2 e^{2t} + n(pe^t + q)^{n-1} pe^t \right]_{t=0}$$

$$= n(n-1)p^2 + np = n^2p^2 - np^2 + np$$

$$= n^2p^2 + np(1-p) = n^2p^2 + npq$$

$$\therefore \text{Variance } \mu_2 = \mu_2' - \mu_1'^2 = n^2p^2 + npq - n^2p^2 = npq$$

$$\mu_3' = n^3p^3 - 3n^2p^3 + 2np^3 + 3n^2p^2 - 3np^2 + np$$

$$\mu_3 = npq(q-p)$$

$$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

$$\mu_4 = 3n^2p^2q^2 + npq(1-6pq)$$

1) The mean of a Binomial distribution is 20 and S.D is 4.

Determine the parameters of the distribution.

Soln: Given: Mean = $np = 20 \rightarrow \textcircled{1}$

$$S.D = \sqrt{npq} = 4$$

$$npq = 16 \rightarrow \textcircled{2}$$

$$\frac{\textcircled{2}}{\textcircled{1}} \Rightarrow \frac{npq}{np} = \frac{16}{20} = \frac{4}{5}$$

$$\text{ie) } q = \frac{4}{5}$$

$$p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}$$

$$\textcircled{1} \Rightarrow n\left(\frac{1}{5}\right) = 20 \Rightarrow n = 100$$

$\therefore$ The parameters of the binomial distribution are $n = 100, p = \frac{1}{5}$.

- 2) Find the probability that in tossing a fair coin 5 times,
 (a) there will appear 3 heads (b) 3 tails and 2 heads
 (c) atleast 1 head and (d) not more than 1 tail.



Soln: Given: $p = \frac{1}{2}$, $q = \frac{1}{2}$, $n = 5$. $nC_x p^x q^{n-x}$

Let X denote the number of heads.

$$a) P[3 \text{ heads}] = P[X=3] = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = 10 \left(\frac{1}{2}\right)^5 = \frac{5}{16}$$

$$b) P[3 \text{ tails and 2 heads}] = P[X=2] = {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = \frac{5}{16}$$

$$c) P[\text{at least 1 head}] = P[X \geq 1] = 1 - P[X < 1] = 1 - P[X=0] \\ = 1 - {}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 = \frac{31}{32}$$

$$d) P[\text{not more than 1 tail}] = P[0 \text{ tail}] + P[1 \text{ tail}] \\ = P[X=5] + P[X=4] \\ = {}^5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 + {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 \\ = \left(\frac{1}{2}\right)^5 + 5 \left(\frac{1}{2}\right)^5 = 6 \left(\frac{1}{2}\right)^5 = \frac{3}{16}$$

③ In a large consignment of electric bulb 10% are defective. A random sample of 20 is taken for inspection. Find the probability that (i) All are good bulbs (ii) At most there are 3 defective bulbs (iii) Exactly there are 3 defective bulbs.

Soln: p - probability of an electric bulb is defective = $\frac{1}{10}$.

$$q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$$

Let X denote the number of defectives.

$$(i) P(\text{All are good}) = P(\text{None of them are defective}) \\ = P[X=0] = {}^{20}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{20} = \left(\frac{9}{10}\right)^{20} = 0.1216$$

$$(ii) P(\text{At most three are defective}) = P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3) \\ = {}^{20}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{20} + {}^{20}C_1 \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^{19} + {}^{20}C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^{18} + {}^{20}C_3 \left(\frac{1}{10}\right)^3 \left(\frac{9}{10}\right)^{17} = 0.8666$$

$$(iii) P[\text{Exactly 3 are defective}] = P[X=3] = {}^{20}C_3 \left(\frac{1}{10}\right)^3 \left(\frac{9}{10}\right)^{17} = 0.14$$

Poisson distribution.

A random variable X is said to follow Poisson distribution if it assumes only non-negative values and its p.m.f is given by

$$p(X=x) = p(x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!}, & x=0, 1, 2, \dots, \lambda > 0, \lambda = np \\ 0, & \text{otherwise} \end{cases}$$

λ is known as the parameter of the Poisson distribution.

Mean and variance.

$$E[X] = \text{Mean} = \lambda, \text{ Variance} = (\mu_2) = \lambda.$$

$$\mu_3 = \lambda, \mu_4 = 3\lambda^2 + \lambda.$$

Problems:

① If X is a Poisson variate such that $p(X=2) = 9p(X=4) + 90p(X=6)$. Find the variance.

Soln: The probability distribution for the Poisson R.V. X is given by $p(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $x=0, 1, \dots, n$ and $\lambda > 0$.

$$p(X=2) = 9p(X=4) + 90p(X=6)$$

$$\frac{e^{-\lambda} \lambda^2}{2!} = \frac{9e^{-\lambda} \lambda^4}{4!} + \frac{90e^{-\lambda} \lambda^6}{6!}$$

$$\therefore \text{by } \lambda^2 e^{-\lambda}, \text{ we get } \frac{1}{2!} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}$$

$$\Rightarrow \frac{1}{2} = \frac{3}{8}\lambda^2 + \frac{1}{8}\lambda^4$$

$$\text{ie) } \frac{3\lambda^2 + \lambda^4}{8} = \frac{1}{2}$$

$$\text{ie) } \frac{3\lambda^2}{4} + \frac{\lambda^4}{4} = 1 \Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

$$(\lambda^2 + 4)(\lambda^2 - 1) = 0$$

$$\lambda^2 = -4 \text{ (or) } \lambda^2 = 1 \Rightarrow \lambda = 1, [\because \lambda > 0]$$

For a Poisson distribution, $\text{var}(X) = \lambda = 1$.

2) Derive the poisson distribution as a limiting case of Binomial distribution.

Solution:

The Binomial probability law for x successes in a series of n independent trials is

$$P[X = x] = p(x) = {}^n C_x p^x q^{n-x}, \quad x = 0, 1, 2, \dots, n.$$

i) n is indefinitely large (i.e.) $n \rightarrow \infty$

ii) p is very small such that $p \rightarrow 0$

(iii) $np = \lambda$ (a finite quantity)

$$\Rightarrow p = \lambda/n \text{ and } q = 1 - p = 1 - \lambda/n$$

$$P[X = x] = {}^n C_x p^x q^{n-x}$$

$$= \frac{n!}{(n-x)! x!} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2) \dots (n-x+1)(n-x) \dots 2 \cdot 1}{[1 \cdot 2 \cdot 3 \dots (n-x)] x!} p^x (1-p)^{n-x}$$

$$= \frac{n(n-1)(n-2) \dots (n-x+1)}{x!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n^x \left[1 - \frac{1}{n}\right] \left[1 - \frac{2}{n}\right] \dots \left[1 - \frac{x-1}{n}\right]}{x!} \frac{\lambda^x}{n^x} \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \left[\left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{x-1}{n}\right) \right] \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} e^{-\lambda} \text{ as } n \rightarrow \infty \left[\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^{n-x} = e^{-\lambda} \right]$$

$$(\text{or}) P[X = x] = \frac{n^x}{x!} \frac{\lambda^x}{n^x} \frac{e^{-\lambda}}{1} = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \dots, \infty$$

where λ is known as the parameter of the distribution.

3) Messages arrive at a switch board in a poisson manner at an average rate of six per hour. Find the probability for each of the following events:

(1) exactly two messages arrive within one hour.

(2) no message arrives within one hour.

(3) atleast three messages arrive within one hour.

Solution:

Mean $\lambda = 6$ per hour

We know that $P[X=x] = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-6} 6^x}{x!}$

$$(i) P[X=2] = \frac{e^{-6} 6^2}{2!} = 0.0446$$

$$(ii) P[X=0] = \frac{e^{-6} 6^0}{0!} = e^{-6} = 0.0025$$

$$\begin{aligned} (iii) P[X \geq 3] &= 1 - P[X < 3] \\ &= 1 - [P(X=0) + P(X=1) + P(X=2)] \\ &= 1 - \left[0.0025 + \frac{e^{-6} 6}{1!} + 0.0446 \right] \\ &= 1 - [0.0025 + 0.1487 + 0.0446] \\ &= 0.8042 = 0.8042 \end{aligned}$$

4) The manufacturer of pins knows that 2% of his products are defective. If he sells pins in boxes of 100 and guarantees that not more than 4 pins will be defective. What is the probability that a box will fail to meet the guaranteed quality?

Soln: Given $n=100$ and $p=2\% = \frac{2}{100}$, $\lambda = np = \frac{2}{100} \times 100 = 2$.

$$\begin{aligned} P[X > 4] &= 1 - P[X \leq 4] \\ &= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)] \\ &= 1 - e^{-2} \left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} \right] = 0.05265. \end{aligned}$$

Geometric distribution

$$P[X=x] = (1-p)^{x-1} p = q^{x-1} p, \quad x=1, 2, \dots$$

$$\sum_{n=1}^{\infty} P\{X=x\} = p \sum_{n=1}^{\infty} (1-p)^{n-1} = \frac{p}{1-(1-p)} = \frac{p}{p} = 1$$

Recurrence formula for Geometric distribution

$$P[X=x] = pq^{x-1}, \quad p+q=1, \quad x=1, 2, \dots$$

$$P[X=x+1] = pq^x$$

$$\text{So } \frac{P[X=x+1]}{P[X=x]} = q$$

$$P[X=x+1] = q P[X=x]$$

$$P[X=x+1] = q P[X=x]$$

Problems:

1) If the probability that a target is destroyed on any one shot is 0.5. What is the probability that it would be destroyed on 6th attempt?

Soln: Given $p=0.5$, $q=1-0.5=0.5$

$$P(X=r) = q^{r-1} p$$

$$P(X=6) = q^5 p = \left(\frac{1}{2}\right)^5 \cdot \frac{1}{2} = \frac{1}{2^6} = \frac{1}{64}$$

2) Establish the memoryless property of geometric distribution;

Soln: If X has a geometric distribution, then for any two positive integers 'm' and 'n', $P[X > m+n / X > m] = P[X > n]$

proof:

$$P[X > m+n / X > m] = \frac{P[X > m+n \cap X > m]}{P[X > m]} = \frac{P[X > m+n]}{P[X > m]}$$

$$\text{Taking } P[X=r] = q^{r-1} p, \quad r=1, 2, 3, \dots$$

$$P[X > k] = \sum_{k+1}^{\infty} q^{r-1} p$$

$$= q^k p + q^{k+1} p + p q^{k+2} + \dots$$

$$= q^K p [1 + q + q^2 + \dots] = q^K p [1 - q]^{-1}$$

$$= \frac{q^K p}{1 - q} = \frac{q^K p}{p} = q^K \quad [\because 1 - q = p]$$

$$\therefore P[X > m+n] = q^{m+n} \text{ and } P[X > m] = q^m$$

$$\therefore P[X > m+n | X > m] = \frac{q^{m+n}}{q^m} = q^n$$

$$P[X > n] = q^n$$

$$\text{Hence, } P[X > m+n | X > m] = P[X > n]$$

3) If the probability that an applicant for a driver's licence will pass the road test on any given trial is 0.8; what is the probability that he will finally pass the test

(a) On the fourth trial and (b) in fewer than 4 trials.

Solution:

Let X denote the number of trials required to achieve the first success.

Then X is a geometric distribution given by

$$P(X=r) = q^{r-1} p; \quad r=1, 2, 3, \dots, \infty$$

Here $p=0.8$ and $q=0.2$

$$(a) P(X=4) = 0.8 \times (0.2)^3 = 0.8 \times (0.2)^3 = 0.8 \times 0.008 = 0.0064$$

$$(b) P(X < 4) = \sum_{r=1}^3 0.8 \times (0.2)^{r-1} = 0.8 [(0.2)^0 + (0.2)^1 + (0.2)^2] = 0.992$$

Uniform distribution (or) Rectangular distribution

A random variable X is said to have a continuous uniform distribution over an interval (a, b) if its p.d.f is a constant $= K$, over the entire range of X

$$i) f(x) = K, \quad a < x < b \\ = 0, \text{ otherwise}$$

Since, the total probability is always unity, $\int_a^b f(x) dx = 1$

$$\Rightarrow \int_a^b k dx = 1 \Rightarrow k[x]_a^b = 1 \Rightarrow k[b-a] = 1 \Rightarrow k = \frac{1}{b-a}$$

$$\therefore f(x) = \frac{1}{b-a}, a < x < b$$

= 0 otherwise

Distribution function of the uniform distribution

The distribution function $F(x)$ is given by

$$F(x) = \begin{cases} 0 & \text{if } -\infty < x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b < x < \infty \end{cases}$$

Since $F(x)$ is not continuous at $x=a$ and $x=b$, it is not differentiable at these points.

$$\text{Thus } \frac{d}{dx} [F(x)] = f(x) = \frac{1}{b-a} \neq 0 \text{ exists.}$$

everywhere except at $x=a$ and $x=b$.

Characteristic function of a uniform distribution:

$$\phi(t) = E[e^{itx}] = \int_a^b e^{itx} f(x) dx = \frac{e^{ibt} - e^{iat}}{it(b-a)}$$

Moments of uniform distribution:

$$\text{Moments are given by } \mu_r' = E[x^r] = \int_a^b x^r f(x) dx$$

$$\mu_r' = \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}$$

Mean, Deviation about the mean of uniform distribution.

The mean deviation of the uniform distribution

$$f(x) = \frac{1}{b-a}, a < x < b \text{ is given by } \frac{b-a}{4}$$

Note: The pdf of a uniform variable 'x' in $(-a, a)$ is given by

$$f(x) = \begin{cases} \frac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

1) X is uniformly distributed with mean 1 and Variance $4/3$, find $P[X < 0]$

Solution:

Given mean = $\frac{b+a}{2} = 1 \rightarrow$ ①

Variance = $\frac{(b-a)^2}{12} = 4/3 \rightarrow$ ②

Solving ① & ②, we get $b=3$ and $a=-1$

$\therefore f(x) = \frac{1}{b-a} = \frac{1}{4}$

Hence, $P[X < 0] = \int_{-1}^0 \frac{1}{4} dx = \frac{1}{4} \int_{-1}^0 dx = \frac{1}{4} [x]_{-1}^0 = \frac{1}{4} [0 - (-1)] = \frac{1}{4}$

2) A random variable X has a uniform distribution over $(-3, 3)$. compute 1) $P(X < 2)$, $P(|X| < 2)$, $P(|X-2| < 2)$.

2) Find K for which $P(X > K) = 1/3$.

Solution:

We know that the p.d.f of a random variable ' X ' which is distributed uniformly in $(-a, a)$ is

$$f(x) = \begin{cases} \frac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{6}, & -3 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

1) $P(X < 2) = \int_{-3}^2 f(x) dx = \int_{-3}^2 \frac{1}{6} dx = \frac{1}{6} [x]_{-3}^2 = \frac{1}{6} [2+3] = \frac{5}{6}$

$P(|X| < 2) = P(-2 < X < 2) = \int_{-2}^2 f(x) dx = \frac{1}{6} \int_{-2}^2 dx = \frac{1}{6} [x]_{-2}^2 = \frac{4}{6} = \frac{2}{3}$

$P(|X-2| < 2) = P(-2 < X-2 < 2)$

$= P(0 < X < 4) = \int_0^4 f(x) dx = \int_0^4 \frac{1}{6} dx = \frac{1}{6} [x]_0^4 = \frac{1}{6} [4-0] = \frac{2}{3}$

2) $P(X > K) = 1/3 \Rightarrow \int_K^3 f(x) dx = 1/3 \Rightarrow \int_K^3 \frac{1}{6} dx = 1/3 \Rightarrow \frac{1}{6} [x]_K^3 = 1/3$
 $\Rightarrow \frac{1}{6} [3-K] = 1/3 \Rightarrow 3-K = 2 \Rightarrow K = 1$

Exponential distribution

A continuous random variable X is said to follow exponential distribution if its p.d.f is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

Distribution function of exponential distribution

$$F(x) = \int_{-\infty}^x f(x) dx = \int_{-\infty}^0 0 dx + \int_0^x \lambda e^{-\lambda x} dx = \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^x$$

$$F(x) = 1 - e^{-\lambda x}$$

Memoryless property of exponential distribution:

If X is exponentially distributed, then

$$P(X > s+t | X > s) = P(X > t), \text{ for any } s, t > 0$$

$$\text{Proof: } P(X > k) = \int_k^{\infty} \lambda e^{-\lambda x} dx$$

$$P(X > k) = \lambda \left[-\frac{e^{-\lambda x}}{\lambda} \right]_k^{\infty} = e^{-\lambda k} \rightarrow \textcircled{1}$$

$$\begin{aligned} P(X > s+t | X > s) &= \frac{P(X > s+t \cap X > s)}{P(X > s)} = \frac{P(X > s+t)}{P(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \text{ (by } \textcircled{1}) = e^{-\lambda t} = P(X > t) \end{aligned}$$

Hence, $P(X > s+t | X > s) = P(X > t)$

The converse of this result is also true.

ie) If $P(X > s+t | X > s) = P(X > t)$ then X follows an exponential distribution.

i) If a continuous random variable X follows uniform distribution in the interval $(0, 2)$ and a continuous random

variable Y follows exponential distribution with parameter α ,

find α such that $P(X < 1) = P(Y < 1)$.

Solution:

Given $X \sim U(0, 2) = f(x) = \frac{1}{b-a} = \frac{1}{2}$

Given $Y \sim$ Exponential distribution with parameter α

$$\Rightarrow f(y) = \alpha e^{-\alpha y}$$

Given that $P(X < 1) = P(Y < 1)$

$$\Rightarrow \int_0^1 \frac{1}{2} dx = \int_0^1 \alpha e^{-\alpha y} dy \Rightarrow \frac{1}{2} [x]_0^1 = \alpha \left[\frac{e^{-\alpha y}}{-\alpha} \right]_0^1$$

$$\Rightarrow \frac{1}{2} = 1 - e^{-\alpha} \Rightarrow e^{-\alpha} = \frac{1}{2} \Rightarrow \alpha = \log_e 2$$

2) If X has an exponential distribution with parameter α , find the p.d.f of $Y = \log X$

Solution: $f_X(x) = \alpha e^{-\alpha x}$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$F_Y(y) = P(Y \leq y) = P[\log X \leq y]$$

$$= P[X \leq e^y] = 1 - P[X > e^y]$$

$$= 1 - e^{-\alpha e^y}$$

$$[\because f(x) = \lambda e^{-\lambda x}]$$

$$\Rightarrow P[X > k] = e^{-\lambda k}$$

$$f_Y(y) = \frac{d}{dy} [1 - e^{-\alpha e^y}]$$

$$= -e^{-\alpha e^y} (-\alpha e^y)$$

$$= \alpha e^y e^{-\alpha e^y}, -\infty < y < \infty$$

Gamma distribution

A continuous random variable X taking non-negative values is said to follow gamma distribution, if its

p.d.f is given by $f(x) = \frac{e^{-x} x^{\alpha-1}}{\Gamma(\alpha)}$, $\alpha > 0$, $0 < x < \infty$

and $\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx = 0$, elsewhere

2) Generalised form of the gamma distribution: [Erlang distribution]

A continuous random variable X having the following density function is said to follow generalised gamma distribution with parameters α and λ .

$$f(x) = \begin{cases} e^{-\lambda x} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}, & \alpha, \lambda > 0, 0 < x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Note:

When $\lambda = 1$, Erlang distribution becomes single gamma distribution

When $\lambda = 1$, Erlang distribution becomes Exponential distribution

Formulae: 1. $\Gamma_{1/2} = \sqrt{\pi}$ 2) $\Gamma_{n+1} = n \Gamma_n$ 3. $\Gamma_n = (n-1)!$

Problems:

1) In a certain city, the daily consumption of electric power in millions of kilowatt hours can be treated as a random Variable having an Erlang distribution with parameters $(1/2, 3)$. If the power plant of this city has a daily capacity of 12 millions kilowatt hours, what is the probability that this power supply will be inadequate on any given day?

Solution:

Let X represent the daily consumption of electric power in millions of kilowatt hours.

Then the density of X is given as

$$f(x) = \frac{e^{-\lambda x} \lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} = \frac{e^{-x/2} \left(\frac{1}{2}\right)^3 x^2}{\Gamma(3)}$$

$$P(\text{the power supply is inadequate}) = P(X > 12) \\ = \int_{12}^{\infty} \frac{x^2 e^{-x/2}}{2^3 \Gamma(3)} dx$$

$$= \frac{1}{16} \left[\frac{x^2 e^{-x/2}}{-1/2} - \frac{2x e^{-x/2}}{1/4} + \frac{2 e^{-x/2}}{-1/8} \right]_{12}^{\infty}$$

$$= \frac{1}{16} e^{-6} (288 + 96 + 16) = 0.0625.$$

2) If $X_1, X_2, \dots, X_n$ are Independent Exponential random variables each with parameter λ , then prove that $X_1 + X_2 + \dots + X_n$ is a Gamma random variable with parameter λ, n .

Soln: We know that $M_X(t) = \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1}$

Now, $M_{X_1 + X_2 + \dots + X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \dots M_{X_n}(t)$

Since $X_1, X_2, \dots, X_n$ are independent random variables

$$\therefore M_{X_1 + X_2 + \dots + X_n}(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} \cdot \left(1 - \frac{t}{\lambda}\right)^{-1} \dots \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$= \left(1 - \frac{t}{\lambda}\right)^{-n}$$

$\therefore X_1 + X_2 + \dots + X_n$ is a Gamma R-V with parameters λ and n
Normal distribution.

A normal distribution is a continuous distribution given by

$$y = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x - \mu}{\sigma}\right)^2} \text{ where } x \text{ is a continuous normal}$$

variable distributed with density function $f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x - \mu}{\sigma}\right)^2}$ with mean μ and standard deviation σ .

Standard normal probability distribution:

If x is a normally distributed random variable μ and σ are respectively its mean and standard deviation,

then $z = \frac{x - \mu}{\sigma}$ is called standard normal random variable.

1) In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and variance of the distribution.

Solution: We know that, $Z = \frac{X - \mu}{\sigma}$

Let $Z = Z_1$, where $X = 45$ and $Z = Z_2$, when $X = 64$.

$$P(0 < Z < Z_1) = 0.31$$

$$\therefore P(Z_1 < Z < 0) = 0.19$$

$$\therefore \text{From tables, } Z_1 = -0.49$$

$$\text{i.e.) } \frac{45 - \mu}{\sigma} = -0.49 \text{ (or) } 45 - \mu = -0.49\sigma \rightarrow \textcircled{1}$$

$$P(Z > Z_1) = 0.8 \text{ or } P(0 < Z < Z_2) = 0.42$$

$$\therefore \text{From tables, } Z_2 = 1.40$$

$$\therefore \frac{64 - \mu}{\sigma} = 1.40 \text{ (or) } 64 - \mu = 1.40\sigma \rightarrow \textcircled{2}$$

$$\textcircled{1} - \textcircled{2} \Rightarrow -19 = -1.89\sigma$$

$$\sigma = \frac{-19}{-1.89} = 10$$

$$\text{From } \textcircled{1} \mu = 45 + (0.49 \times 10) = 45 + 4.9 = 49.9 = 50$$

$$\therefore \text{Mean} = 50 \text{ and S.D} = 10$$

$$\mu = 50, \sigma = 10, \text{Variance } (\sigma^2) = 100.$$

2) The peak temperature T , as measured in degrees Fahrenheit, on a particular day is the Gaussian $(85, 10)$ random variable. What is $P(T > 100)$, $P(T < 60)$ and $P(70 \leq T \leq 100)$?

Solution:

$$\text{Given: } \mu = 85, \sigma = 10$$

$$\text{We know that } Z = \frac{T - \mu}{\sigma}$$

$$(i) \text{ When } T = 100, Z = \frac{100 - 85}{10} = \frac{15}{10} = 1.5$$

$$P[T > 100] = P[Z > 1.5] = 0.5 - P[0 < Z < 1.5] \\ = 0.5 - 0.4332 \\ = 0.0668$$

ii) When $T = 60$, $Z = \frac{60 - 85}{10} = -2.5$

$$P[T < 60] = P[Z < -2.5] = 0.5 - P[0 < Z < 2.5] \\ = 0.5 - 0.4938 \\ = 0.0062$$

iii) When $T = 70$, $Z = \frac{70 - 85}{10} = -1.5$

$$P[70 \leq T \leq 100] = P[-1.5 \leq Z \leq 1.5]$$

$$= P[-1.5 \leq Z \leq 0] + P[0 \leq Z \leq 1.5]$$

$$= 0.4332 + 0.4332 = 0.8664$$

$$\text{②} \leftarrow P(A \cdot 1) = 11 - P(A \cdot 0) \quad P(A \cdot 1) = \frac{11 - 10}{10}$$

$$P(B \cdot 1) = P(1) - \text{①} \\ 0.8 = \frac{10}{10} - P(1)$$

$$P(1) = 11 - 10 = 1$$

$$P(2) = 10 - 10 = 0$$

$$P(3) = 10 - 10 = 0$$

From ① $11 = \sqrt{2} \times 10 \times 10 = 11$
 $10 = 10 \times 10 \times 10 = 10$
 $10 = 10 \times 10 \times 10 = 10$
 $10 = 10 \times 10 \times 10 = 10$

UNIT - 2

Two dimensional Random Variables

Joint distributions - Marginal and conditional distributions.

Joint Probability distribution:

The probabilities of two events $A = \{X \leq x\}$ and $B = \{Y \leq y\}$ defined as functions of x and y , respectively, are called probability distribution functions.

$$F_X(x) = P(X \leq x) ; F_Y(y) = P(Y \leq y).$$

Joint probability distribution of two random variables X and Y .

We define the probability of the joint event $\{X \leq x, Y \leq y\}$ which is a function of the numbers x and y , by a joint probability distribution function and denote it by the symbol $F_{X,Y}(x,y)$.

$$\text{Hence } F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

properties of the Joint distribution:

$$1) F_{X,Y}(-\infty, -\infty) = 0, F_{X,Y}(-\infty, y) = 0 \text{ \& } F_{X,Y}(x, -\infty) = 0$$

$$2) F_{X,Y}(\infty, \infty) = 1$$

$$3) 0 \leq F_{X,Y}(x,y) \leq 1$$

4) $F_{X,Y}(x,y)$ is a non-decreasing function of x and y .

$$5) F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_2, y_1) = P\{x_1 < X \leq x_2; y_1 < Y \leq y_2\} \geq 0$$

$$6) F_{X,Y}(x, \infty) = F_X(x) \text{ and } F_{X,Y}(\infty, y) = F_Y(y).$$

Joint probability function of the discrete random variables X and Y .

If (X, Y) is a two-dimensional discrete random variable such that $f(x_i, y_j) = P(X = x_i, Y = y_j) = p_{ij}$ is called the joint probability function or joint probability mass function of (X, Y) provided the following conditions are satisfied.

$$1) p_{ij} \geq 0 \forall i \text{ and } j$$

Marginal probability mass function of X :

If the joint probability distribution of two random variables X and Y is given, then the marginal probability function of X is given by

$$f(x) = P_X(x_i) = P(X = x_i)$$

$$= P[X = x_i, Y = y_1] + P[X = x_i, Y = y_2] + \dots + P[X = x_i, Y = y_j] + \dots + P[X = x_i, Y = y_m]$$

$$= P_{i1} + P_{i2} + \dots + P_{ij} + \dots + P_{im}$$

$$= \sum_{j=1}^m P_{ij} = \sum_{j=1}^m P(x_i, y_j) = P_{i\cdot}$$

Marginal probability mass function of Y :

If the joint probability distribution of two random variables X and Y is given, then the marginal probability function of Y is given by $f_y = P_Y(y_j) = P(Y = y_j) = P_{\cdot j}$

$$\text{Here } P_{\cdot j} = \sum_i P_{ij} = P_{1j} + P_{2j} + \dots$$

Conditional probability distribution:

$$P\{X = x_i / Y = y_j\} = \frac{P\{X = x_i \cap Y = y_j\}}{P(Y = y_j)} = \frac{P_{ij}}{P_{\cdot j}}$$

is called the conditional probability function of X , given $Y = y_j$

Let (X, Y) be the two dimensional continuous R.V.

The conditional probability density function of X given Y is denoted by $f(x/y)$ and is defined as

$$f(x/y) = \frac{f(x, y)}{f(y)}$$

Similarly, the conditional probability density function of Y given X is denoted by $f(y/x)$ and is defined as,

$$f(y/x) = \frac{f(x, y)}{f(x)}$$

Independent random variables:

Two R.v.s X and Y are said to be independent if $f(x, y) = f(x) \cdot f(y)$ where $f(x, y)$ is the J.P.d.f of (X, Y) , $f(x)$ is the m.d.f of X and $f(y)$ is the m.d.f of Y .

Joint probability density function:

If (X, Y) is a two-dimensional continuous R.v such that $P\left\{x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}, y - \frac{dy}{2} \leq Y \leq y + \frac{dy}{2}\right\} = f(x, y) dx dy$, then $f(x, y)$ is called the joint p.d.f of (X, Y) , provided $f(x, y)$ satisfies the following conditions.

i) $f(x, y) \geq 0 \forall (x, y) \in R$, (ii) $\iint_R f(x, y) dx dy = 1$.

Cumulative distribution function:

If (X, Y) is a two-dimensional continuous random variable, then $F(x, y) = P(X \leq x \text{ and } Y \leq y)$ is called the cdf of (X, Y) and is defined as $F(x, y) = \int_{-\infty}^y \int_{-\infty}^x f(x, y) dx dy$.

Marginal density function:

Let (X, Y) be the two dimensional random variable. Then the m.p.d.f of X denoted by $f(x)$ and is defined as

$$f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy.$$

Similarly, the marginal probability density function of Y denoted by $f(y)$ and is defined as

$$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx.$$

Joint probability density function:

Let (X, Y) be the two dimensional random variable and $F(x, y)$ be the joint probability distribution function. Then the joint probability density function of X and Y is denoted by $f(x, y)$ and is defined as,

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}.$$

Problems:

The Joint probability mass function of (X, Y) is given by $P(x, y) = K(2x + 3y)$, $x = 0, 1, 2$; $y = 1, 2, 3$. Find all the marginal and conditional probability distributions. Also, find the probability distribution of $X+Y$ and $P[X+Y > 3]$.

Solution:

X \ Y	1	2	3	Total
0	3K	6K	9K	18K
1	5K	8K	11K	24K
2	7K	10K	13K	30K

Total probability = 1
 $72K = 1$
 $K = 1/72$

X \ Y	1	2	3	$P_X(x) = P_{i*}$
0	$3/72$ $P(0,1)$	$6/72$ $P(0,2)$	$9/72$ $P(0,3)$	$P(X=0) = 18/72$
1	$5/72$ $KP(1,1)$	$8/72$ $P(1,2)$	$11/72$ $P(1,3)$	$P(X=1) = 24/72$
2	$7/72$ $P(2,1)$	$10/72$ $P(2,2)$	$13/72$ $P(2,3)$	$P(X=2) = 30/72$

$$P_Y(y) = P_{*j}$$

$$P(Y=1) = 15/72 \quad P(Y=2) = 24/72 \quad P(Y=3) = 33/72$$

The Marginal distribution of X:

$$P(X=0) = 18/72; P(X=1) = 24/72; P(X=2) = 30/72$$

The Marginal distribution of Y:

$$P(Y=1) = 15/72; P(Y=2) = 24/72; P(Y=3) = 33/72$$

The conditional distribution of X , given Y is $P\{X=x_i | Y=y_j\}$

$$P(X=0|Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{3/12}{15/12} = 3/15 = 1/5$$

$$P(X=1|Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{5/15}{15/12} = 5/15 = 1/3$$

$$P(X=2|Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = 7/15$$

$$P(X=0|Y=2) = \frac{P(X=0, Y=2)}{P(Y=2)} = \frac{6/24}{24/24} = 6/24 = 1/4$$

$$P(X=1|Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{8/24}{24/24} = 8/24 = 1/3$$

$$P(X=2|Y=2) = \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{10/24}{24/24} = 10/24 = 5/12$$

$$P(X=0|Y=3) = \frac{9/33}{33/33} = 9/33$$

$$P(X=1|Y=3) = \frac{12/33}{33/33} = 12/33 = 4/11$$

$$P(X=2|Y=3) = \frac{12/33}{33/33} = 12/33 = 4/11$$

The conditional distribution of Y , given X is $P\{Y=y_j | X=x_i\}$

$$P(Y=1|X=0) = 1/6 \quad P(Y=2|X=0) = 1/3 \quad P(Y=3|X=0) = 1/2$$

$$P(Y=1|X=1) = 5/24 \quad P(Y=2|X=1) = 1/3 \quad P(Y=3|X=1) = 11/24$$

$$P(Y=1|X=2) = 7/30 \quad P(Y=2|X=2) = 1/3 \quad P(Y=3|X=2) = 13/30$$

Probability distribution of $X+Y$

$X+Y$

1 $P(0,1)$

2 $P(0,2) + P(1,1)$

3 $P(0,3) + P(1,2) + P(2,1)$

4 $P(1,3) + P(2,2)$

5 $P(2,3)$

Total = 1

P

$3/12$

$6/12 + 5/12 = 11/12$

$9/12 + 8/12 + 7/12 = 24/12$

$11/12 + 10/12 = 21/12$

$13/12$



$$P[X+Y \geq 3] = P[X+Y=4] + P[X+Y=5] = \frac{2}{72} + \frac{13}{72} = \frac{34}{72}$$

2) The joint p.d.f of the random variable (X, Y) is given by $f(x, y) = kxy e^{-(x^2+y^2)}$, $x > 0, y > 0$. Find the value of k and also prove that X and Y are Independent.

Solution:

We know that $\int_0^{\infty} \int_0^{\infty} kxy e^{-(x^2+y^2)} dx dy = 1 \rightarrow (1)$

$$k \int_0^{\infty} \int_0^{\infty} xy e^{-x^2} e^{-y^2} dx dy = 1$$

$$k \left[\int_0^{\infty} x e^{-x^2} dx \right] \left[\int_0^{\infty} y e^{-y^2} dy \right] = 1 \rightarrow (2)$$

$$\int_0^{\infty} x e^{-x^2} dx \rightarrow (3)$$

Put $x^2 = t$, $x \rightarrow 0 \Rightarrow t \rightarrow 0$

$2x dx = dt$, $x \rightarrow \infty \Rightarrow t \rightarrow \infty$

$$x dx = \frac{1}{2} dt$$

$$(3) \Rightarrow \int_0^{\infty} e^{-t} \frac{1}{2} dt = \frac{1}{2} \left[\frac{e^{-t}}{-1} \right]_0^{\infty} = -\frac{1}{2} [e^{-t}]_0^{\infty} = -\frac{1}{2} [0 - 1] = \frac{1}{2}$$

$$\therefore \int_0^{\infty} x e^{-x^2} dx = \int_0^{\infty} y e^{-y^2} dy = \frac{1}{2} \rightarrow (4)$$

$$\therefore (2) \Rightarrow k \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) = 1$$

$$\Rightarrow \frac{k}{4} = 1 \Rightarrow k = 4$$

To prove: X and Y are Independent.

To prove: $f(x) f(y) = f(x, y)$

The marginal density of X

is given by

$$f(x) = \int_0^{\infty} f(x, y) dy = \int_0^{\infty} 4xy e^{-(x^2+y^2)} dy = 4 \int_0^{\infty} xy e^{-x^2} e^{-y^2} dy$$

$$= 4x e^{-x^2} \int_0^{\infty} y e^{-y^2} dy = 4x e^{-x^2} \left(\frac{1}{2} \right) \text{ (by (4))} = 2x e^{-x^2}, x > 0.$$

The marginal density of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^{\infty} 4xy e^{-(x^2+y^2)} dx = 4 \int_0^{\infty} xy e^{-x^2} e^{-y^2} dx$$

$$= 4y e^{-y^2} \int_0^{\infty} x e^{-x^2} dx = 4y e^{-y^2} \left(\frac{1}{2} \right) \text{ (by ④)} = 2y e^{-y^2}, y > 0$$

$$f(x)f(y) = (2x e^{-x^2})(2y e^{-y^2}) = 4xy e^{-(x^2+y^2)} = f(x, y)$$

$\therefore X$ and Y are independent.

- ③ Given $f_{xy}(x, y) = c x(x-y)$, $0 < x < 2$, $-x < y < x$ and 0, elsewhere (a) Evaluate c ; (b) find $f_x(x)$; (c) $f_{y|x}(y/x)$ and (d) $f_y(y)$.

- ④ The joint pdf of (X, Y) is given by $f(x, y) = e^{-(x+y)}$, $0 \leq x, y < \infty$. Are X and Y independent? Why?

- ⑤ If the joint pdf of a two-dimensional random variable (X, Y) is given by $f(x, y) = x^2 + xy/3$, $0 < x < 1$, $0 < y < 2$. = 0 elsewhere.

Find (i) $P(X > 1/2)$; (ii) $P(Y < X)$; (iii) $P(Y < 1/2 | X < 1/2)$. Check whether the conditional density functions are valid.

Solution: Given $0 < x < 1$, $0 < y < 2$.

The marginal density of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

$$= \int_0^2 (x^2 + xy/3) dy$$

$$= \left[x^2 y + \frac{xy^2}{6} \right]_0^2$$

$$= 2x^2 + \frac{2x}{3}, 0 < x < 1$$

The marginal density of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

$$= \int_0^1 (x^2 + xy/3) dx$$

$$= \left(\frac{x^3}{3} + \frac{y}{6} x^2 \right)_0^1$$

$$= \left(\frac{1}{3} + \frac{y}{6} \right) - (0 + 0)$$

$$= \frac{1}{3} + \frac{y}{6} = \frac{2+y}{6}, 0 < y < 2.$$

$$i) P(X > 1/2) = \int_{1/2}^1 f(x) dx = \int_{1/2}^1 (2x^2 + 2x/3) dx = \left[\frac{2x^3}{3} + \frac{x^2}{3} \right]_{1/2}^1$$

$$= \frac{2}{3} + \frac{1}{3} - \frac{1}{12} - \frac{1}{12} = \frac{5}{6}.$$

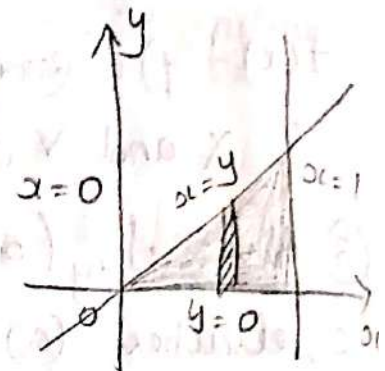
ii) To find: $P[Y < X]$

$$\text{Let } P[Y < X] = \iint_R (x^2 + xy/3) dy dx \rightarrow \textcircled{1}$$

Here outer limit x is $0 < x < 1$

ii) x limit varies from $x=0$ to $x=1$;

inner limit y , Here $Y < X$, $0 < y < x$



$$(1) \Rightarrow P[Y < X] = \int_0^1 \int_0^x (x^2 + xy/3) dy dx$$

$$= \int_0^1 \left(x^2 y + \frac{xy^2}{6} \right)_0^x dx$$

$$= \int_0^1 \left(x^3 + x^3/6 \right) dx$$

$$= \frac{7}{6} \int_0^1 x^3 dx = \frac{7}{6} \left[\frac{x^4}{4} \right]_0^1$$

$$= \frac{7}{24}$$

$$(iii) P(Y < 1/2 | X < 1/2) = \frac{P(X < 1/2, Y < 1/2)}{P(X < 1/2)} \rightarrow \textcircled{2}$$

$$iii) P(Y < 1/2 | X < 1/2)$$

$$P(X < 1/2, Y < 1/2)$$

$$= \int_0^{1/2} \int_0^{1/2} f(x, y) dy dx$$

$$= \int_0^{1/2} \int_0^{1/2} (x^2 + xy/3) dy dx$$

$$= \int_0^{1/2} \left[x^2 y + \frac{xy^2}{6} \right]_0^{1/2} dx$$

$$= \int_0^{1/2} \left[\frac{x^2}{2} + \frac{x}{24} \right] dx$$

$$= \left[\frac{x^3}{6} + \frac{x^2}{48} \right]_0^{1/2}$$

$$= \frac{5}{192}$$

$$P(X < 1/2)$$

$$= \int_0^{1/2} f(x) dx$$

$$= \int_0^{1/2} (2x^2 + 2x/3) dx$$

$$= \left[\frac{2x^3}{3} + \frac{x^2}{3} \right]_0^{1/2}$$

$$= \frac{1}{12} + \frac{1}{12}$$

$$= \frac{2}{12}$$

$$= \frac{1}{6}$$

$$\therefore \textcircled{2} \Rightarrow P(Y < 1/2 | X < 1/2) = \frac{5/192}{1/6} = \frac{5}{32}$$

checking the conditional density functions are valid.

$$\begin{aligned} \int_0^1 f(x|y) dx &= \int_0^1 \frac{f(x,y)}{f(y)} dx \\ &= \int_0^1 \left(\frac{6x^2 + 2xy}{2+y} \right) dx \\ &= \frac{1}{2+y} \left[\frac{6x^3}{3} + 2y \frac{x^2}{2} \right]_0^1 \\ &= \frac{1}{2+y} \left[(2+y) - (0+0) \right] \\ &= \frac{1}{2+y} (2+y) = 1 \end{aligned}$$

$$\begin{aligned} \int_0^2 f(y|x) dy &= \int_0^2 \frac{f(y,x)}{f(x)} dy \\ &= \int_0^2 \frac{f(x,y)}{f(x)} dy \\ &= \int_0^2 \left(\frac{3x+y}{6x+2} \right) dy \\ &= \frac{1}{6x+2} \int_0^2 (3x+y) dy \\ &= \frac{1}{6x+2} \left[3xy + \frac{y^2}{2} \right]_0^2 \\ &= \frac{1}{6x+2} (6x+2) = 1. \end{aligned}$$

⑥ If the joint distribution function of X and Y is given by $F(x,y) = (1 - e^{-x})(1 - e^{-y})$ for $x > 0, y > 0$
 $= 0$ otherwise.

- (i) Find the marginal density of X and Y .
 (ii) Are X and Y independent? (iii) $P(1 < X < 3, 1 < Y < 2)$.

Solution: Given: $F(x,y) = (1 - e^{-x})(1 - e^{-y}) = 1 - e^{-x} - e^{-y} + e^{-(x+y)}$

The joint p.d.f is given by $f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} [1 - e^{-x} - e^{-y} + e^{-(x+y)}] = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} [1 - e^{-x} - e^{-y} + e^{-(x+y)}] \right]$$

$$= \frac{\partial}{\partial x} [0 - 0 + e^{-y} - e^{-(x+y)}] = e^{-y} - e^{-(x+y)}$$

$$\therefore f(x,y) = e^{-(x+y)}, x > 0, y > 0 = 0, \text{ otherwise}$$

The marginal density of X is $f(x) = \int_{-\infty}^{\infty} f(x,y) dy = e^{-x}, x > 0$.

The marginal density of Y is $f(y) = \int_{-\infty}^{\infty} f(x,y) dx = e^{-y}, y > 0$.

$$\text{Also } f(x) \cdot f(y) = e^{-x} \cdot e^{-y} = e^{-(x+y)} = f(x,y)$$

$\Rightarrow X$ & Y are independent.

$$\begin{aligned} \text{iii) } P[1 < X < 3, 1 < Y < 2] &= \int_1^3 f(x) dx \cdot \int_1^2 f(y) dy \\ &= \int_1^3 e^{-x} dx \cdot \int_1^2 e^{-y} dy = \left(-e^{-x} \right)_1^3 \left(-e^{-y} \right)_1^2 = (e^{-1} - e^{-3}) (e^{-1} - e^{-2}) \\ &= \left(\frac{1}{e} - \frac{1}{e^3} \right) \left(\frac{1}{e} - \frac{1}{e^2} \right) \\ &= \left(\frac{1 - e^{-2}}{e^3} \right) \left(\frac{1 - e^{-1}}{e^2} \right) \end{aligned}$$

Covariance, Correlation and Regression

Covariance:

If X and Y are random variables, then co-variance between them is defined as

$$\text{cov}(X, Y) = E[XY] - E[X]E[Y].$$

Note: If X and Y are independent, then $\text{cov}(X, Y) = 0$

If X and Y are independent, then $E[XY] = E[X] \cdot E[Y]$
 $\Rightarrow \text{cov}(X, Y) = 0.$

$$(i) \text{cov}(aX, bY) = ab \text{cov}(X, Y)$$

$$(ii) \text{cov}(X+a, Y+b) = \text{cov}(X, Y).$$

$$(iii) \text{cov}(aX+b, cY+d) = ac \text{cov}(X, Y).$$

$$(iv) V(X_1 + X_2) = V(X_1) + V(X_2) + 2 \text{cov}(X_1, X_2)$$

$$(v) V(X_1 - X_2) = V(X_1) + V(X_2) - 2 \text{cov}(X_1, X_2).$$

If X_1 and X_2 are independent, then $V(X_1 \pm X_2) = V(X_1) \pm V(X_2)$

Problems:

1) Calculate the correlation co-efficient for the following heights of fathers X and their sons Y .

$X :$ 65 66 67 67 68 69 70 72.

$Y :$ 67 68 65 68 72 72 69 71

Soln:	X	Y	XY	X^2	Y^2
	65	67	4355	4225	4489
	66	68	4488	4356	4624
	67	65	4355	4489	4225
	67	68	4556	4489	4624
	68	72	4896	4624	5184
	69	72	4968	4761	5184
	70	69	4830	4900	4761
	72	71	5112	5184	5041

$$\sum X = 544, \sum Y = 563, \sum XY = 37560, \sum X^2 = 31032, \\ \sum Y^2 = 38133$$

$$\bar{X} = \frac{\sum X}{n} = \frac{544}{8} = 68, \quad \bar{Y} = \frac{\sum Y}{n} = \frac{563}{8} = 69$$

$$\bar{X}\bar{Y} = (68 \times 69) = 4692$$

$$\sigma_x = \sqrt{\frac{1}{n} \sum X^2 - (\bar{X})^2} = \sqrt{\frac{1}{8}(31032) - 68^2} = \sqrt{4638.5 - 4624} = 2.131$$

$$\sigma_y = \sqrt{\frac{1}{n} \sum Y^2 - (\bar{Y})^2} = \sqrt{\frac{1}{8}(38133) - 69^2} = \sqrt{4766.5 - 4761} = 2.345$$

$$\text{cov}(X, Y) = \frac{1}{n} \sum XY - \bar{X}\bar{Y} = \frac{1}{8}(37560) - 68 \times 69 \\ = 4695 - 4692 = 3$$

The correlation coefficient of X and Y is given by

$$r(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_x \sigma_y} = \frac{3}{(2.131)(2.345)} = \frac{3}{4.973} = 0.6032$$

2) Two random variables X and Y have the joint density

$$f(x, y) = \begin{cases} 2-x-y, & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{Show that } \text{cov}(X, Y) = -\frac{1}{144}$$

Soln: The marginal density function of X is

$$f(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 (2-x-y) dy = \frac{3}{2} - x, \quad 0 < x < 1$$

The marginal density function of Y is

$$f(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^1 (2-x-y) dx = \frac{3}{2} - y, \quad 0 < y < 1$$

$$\text{Now, } E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x \left(\frac{3}{2} - x \right) dx = \int_0^1 \left(\frac{3}{2}x - x^2 \right) dx \\ = \left[\frac{3x^2}{4} - \frac{x^3}{3} \right]_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}$$

$$E[Y] = \int_{-\infty}^{\infty} y f(y) dy = \int_0^1 y \left(\frac{3}{2} - y \right) dy = \int_0^1 \left(\frac{3}{2}y - y^2 \right) dy \\ = \left[\frac{3y^2}{4} - \frac{y^3}{3} \right]_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}$$

$$\begin{aligned}
 E[XY] &= \int_0^1 \int_0^1 xy f(x,y) dx dy = \int_0^1 \int_0^1 xy(2-x-y) dx dy \\
 &= \int_0^1 \int_0^1 (2xy - x^2y - xy^2) dx dy \\
 &= \int_0^1 \left[x^2y - \frac{x^3y}{3} - \frac{x^2y^2}{2} \right]_0^1 dy = \int_0^1 \left(y - \frac{y}{3} - \frac{y^2}{2} \right) dy \\
 &= \int_0^1 \left(\frac{2y}{3} - \frac{y^2}{2} \right) dy = \left[\frac{y^2}{3} - \frac{y^3}{6} \right]_0^1 = \frac{1}{3} - \frac{1}{6} = \frac{2-1}{6} = \frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{COV}(X,Y) &= E[XY] - E[X] \cdot E[Y] \\
 &= \frac{1}{6} - \frac{5}{12} \cdot \frac{5}{12} = \frac{1}{6} - \frac{25}{144} = \frac{24-25}{144} = \frac{-1}{144}
 \end{aligned}$$

③ Suppose that the 2D RV (X,Y) has the joint p.d.f

$$f(x,y) = \begin{cases} x+y, & 0 \leq x < 1, 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Obtain the correlation coefficient between X and Y .
Check whether X and Y are independent.

4) Two R.V X and Y are defined by

$$f(x) = 4ax, 0 \leq x \leq 1 \quad f(y) = 4by, 0 \leq y \leq 1 \\
 \begin{matrix} 1 - (x,y) \\ 0 \end{matrix} \quad \begin{matrix} \text{otherwise} \\ 0 \end{matrix} \quad \begin{matrix} \text{otherwise} \\ 0 \end{matrix}$$

Show that $U = X+Y$ and $V = X-Y$ are uncorrelated.

Solution:

$$(i) f(x) = 4ax, 0 \leq x \leq 1 \quad f(y) = 4by, 0 \leq y \leq 1$$

$$\int_0^1 f(x) dx = 1 \Rightarrow a = \frac{1}{2} \quad \int_0^1 f(y) dy = 1 \Rightarrow b = \frac{1}{3}$$

To Prove: $\text{COV}(U,V) = 0$.

$$\text{COV}(U,V) = E[UV] - E[U]E[V]$$

$$E[U] = E[X+Y] = E[X] + E[Y]$$

$$E[V] = E[X-Y] = E[X] - E[Y]$$

$$E[UV] = E[X^2 - Y^2]$$

$$E[X] = \frac{2}{3}, \quad E[Y] = \frac{2}{3}, \quad E[XY] = \frac{4}{9}$$

$$E[U] = 1/3, E[V] = 0, E[UV] = E[X^2] - E[Y^2] = 1/3 - 1/3 = 0$$

$$\text{Cov}(U, V) = E[UV] - E[U]E[V] = 0 - 1/3(0) = 0$$

∴ U and V are independent.

Regression

Regression: Regression is a mathematical measure of the average relationship between two or more variables in terms of the original limits of the data.

Lines of Regression:

(1) The line of regression of y on x is given by

$$y - \bar{y} = r \frac{\sigma_y}{\sigma_x} (x - \bar{x}) \rightarrow \textcircled{1}$$

(2) The line of regression of x on y is given by

$$x - \bar{x} = r \frac{\sigma_x}{\sigma_y} (y - \bar{y}) \rightarrow \textcircled{2}$$

Regression coefficients

(1) Regression coefficient of y on x is $r \frac{\sigma_y}{\sigma_x} = b_{yx}$.

(2) Regression coefficient of x on y is $r \frac{\sigma_x}{\sigma_y} = b_{xy}$.

$$\text{Correlation coefficient } r = \pm \sqrt{b_{yx} b_{xy}}$$

$$\text{where } b_{yx} = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2}; \quad b_{xy} = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (y - \bar{y})^2}$$

Angle between the regression lines

The slopes of the regression lines are $m_1 = r \frac{\sigma_y}{\sigma_x}$, $m_2 = \frac{1}{r} \frac{\sigma_y}{\sigma_x}$

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\frac{\sigma_y}{\sigma_x} \left(\frac{1}{r} - r \right)}{1 + \left(\frac{\sigma_y}{\sigma_x} \right)^2} = \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2} \left(\frac{1}{r} - r \right)$$

$$\tan \theta = \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2} \left(\frac{1 - r^2}{r} \right)$$

Correlation of Grouped data:

The formula for calculating the co-efficient of correlation is

$$r = \frac{\rho}{\sigma_x \sigma_y}$$

$$\text{where } \rho = \frac{\sum xy f_{xy}}{N} - \left(\frac{\sum x f_x}{N} \right) \left(\frac{\sum y f_y}{N} \right)$$

$$\sigma_x^2 = \frac{\sum x^2 f_x}{N} - \left(\frac{\sum x f_x}{N} \right)^2$$

$$\sigma_y^2 = \frac{\sum y^2 f_y}{N} - \left(\frac{\sum y f_y}{N} \right)^2$$

Problems:

- i) The two lines of regression are $8x - 10y + 66 = 0 \rightarrow (A)$
 $40x - 18y - 214 = 0 \rightarrow (B)$

The variance of x is 9. Find (i) The mean value of x and y

- (ii) The correlation coefficient between x and y .

Solution:

- i) Since both the lines of regression pass through the mean values $\bar{x}$ and $\bar{y}$, the point $(\bar{x}, \bar{y})$ must satisfy the two given regression lines

$$8\bar{x} - 10\bar{y} = -66 \rightarrow (1)$$

$$40\bar{x} - 18\bar{y} = 214 \rightarrow (2)$$

$$(1) \times 5 \Rightarrow 40\bar{x} - 50\bar{y} = -330 \rightarrow (3)$$

$$(2) \times 1 \Rightarrow 40\bar{x} - 18\bar{y} = 214 \rightarrow (4)$$

$$(3) - (4) \Rightarrow -32\bar{y} = -544$$

$$\bar{y} = 17$$

$$\text{Sub in (1), we get } 8\bar{x} - 10(17) = -66$$

$$\bar{x} = 13$$

Hence the mean values are given by $\bar{x} = 13, \bar{y} = 17$.



(ii)

$$(A) \Rightarrow 8x - 10y = -66$$

$$\Rightarrow x = \frac{10}{8}y - \frac{66}{8}$$

$$\text{ie) } b_{xy} = 10/8$$

$$(A) \Rightarrow 10y = 8x + 66$$

$$\Rightarrow y = 8/10x + 66/10$$

$$\text{ie) } b_{yx} = 8/10$$

$$(B) \Rightarrow 18y = 40x - 214$$

$$\Rightarrow y = \frac{40}{18}x - \frac{214}{18}$$

$$\text{ie) } b_{yx} = 40/18$$

$$(B) \Rightarrow 40x = 18y + 214$$

$$\Rightarrow x = \frac{18}{40}y + \frac{214}{40}$$

$$\text{ie) } b_{xy} = 18/40$$

$$r^2 = b_{xy} b_{yx} = \left(\frac{10}{8}\right) \left(\frac{40}{18}\right)$$

$$= 2.77$$

$$r = 1.66 \neq 1$$

$$r^2 = b_{yx} b_{xy}$$

$$= \left(\frac{8}{10}\right) \left(\frac{18}{40}\right)$$

$$= 0.36$$

$$r = \pm 0.6$$

Since both the regression coefficients are positive r must be positive $r = 0.6$.

② The equations of two regression lines are $3x + 12y = 19$ and $3y + 9x = 46$. Find $\bar{x}$, $\bar{y}$ and the correlation coefficient between X and Y .

3) The regression equation of X on Y is $3Y - 5X + 108 = 0$. If the mean value of Y is 44 and the variance of X is $(9/16)^{\text{th}}$ of the variance of Y . Find the mean value of X and the correlation coefficient.

Solution: Given: $\bar{y} = 44$

$$3\bar{y} - 5\bar{x} = -108 \rightarrow (1)$$

$$\Rightarrow 3(44) - 5\bar{x} = -108 \Rightarrow -5\bar{x} = -108 - 132 \Rightarrow -5\bar{x} = -240$$

$$\Rightarrow \bar{x} = 48 \quad (1) \Rightarrow 5\bar{x} = 3\bar{y} + 108$$

$$\Rightarrow \bar{x} = \frac{3}{5}\bar{y} + \frac{108}{5} \Rightarrow b_{xy} = 3/5$$

$$\Rightarrow r \frac{\sigma_x}{\sigma_y} = 3/5 \rightarrow (2)$$

Given: $\sigma_x^2 = 9/16 \sigma_y^2$

$$\Rightarrow \sigma_x = 3/4 \sigma_y \rightarrow (3)$$

$$\Rightarrow \frac{\sigma_x}{\sigma_y} = 3/4$$

$$(2) \Rightarrow r \left(\frac{3}{4}\right) = 3/5 \Rightarrow r = 4/5 = 0.8$$

Transformation of Random Variables

Two functions of two random variables.

If (x, y) is a two dimensional random variable with joint p.d.f $f_{xy}(x, y)$ and if $z = g(x, y)$ and $w = h(x, y)$ are two other random variables then the joint p.d.f of (z, w) is given by,

$$f_{zw}(z, w) = \frac{f_{xy}(x, y)}{|J|} \quad \text{where } |J| = \frac{\partial(z, w)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix}$$

$$(or) f_{zw}(z, w) = |J| f_{xy}(x, y), \quad \text{where } J = \frac{\partial(x, y)}{\partial(z, w)}$$

1) If X and Y are independent random variables with pdf $e^{-x}, x \geq 0$ and $e^{-y}, y \geq 0$ respectively. Find the density function of $U = \frac{X}{X+Y}$ and $V = X+Y$. Are u and v independent?

Soln: Since X and Y are independent,

$$f(x, y) = f(x)f(y) = e^{-x} \cdot e^{-y} = e^{-(x+y)}, \quad x, y \geq 0$$

The density function of U is

$$f(u) = \int_{-\infty}^{\infty} f(u, v) dv \rightarrow (1)$$

The density function of V is

$$f(v) = \int_{-\infty}^{\infty} f(u, v) du \rightarrow (2)$$

$$\text{where } f(u, v) = |J| f(x, y) \rightarrow (3)$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \rightarrow (4)$$

Given:

$$u = \frac{x}{x+y}$$

$$\Rightarrow u = \frac{x}{v}$$

$$\Rightarrow x = uv$$

$$\frac{\partial x}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = u$$

$$v = x+y$$

$$y = v - x$$

$$y = v - uv$$

$$\frac{\partial y}{\partial u} = -v$$

$$\frac{\partial y}{\partial v} = 1 - u$$

$$x \geq 0 \quad y \geq 0$$

$$uv \geq 0 \quad v - uv \geq 0$$

$$v \geq 0 \Rightarrow v \geq uv$$

$$\Rightarrow 1 \geq u$$

$$\Rightarrow u \leq 1$$

$$(4) \Rightarrow J = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v - uv + uv = v$$

$$(3) \Rightarrow f(u, v) = v e^{-v}$$

$$(1) \Rightarrow f(u) = \int_0^{\infty} v e^{-v} dv$$

$$= \left[v \left(\frac{e^{-v}}{-1} \right) - (1) \left(\frac{e^{-v}}{(-1)^2} \right) \right]_0^{\infty} = \left[(-v - 1) e^{-v} - (-1) \right]_0^{\infty} = 1$$

$$(2) \rightarrow f(u) = \int v e^{-v} dv = \left[-v e^{-v} - \int -e^{-v} dv \right]_0^{\infty} = \left[-v e^{-v} + e^{-v} \right]_0^{\infty} = 1$$

$$\text{Now } f(u)f(v) = (1)(e^{-v}) = e^{-v} = f(u,v)$$

u and v are independent random variables.

(2) The joint p.d.f of X and Y is given by $f(x,y) = e^{-x-y}$, $x > 0, y > 0$. Find the probability density function of $U = \frac{x+y}{2}$.

2) If X and Y are independent exponential distributions with parameter 1, then find the p.d.f of $U = X + Y$.

3) If the p.d.f of a two dimensional R.V. (X,Y) is given by

$$f(x,y) = x+y, \quad 0 \leq x, y \leq 1. \text{ Find the p.d.f of } U = XY.$$

Soln:

The density function of u is given by

$$f(u) = \int_{-\infty}^{\infty} f(x,y) dy \rightarrow (1) \quad u = xy \rightarrow y = \frac{u}{x}$$

$$f(u,v) = |J| f(x,y) \rightarrow (2) \quad \begin{matrix} u = xv \\ x = \frac{u}{v} \end{matrix} \quad \frac{\partial x}{\partial u} = \frac{1}{v}, \quad \frac{\partial x}{\partial v} = -\frac{u}{v^2}$$

$$\text{where } J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ \frac{1}{x} & \frac{1}{v} \end{vmatrix} = \frac{1}{v} - \frac{u}{v^2} = \frac{v-u}{v^2}$$

$$(3) \Rightarrow J = \left| \frac{1}{v} - \frac{u}{v^2} \right| = \frac{1}{v}$$

$$(2) \Rightarrow f(u,v) = \left(\frac{1}{v} \right) \left(\frac{u}{v} + v \right) = \frac{u}{v^2} + 1$$

$$(1) \Rightarrow f(u) = \int_0^1 \left(\frac{u}{v^2} + 1 \right) dv = \left[-\frac{u}{v} + v \right]_0^1 = -\frac{u}{1} + 1 = 1 - u$$

Analytic Functions.

Complex Variable:

$x+iy$ is a complex variable and it is denoted by z .

(e) $z = x+iy$, where $i = \sqrt{-1}$.

Function of a complex variable:

If $z = x+iy$ and $w = u+iv$ are two complex variables, and if for each value of z in a given region R of the complex plane there corresponds one or more values of w , then w is said to be a function of z and is denoted by $w = f(z)$.
 $w = f(z) = f(x+iy) = u(x,y) + iv(x,y)$, where $u(x,y)$ and $v(x,y)$ are real functions of the real variable x and y .

Note:

1) Single valued Function:

If for each value of z in R there is correspondingly only one value of w , then w is called a single valued function of z .

2) Multiple-valued function:

If there is more than one value of w corresponding to a given value of z , then w is called a multiple-valued function.

Necessary conditions - Cauchy - Riemann Equations =

Sufficient conditions

The necessary condition for $f(z)$ to be analytic [Cauchy - Riemann Equations]

The necessary condition for a complex function $f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{e) } u_x = v_y \quad \text{and} \quad v_x = -u_y$$

Sufficient conditions for $f(z)$ to be analytic:

If the partial derivatives u_x, u_y, v_x and v_y are all continuous in D and $u_x = v_y$ and $u_y = -v_x$, then the function $f(z)$ is analytic in a domain D .

Polar form of C-R equations:

In Cartesian co-ordinates any point z is $z = x + iy$

In polar co-ordinates $z = re^{i\theta}$, where r is the modulus and θ is the argument.

Theorem:

If $f(z) = u(r, \theta) + iv(r, \theta)$ is differentiable at $z = re^{i\theta}$,

then $u_r = \frac{1}{r} v_\theta$, $v_r = -\frac{1}{r} u_\theta$ (or)

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

problems:

1) Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Soln:

Given: $f(z) = \bar{z} = x - iy$

$u = x, v = -y$

$\frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial y} = -1$

$\frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0$

$\therefore u_x \neq v_y$

C-R eqns are not satisfied anywhere

Hence, $f(z) = \bar{z}$ is not differentiable anywhere

2) If $w = e^z$, find $\frac{dw}{dz}$ (or) Show that the function $w = e^z$ is analytic everywhere in the complex plane.

Soln: Let $z = x + iy$

$f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x [\cos y + i \sin y]$

$u + iv = e^x [\cos y + i \sin y] = \frac{u}{e^x} \text{ and } \frac{v}{e^x} = \frac{v}{e^x}$

$$u = e^x \cos y \quad v = e^x \sin y$$

$$u_x = e^x \cos y \quad v_x = e^x \sin y$$

$$u_y = -e^x \sin y \quad v_y = e^x \cos y$$

$$\text{Here } u_x = v_y \text{ and } u_y = -v_x$$

$\Rightarrow$ C.R equations are satisfied.

Further e^x , $\cos y$, $\sin y$ are continuous and so u_x, v_x, u_y and v_y are continuous everywhere.

Thus the sufficiency conditions for the existence of derivative of e^z are satisfied everywhere

$$\begin{aligned} \therefore f'(z) &= u_x + i v_x = e^x \cos y + i e^x \sin y \\ &= e^x [\cos y + i \sin y] = e^x e^{iy} \\ &= e^{x+iy} = e^z \quad [\text{itself}] \end{aligned}$$

2) Find $\frac{dw}{dz}$ given $w = \tan z$

$$\text{Soln: } w = \tan z = \tan(x+iy)$$

$$= \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{2 \sin(x+iy) \cos(x-iy)}{2 \cos(x+iy) \cos(x-iy)}$$

$$= \frac{\sin[(x+iy) + (x-iy)] + \sin[(x+iy) - (x-iy)]}{\cos[(x+iy) + (x-iy)] + \cos[(x+iy) - (x-iy)]}$$

$$= \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$$

$$\therefore u = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

$$v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

$$u_x = \frac{[\cos 2x + \cosh 2y] 2 \cos 2x - \sin 2x [-2 \sin 2x]}{(\cos 2x + \cosh 2y)^2}$$

$$= \frac{2 \cos^2 2x + 2 \cos 2x \cosh 2y + 2 \sin^2 2x}{(\cos 2x + \cosh 2y)^2}$$

$$= \frac{2 + 2 \cos 2x \cosh 2y}{(\cos 2x + \cosh 2y)^2}$$

$$u_y = \frac{(\cos 2x + \cosh 2y)(0) - \sin 2x \cdot 2 \sinh 2y}{(\cos 2x + \cosh 2y)^2}$$

$$= \frac{-2 \sin 2x \sinh 2y}{(\cos 2x + \cosh 2y)^2}$$

$$v_x = \frac{(\cos 2x + \cosh 2y)(0) - \sinh 2y (-2 \sin 2x)}{(\cos 2x + \cosh 2y)^2}$$

$$= \frac{2 \sin 2x \sinh 2y}{(\cos 2x + \cosh 2y)^2}$$

$$v_y = \frac{(\cos 2x + \cosh 2y)(2 \cosh 2y) - \sinh 2y (2 \sinh 2y)}{(\cos 2x + \cosh 2y)^2}$$

$$= \frac{2 \cos 2x \cosh 2y + 2 \cosh^2 2y - 2 \sinh^2 2y}{(\cos 2x + \cosh 2y)^2}$$

$$= \frac{2 \cos 2x \cosh 2y + 2 [\cosh^2 2y - \sinh^2 2y]}{(\cos 2x + \cosh 2y)^2}$$

$$= \frac{2 + 2 \cos 2x \cosh 2y}{(\cos 2x + \cosh 2y)^2}$$

$$u_x = v_y \text{ and } u_y = -v_x$$

So $\tan z$ satisfies the C.R eqns

$$\therefore f'(z) = u_x + i v_x$$

$$= \frac{2 + 2 \cos 2x \cosh 2y}{(\cos 2x + \cosh 2y)^2} + i \frac{2 \sin 2x \sinh 2y}{(\cos 2x + \cosh 2y)^2}$$

$$= \frac{2 + 2 \cos 2x \cosh 2y + i 2 \sin 2x \sinh 2y}{(\cos 2x + \cosh 2y)^2}$$

$$= \sec^2 z + c$$

check for the analyticity of $\log z$.

Soln: let $z = re^{i\theta}$

$$f(z) = \log z = \log(re^{i\theta}) = \log r + i \log e^{i\theta} = \log r + i\theta$$

But, at the origin, $r=0$, thus at the origin

$$f(z) = \log 0 + i\theta = -\infty + i\theta$$

So, $f(z)$ is not defined at the origin and hence is not differentiable there.

At points other than the origin, we have

$$u(r, \theta) = \log r \quad v(r, \theta) = \theta$$

$$u_r = \frac{1}{r} \quad v_r = 0$$

$$u_\theta = 0 \quad v_\theta = 1$$

So, $\log z$ satisfies the C-R eqns

Further $\frac{1}{r}$ is not continuous at $z=0$.

So, $u_r, u_\theta, v_r, v_\theta$ are continuous everywhere except

at $z=0$. Thus $\log z$ satisfies all the sufficient conditions for the existence of the derivative except at the origin.

$$\text{The derivative is } f'(z) = \frac{u_r + i v_r}{e^{i\theta}} = \frac{(\frac{1}{r}) + i(0)}{e^{i\theta}} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

5) Check the analyticity of the function $w = \sin z$

Soln: let $w = f(z) = \sin z$

$$u + iv = \sin(\alpha + i\gamma)$$

$$u + iv = \sin \alpha \cosh \gamma + i \cos \alpha \sinh \gamma$$

$$u + iv = \sin \alpha \cosh \gamma + i \cos \alpha \sinh \gamma$$

Equating real & imaginary parts, we get

$$u = \sin \alpha \cosh \gamma \quad v = \cos \alpha \sinh \gamma$$

$$u_x = \cos \alpha \cosh \gamma \quad v_x = -\sin \alpha \sinh \gamma$$

$$u_y = \sin \alpha \sinh \gamma \quad v_y = \cos \alpha \cosh \gamma$$

$$\therefore u_x = v_y \text{ and } u_y = -v_x$$



C-R eqns are satisfied. $\therefore$ the function is analytic.

Show that $f(z) = |z|^2$ is differentiable at $z=0$, but not analytic at $z=0$.

Soln: Let $z = x+iy$, $\bar{z} = x-iy$

$$|z|^2 = z\bar{z} = x^2 + y^2$$

$$f(z) = |z|^2 = (x^2 + y^2) + i \cdot 0$$

$$u = x^2 + y^2, \quad v = 0$$

$$u_x = 2x, \quad v_x = 0$$

$$u_y = 2y, \quad v_y = 0$$

C-R equations are satisfied everywhere at $z=0$

So, $f(z)$ may be differentiable only at $z=0$.

So, $f(z)$ is differentiable at $z=0$ only and is not analytic there.

7) Show that $w = \frac{z}{z+a}$ where $a \neq 0$ is analytic whereas $\bar{w} = \frac{\bar{z}}{\bar{z}+a}$ is not analytic.

Soln: Let $z = x+iy$

$w = \frac{z}{z+a}$ is a function of z

We know that $f(z)$ is analytic $f(\bar{z})$ is not analytic

$\therefore w = u+iv$ is not analytic function of z

$$\text{Given } u+iv = \frac{z}{z+a} = f(z) \quad (u+iv)(z+a) = v+iu$$

$$f(z) = \frac{z}{z+a}$$

$$f'(z) = \frac{a}{(z+a)^2} \text{ which is analytic everywhere}$$

except at $z = -a$.

$\therefore f(z)$ is analytic, except at $z = -a$

8) Prove that every analytic function $w = u(x, y) + iv(x, y)$ can be expressed as a function of z alone.

proof: Let $z = x + iy$ and $\bar{z} = x - iy$

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}$$

Hence u and v and also w can be considered as a function of z and $\bar{z}$. We have $w = u + iv$

$$\begin{aligned} \text{Consider } \frac{\partial w}{\partial \bar{z}} &= \frac{\partial u}{\partial \bar{z}} + i \frac{\partial v}{\partial \bar{z}} \\ &= \left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} \right) + i \left(\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} \right) \\ &= \left(\frac{1}{2} u_x - \frac{1}{2i} u_y \right) + i \left(\frac{1}{2} v_x - \frac{1}{2i} v_y \right) \\ &= \frac{1}{2} (u_x - v_y) + i \frac{1}{2} (u_y + v_x) \\ &= \frac{1}{2} (v_y - v_y) + i \frac{1}{2} (-v_x + v_x) = 0 \text{ by C-R equations} \end{aligned}$$

This means that w is independent of $\bar{z}$.

ie) w is a function of z alone.

This means that if $w = u(x, y) + iv(x, y)$ is analytic, it can be written as a function of $(x + iy)$.

Equivalently a function of $\bar{z}$ can not be an analytic function of z .

$$\text{Laplace Equation: } \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ is known as Laplace equation in two dimensions.}$$

dimensions.

Laplacian operator:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \text{ is called the Laplacian operator and is denoted by } \nabla^2.$$

by ∇^2 .

Note: i) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$ is known as Laplace equation in three dimensions.

(ii) The Laplace equation in polar coordinates is defined as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0.$$

Properties of analytic functions:

i) Both the real part and the imaginary parts of any analytic function satisfies Laplace's equation. If $f(z) = u + iv$ is an analytic function, then u and v are harmonic functions.

Proof: Let $f(z) = u + iv$ be an analytic function.

We know that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \left(-\frac{\partial v}{\partial x} \right) = -\frac{\partial^2 v}{\partial y \partial x}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

$$\text{Similarly, } \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right) = \frac{\partial}{\partial x} \left(-\frac{\partial u}{\partial y} \right) = -\frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial y \partial x}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Problems:

1) Prove that $u = x^2 - y^2$ and $v = \frac{y}{x^2 + y^2}$ are harmonic but $u + iv$ is not regular.

Let $u = x^2 - y^2$, $v = \frac{-y}{x^2 + y^2}$

$$\frac{\partial u}{\partial x} = 2x \rightarrow (1) \quad \frac{\partial v}{\partial x} = -y \left[\frac{0 - 2x}{(x^2 + y^2)^2} \right] = \frac{2xy}{(x^2 + y^2)^2} \rightarrow (5)$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \rightarrow (2) \quad \frac{\partial^2 v}{\partial x^2} = \frac{(x^2 + y^2)^2 \cdot 2y - 2xy \cdot 2(x^2 + y^2) \cdot 2x}{(x^2 + y^2)^4}$$

$$= \frac{(x^2 + y^2)[(x^2 + y^2)2y - 8x^2y]}{(x^2 + y^2)^4} = \frac{2y^3 - 6x^2y}{(x^2 + y^2)^3} \rightarrow (6)$$

$$\left[\frac{\partial u}{\partial y} = -2y \rightarrow (3) \quad \frac{\partial v}{\partial y} = - \left[\frac{(x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} \right] = \frac{y^2 - x^2}{(x^2 + y^2)^2} \rightarrow (7) \right]$$

$$\frac{\partial^2 u}{\partial y^2} = -2 \rightarrow (4) \quad \frac{\partial^2 v}{\partial y^2} = \frac{(x^2 + y^2)^2 \cdot 2y - (y^2 - x^2) \cdot 2(x^2 + y^2) \cdot 2y}{(x^2 + y^2)^4}$$

$$= \frac{(x^2 + y^2)[(x^2 + y^2)2y - 4y(y^2 - x^2)]}{(x^2 + y^2)^4}$$

$$= \frac{2x^2y + 2y^3 - 4y^3 + 4x^2y}{(x^2 + y^2)^3} = \frac{6x^2y - 2y^3}{(x^2 + y^2)^3} \rightarrow (8)$$

Using (2) and (4), we get

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \therefore u \text{ is harmonic}$$

Using (6) and (8), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \therefore v \text{ is harmonic}$$

From (1), (3), (5) and (7), we get $u_x \neq v_y$ and $u_y \neq -v_x$

$\therefore u$ and v are not regular (or analytic).

$u = \frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial x}$, $v = \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial y}$, show that $u + iv$ is analytic.

$u = \frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial x}$, $v = \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial y}$, show that $u + iv$ is analytic.

$$u = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \quad (\text{Given})$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x^2} \rightarrow \textcircled{1}$$

$$V = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \quad (\text{Given})$$

$$\frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} \quad \left[\because \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \right]$$

From (1) and (2), we get

$$u_x = v_y$$

Similarly, $\frac{\partial u}{\partial y} = \left(\frac{\partial^2 \phi}{\partial y^2} - \mu \frac{\partial^2 \psi}{\partial y \partial x} \right) \Rightarrow \textcircled{3}$

$$\left((V_z) \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) (\text{Given})_{(y+x)} =$$

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x \partial y} \\ &= -\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial y \partial x} \quad \left[\because \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \right] \\ &= -\left[\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial x \partial y} \right] \rightarrow \text{---} \end{aligned}$$

From (3) and (4), we get

$$u_y = -v_x$$

∴ C-R eqns are satisfied and also u_x, u_y, v_x, v_y are continuous. Hence $u+iv$ is analytic. $0 = \frac{v_x}{-y^2} + \frac{v_y}{x^2}$

$x^V \neq y^V$ bzw. $y^V \neq x^V$ top sw., (7) bzw. (2), (4), (1) nicht
 (zusammen mit) regulär für ein v bzw. u .

Construction of Analytic Functions.

Milne-Thomson method:

(i) To find $f(z)$, when u is given

$$f(z) = \int \phi_1(z, 0) dz - i \int \phi_2(z, 0) dz + c$$

where $\phi_1(z, 0) = \left(\frac{\partial u}{\partial x}\right)(z, 0)$, $\phi_2(z, 0) = \left(\frac{\partial u}{\partial y}\right)(z, 0)$, c is a complex constant.

(ii) To find $f(z)$, when v is given

$$f(z) = \int \phi_1(z, 0) dz + i \int \phi_2(z, 0) dz + c$$

where $\phi_1(z, 0) = \left(\frac{\partial v}{\partial y}\right)(z, 0)$, $\phi_2(z, 0) = \left(\frac{\partial v}{\partial x}\right)(z, 0)$, c is a complex constant.

Harmonic Function:

Any function which has continuous second order partial derivatives and which satisfies Laplace's equation is called Harmonic function.

If 'u' is harmonic then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Two harmonic functions u and v which are such that $u+iv$ is an analytic function are called conjugate harmonic functions.

Note: $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$
 $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

1) Show that the function $u = \frac{1}{2} \log(x^2 + y^2)$ is harmonic and determine its conjugate. Also find $f(z)$.

Soln: $u = \frac{1}{2} \log(x^2 + y^2)$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0$$

Hence u satisfies Laplace equation;

$\therefore u$ is harmonic.

To find conjugate of u (i.e. v)

$$\text{We know that } dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$\begin{aligned} &= -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\ &= \frac{-y dx + x dy}{x^2 + y^2} = \frac{x dy - y dx}{x^2 + y^2} = \frac{1}{x^2 + y^2} d\left(\frac{y}{x}\right) \end{aligned}$$

$$\int dv = \int \frac{d(y/x)}{1 + (y/x)^2} = \tan^{-1}\left(\frac{y}{x}\right)$$

$\therefore$ The required analytic function is

$$\begin{aligned} f(z) = u + iv &= \frac{1}{2} \log(x^2 + y^2) + i \tan^{-1}\left(\frac{y}{x}\right) \\ &= \log(x + iy) \\ &= \log z \end{aligned}$$

2) Find the analytic function $f(z) = u + iv$ whose real part u is $\frac{x}{x^2 + y^2}$

$$\text{Soln: } u = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\phi_1(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2} \cdot \frac{y}{x} = \frac{y^2 - x^2}{x(x^2 + y^2)^2}$$

$$\phi_1(z, 0) = \frac{-z^2}{z^4} = -\frac{1}{z^2}$$

$$\phi_2(x, y) = \frac{\partial u(x, y)}{\partial y} = \frac{0 - x \cdot 2y}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$(i) \quad \phi_2(x, 0) = \frac{\partial u(x, 0)}{\partial y} = 0$$

By Milne's method, $f'(z) = \int \phi_1(z, 0) dz - i \int \phi_2(z, 0) dz$
 $= \int -\frac{1}{z^2} dz$
 $= \frac{1}{z} + C$

3) If $u = \frac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x}$, find $w = f(z)$ such that $f(z)$ is

analytic (or) Determine the analytic function whose real part is $\sin 2x$

Soln: Given: $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = \frac{2(\cosh 2y - \cos 2x)(\cos 2x) - 2 \sin 2x(\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2 \cos 2x \cosh 2y - 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2(\cos 2x \cosh 2y - 1)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(x, 0) = \frac{2(\cos 2x - 1)}{(1 - \cos 2x)^2} \rightarrow \textcircled{1}$$

$$\phi_2(x, y) = \frac{\partial u}{\partial y} = \frac{0 - \sin 2x(2 \sinh 2y)}{(\cosh 2y - \cos 2x)^2} = \frac{-2 \sinh 2y \sin 2x}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(x, 0) = 0 \rightarrow \textcircled{2}$$

By Milne's Thomson method, we have

$$f(z) = \int \phi_1(z, 0) dz - i \int \phi_2(z, 0) dz$$

$$= \int \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2} dz$$

$$\begin{aligned}
 &= \int \frac{-2}{1 - \cos 2z} dz = \frac{1}{1 - \cos 2z} \\
 &= - \int \frac{1}{\sin^2 z} \quad \left(\because 1 - \cos 2x = 2 \sin^2 x \right) \\
 &= - \int \sec^2 z = -\cot z + c \\
 \therefore f(z) &= \cot z + c, \quad c \text{ is a constant.}
 \end{aligned}$$

Conformal mappings:

1) A mapping $w = f(z)$ is said to be conformal at $z = z_0$ if $f'(z_0) \neq 0$.

2) A point z_0 at which the mapping $w = f(z)$ is not conformal i.e. $f'(z) = 0$ is called a critical point of the mapping.

If the transformation $w = f(z)$ is conformal at a point, the inverse transformation $z = f^{-1}(w)$ is also conformal at the corresponding point.

The critical points of $z = f^{-1}(w)$ are given by $\frac{dz}{dw} = 0$.

Hence the critical points of the transformation $w = f(z)$ are given by $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$.

Conformal mapping:

A transformation that preserves angles between every pair of curves through a point, both in magnitude and sense, is said to be conformal at that point.

Isogonal:

A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be an isogonal at that point.

problems based on $w = 1/z$.

1) Find the image of $|z - 2i| = 2$ under the transformation $w = 1/z$.

Soln: Given: $|z - 2i| = 2 \rightarrow \textcircled{1}$ is a circle

Centre = $2i$ (i.e.) $(0, 2)$

radius = 2

Given: $w = 1/z \Rightarrow z = 1/w$

$$\textcircled{1} \Rightarrow |1/w - 2i| = 2$$

$$\Rightarrow |1 - 2wi| = 2|w|$$

$$\Rightarrow |1 - 2(u+iv)i| = 2|u+iv|$$

$$\Rightarrow |1 - 2ui + 2v| = 2\sqrt{u^2 + v^2}$$

$$\Rightarrow |1 + 2v - 2ui| = 2\sqrt{u^2 + v^2}$$

$$\Rightarrow \sqrt{(1+2v)^2 + (-2u)^2} = 2\sqrt{u^2 + v^2}$$

$$\Rightarrow (1+2v)^2 + 4u^2 = 4(u^2 + v^2)$$

$$\Rightarrow 1 + 4v^2 + 4v + 4u^2 = 4u^2 + 4v^2$$

$$\Rightarrow 1 + 4v = 0$$

$$\Rightarrow v = -1/4$$

which is a straight line in w -plane

2) Find the image of the circle $|z - 1| = 1$ in the complex plane under the mapping $w = 1/z$.

Soln: Given: $|z - 1| = 1 \rightarrow \textcircled{1}$ is a circle

Centre = 1 , (i.e.) $(1, 0)$

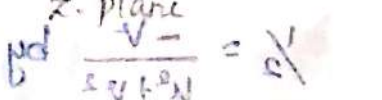
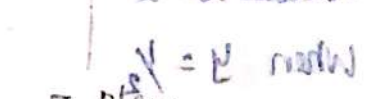
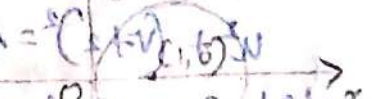
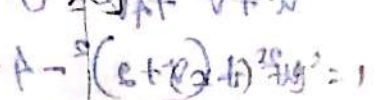
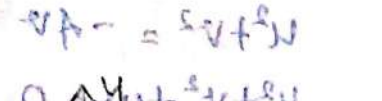
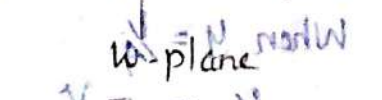
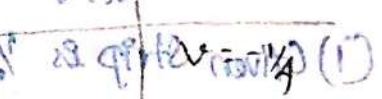
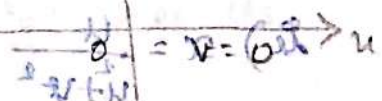
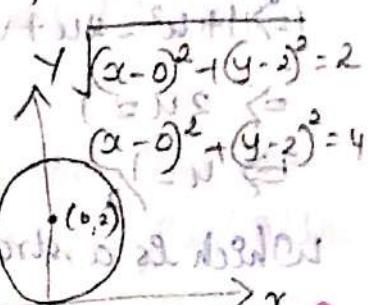
Radius = 1

Given: $w = 1/z \Rightarrow z = 1/w$

$$\textcircled{1} \Rightarrow |1/w - 1| = 1$$

$$\Rightarrow |1 - w| = |w|$$

$$\Rightarrow |1 - (u+iv)| = |u+iv|$$



$$\Rightarrow |1-u-iv| = |u-iv|$$

$$\Rightarrow \sqrt{(1-u)^2 + (-v)^2} = \sqrt{u^2 + v^2}$$

$$\Rightarrow (1-u)^2 + v^2 = u^2 + v^2$$

$$\Rightarrow 1 + u^2 - 2u + v^2 = u^2 + v^2$$

$$\Rightarrow 2u = 1$$

$$\Rightarrow u = \frac{1}{2}$$

Which is a straight line in the w -plane.

3) Find the image of the infinite strips (i) $\frac{1}{4} < y < \frac{1}{2}$
(ii) $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$.

Soln: $w = \frac{1}{z}$ (Given)

$$\text{e) } z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u-iv}{u^2+v^2} = \left[\frac{u}{u^2+v^2} \right] + i \left[\frac{-v}{u^2+v^2} \right]$$

$$\text{e) } x = \frac{u}{u^2+v^2} \rightarrow \textcircled{1}, y = \frac{-v}{u^2+v^2} \rightarrow \textcircled{2}$$

(i) Given Strip is $\frac{1}{4} < y < \frac{1}{2}$

When $y = \frac{1}{4}$

$$\frac{1}{4} = \frac{-v}{u^2+v^2} \text{ by } \textcircled{2}$$

$$u^2+v^2 = -4v$$

$$u^2+v^2+4v = 0$$

$$u^2 + (v+2)^2 - 4 = 0$$

$$u^2 + (v+2)^2 = 4 \rightarrow \textcircled{3}$$

which is a circle whose centre is $(0, -2)$ in the w -plane and radius is 2.

When $y = \frac{1}{2}$

$$\frac{1}{2} = \frac{-v}{u^2+v^2} \text{ by } \textcircled{2}$$

$$u^2 + v^2 = -2v$$

$$u^2 + v^2 + 2v = 0$$

$$u^2 + (v+1)^2 - 1 = 0$$

$$u^2 + (v+1)^2 = 1 \rightarrow (4)$$

which is a circle whose centre is $(0, -1)$ in the w -plane and unit radius.

Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region in between circles $u^2 + (v+1)^2 = 1$ and $u^2 + (v+2)^2 = 4$ in the w -plane.

(ii) Given strip is $0 < y < \frac{1}{2}$

When $y = 0 \Rightarrow v = 0$ by (2)

When $y = \frac{1}{2}$, we get $u^2 + (v+1)^2 = 1$ by (4)

Hence the infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2 + (v+1)^2 = 1$ in the lower half of the w -plane

4) Find the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$.

Soln: Given: $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$$x + iy = \frac{1}{Re^{i\theta}}$$

$$x + iy = \frac{1}{R} e^{-i\theta} = \frac{1}{R} [\cos \theta - i \sin \theta]$$

$$x = \frac{1}{R} \cos \theta, y = -\frac{1}{R} \sin \theta$$

$$\text{Given: } x^2 - y^2 = 1$$

$$\Rightarrow \left(\frac{1}{R} \cos \theta\right)^2 - \left(-\frac{1}{R} \sin \theta\right)^2 = 1$$

$$\frac{\cos^2 \theta - \sin^2 \theta}{R^2} = 1$$

$$\cos 2\theta = R^2$$

i.e) $R^2 = \cos 2\theta$, which is Lemniscate

5) Find the critical points on the map $w^2 = (z-\alpha)(z-\beta)$

Soln: Given: $w^2 = (z-\alpha)(z-\beta) \rightarrow (1)$

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0 \rightarrow (2)$

Differentiating (1) w.r. to z , we get

$$2w \frac{dw}{dz} = (z-\alpha)(1) + (z-\beta)(1)$$

$$= z-\alpha + z-\beta$$

$$2w \frac{dw}{dz} = 2z - (\alpha + \beta) \rightarrow (3)$$

Sub (2) in (3), we get

$$2z - (\alpha + \beta) = 0$$

$$2z = \alpha + \beta$$

$$z = \frac{\alpha + \beta}{2} \rightarrow (4)$$

From (3), $\frac{dz}{dw} = \frac{2w}{2z - (\alpha + \beta)} \rightarrow (5)$

Sub (2) in (5), we get

$$\frac{2w}{2z - (\alpha + \beta)} = 0$$

$$w = 0$$

$$\Rightarrow (z-\alpha)(z-\beta) = 0$$

$$z-\alpha = 0,$$

$$z-\beta = 0$$

$$z = \alpha, \beta$$

$\therefore$ The critical points are $\frac{\alpha+\beta}{2}$, α and β .

Bilinear transformation:

$$w = \frac{az+b}{cz+d} \rightarrow (1)$$

where a, b, c and d are complex constants and $ad-bc \neq 0$

Is known as bilinear transformations.

Differentiating (1), we get

$$\frac{dw}{dz} = \frac{(cz+d)a + (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

If $ad-bc \neq 0$, then $\frac{dw}{dz} \neq 0$

The constant $ad-bc$ is called the determinant of the transformation. The transformation (1) is said to be normalised

If $ad-bc=1$

From (1), we get $w(cz+d) = az+b$

$$cwz + dw - az - b = 0 \rightarrow (2)$$

Eqn (2) is linear both in w and z .

$\therefore$ It is called bilinear transformation.

Definition:

Under the transformation $w = f(z)$ if the image of z is itself, then the point is called the fixed point or an invariant point of the transformation. Thus a fixed point of the transformation $w = f(z)$ is $z = f(z)$.

1) Find the invariant points of the transformation $w = \frac{(2z+4i)}{iz+1}$

Soln:

The invariant points of the transformation is given by

$$z = \frac{-(2z+4i)(1-i) - (0-i)}{iz+1} = \frac{(0-i)(i-w)}{iz+1}$$

$$z(iz+1) = -(2z+4i) \frac{(1-i)}{(1-i)} = \frac{i-w}{1-i}$$

$$\Rightarrow iz^2 + z + 2z + 4i = 0 \Rightarrow iz^2 + 3z + 4i = 0$$

$$\Rightarrow z^2 - 3iz + 4 = 0$$

$$\Rightarrow (z-4i)(z+i) = 0$$

$\therefore z = 4i, -i$ are the invariant points.

$$\frac{z_1}{1+z_1} = \frac{i-w}{w} \therefore$$

To find bilinear transformation which maps points into three specified distinct points

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} \rightarrow (1)$$

(1) is the required bilinear transformation.

Definition:

The cross-ratio of the four points z_1, z_2, z_3, z_4 in the z -plane is defined by

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

1) Find the bilinear transformation which maps the points $z = 0, i, -1$ into $w = i, 1, 0$ respectively.

Soln: Given points $z_1 = 0, z_2 = i, z_3 = -1$

and $w_1 = i, w_2 = 1, w_3 = 0$

The bilinear transformation is

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} \rightarrow (2)$$

Sub (1) in (2), we get

$$\frac{(w - i)(1 - 0)}{(i - 1)(0 - w)} = \frac{(z - 0)(-1 + i)}{(0 - i)(-1 - z)}$$

$$\frac{w - i}{(i - 1)w} = \frac{z}{z + 1} \cdot \frac{(i - 1)}{1}$$

$$\frac{w - i}{w} = \frac{z}{z + 1} (i - 1)$$

$$\frac{w - i}{w} = \frac{z}{z + 1} (i + 1 - 1 + i)$$

$$\therefore \frac{w - i}{w} = \frac{2z}{z + 1}$$

$$(w-i)(z+1) = 2zw$$

$$zw - 2zw + w - i - iz = 0$$

$$w - wz - i(1+z) = 0$$

$$w(1-z) = i(1+z)$$

$$w = \frac{i(1+z)}{1-z} = \frac{1}{i} \left(\frac{z+1}{z-1} \right)$$

2) Find the bilinear transformation which maps the points $z_1 = -1, z_2 = 0, z_3 = 1$ into the points $w_1 = 0, w_2 = i, w_3 = 3i$ respectively.

Soln: Let the bilinear transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \quad \text{--- (1)}$$

$$\text{Given } z_1 = -1, z_2 = 0, z_3 = 1 \quad w_1 = 0, w_2 = i, w_3 = 3i \quad \text{--- (2)}$$

Sub (2) in (1), we get

$$\frac{(w-0)(i-3i)}{(0-i)(3i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\frac{w(-2i)}{-i(3i-w)} = \frac{z+1}{1-z}$$

$$\frac{-2iw}{(w-3i)i} = -\left(\frac{z+1}{z-1} \right)$$

$$\frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$2w(z-1) = (z+1)(w-3i)$$

$$= 2wz - 3iz + w - 3i$$

$$2w(z-1) = w(z+1) - 3i(z+1)$$

$$w[2(z-1) - (z+1)] = -3i(z+1)$$

$$w = \frac{-3i(z+1)}{z-3}$$

which is the required bilinear transformation.

Complex Integration

Cauchy's Integral theorem:

(1) If $f(z)$ is analytic and $f'(z)$ is continuous inside and on a simple closed curve C , then $\int_C f(z) dz = 0$.

Connected Region:

A connected region is one in which any two points in it can be connected by a curve which lies entirely within the region.

Simply Connected Region:

A curve which does not cross itself is called a simple closed curve.

A region in which every closed curve in it encloses points of the region only is called a simply connected region.

Contour Integral:

An integral along a simple closed curve is called a contour integral.

Cauchy's Integral formula:

If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and if a is any point within C , then $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$, the integration around C being taken in the positive direction.

Cauchy's Integral formula for derivative:

If a function $f(z)$ is analytic within and on a simple closed curve C and a is any point lying in it, then $f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$.

1) Evaluate $\int_C \frac{z}{z-2} dz$ where C is the circle $|z|=1$.

Soln:

$$\text{Let } f(z) = \frac{z}{z-2}$$

Here $z=2$ lies outside C

$\therefore f(z)$ is analytic inside and on C .

$f'(z)$ is continuous inside C .

Hence by Cauchy's Integral theorem $\int_C f(z) dz = 0$.

2) Evaluate $\int_C e^{1/z} dz$ where C is $|z|=1$

Soln:

$$\text{Given: } \int_C e^{1/z} dz$$

Here $f(z) = e^{1/z}$ which is analytic in $|z|=1$

Here C is a closed curve

$\therefore$ By Cauchy's Integral theorem, $\int_C e^{1/z} dz = 0$

Problems based on Cauchy's Integral formula:

1) Evaluate $\int_C \frac{z}{z-2} dz$, where C is $|z|=3$.

Soln:

We know that Cauchy Integral formula is

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz = f(a)$$

$$\Rightarrow \int_C \frac{f(z)}{z-a} dz = 2\pi i f(a) \quad \text{--- (1)}$$

Given: $f(z) = z$, $a = 2$, C is $|z|=3$

$a=2$ lies inside $|z|=3$

$$\therefore f(a) = f(2) = 2$$

$$\therefore \text{--- (1)} \Rightarrow \int_C \frac{z}{z-2} dz = 2\pi i (2) = 4\pi i$$

2) Evaluate using Cauchy's Integral formula,

$\int_C \frac{z+1}{(z-3)(z-1)} dz$ where C is the circle $|z|=2$.

Soln:

$$\frac{z+1}{(z-3)(z-1)} = \frac{A}{z-3} + \frac{B}{z-1}$$

$$z+1 = A(z-1) + B(z+3)$$

Put $z=1$, $2 = -2B \Rightarrow B = -1$

Put $z=3$, $4 = 2A \Rightarrow A = 2$

$$\therefore \frac{z+1}{(z-3)(z-1)} = \frac{2}{z-3} - \frac{1}{z-1}$$

$$\therefore \int_C \frac{z+1}{(z-3)(z-1)} dz = \int_{|z|=2} \left(\frac{2}{z-3} - \frac{1}{z-1} \right) dz$$

$$= 2 \int_{|z|=2} \frac{1}{z-3} dz - \int_{|z|=2} \frac{1}{z-1} dz$$

In the first integral, $a=3$

In the second integral, $a=1$

Since $a=3$ lies outside the circle $|z|=2$, $\int_{|z|=2} \frac{1}{z-3} dz = 0$

$\therefore$ By Cauchy's Integral theorem, the 1st integral is zero.

$$\therefore \int_C \frac{z+1}{(z-3)(z-1)} dz = 0 - \int_{|z|=2} \frac{1}{z-1} dz$$

$$= -2\pi i f(1)$$

$$= -2\pi i$$

3) Evaluate, using Cauchy's Integral formula $\frac{1}{2\pi i} \int_C \frac{z^2+5}{z-3} dz$

on the circle (i) $|z|=4$ (ii) $|z|=1$.

Soln:

i) Cauchy's Integral formula is

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz = f(a)$$

$$f(z) = z^2+5, a=3$$

$$\therefore \frac{1}{2\pi i} \int_C \frac{z^2+5}{z-3} dz = f(3) = 3^2+5 = 14$$

(i) The point $a = 3$ lies outside the circle $|z| = 1$

∴ By Cauchy's Integral theorem, we get: $\frac{1}{2\pi i} \int_C \frac{z^2 + 5}{z - 3} dz = 0$.

f) Evaluate $\int_C \frac{dz}{ze^z}$ where C is $|z| = 1$

Soln: W. K. + Cauchy's Integral formula is

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Given: $\int_C \frac{dz}{ze^z} = \int_C \frac{e^{-z} dz}{z-0}$

Here $f(z) = e^{-z}$

$a = 0$ lies inside $|z| = 1$

By Cauchy's Integral formula, we get

$$\frac{1}{2\pi i} \int_C \frac{e^{-z}}{z} dz = \frac{2\pi i f(0)}{2\pi i} = f(0) = e^{-0} = 1$$

g) Evaluate $\int_C \frac{1}{2z+3} dz$ where C is $|z| = 2$.

Soln:

W. K. + Cauchy's Integral formula is $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$

Given: $\int_C \frac{1}{2z+3} dz = \int_C \frac{1}{2(z + 3/2)} dz = \frac{1}{2} \int_C \frac{1}{z + 3/2} dz$

Here $f(z) = 1$, $a = -3/2$.

$a = -3/2 = -1.5$ lies inside $|z| = 2$.

∴ $\frac{1}{2} \int_C \frac{1}{z + 3/2} dz = \frac{1}{2} \cdot 2\pi i f(-3/2) = \frac{1}{2} \cdot 2\pi i (1) = \pi i$

h) Evaluate $\int_C \frac{z}{(z-1)^3} dz$ where C is $|z| = 2$, using Cauchy's

Integral formula.

Soln: Given: $\int_C \frac{z dz}{(z-1)^3}$

Here $f(z) = z$, $a = 1$ lies inside $|z| = 2$

By Cauchy's Integral formula, we get

$$\int_C \frac{z}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1)$$

$f(z) = z$, $f'(z) = 1$, $f''(z) = 0$

7) Evaluate $\int_C \frac{z dz}{(z-1)(z-2)^2}$, where C is $|z-2| = \frac{1}{2}$ by using Cauchy's integral formula.

Soln: Cauchy's integral formula is $\int_C \frac{f(z)}{(z-a)^2} dz = \frac{2\pi i}{1!} f'(a)$

Given: $|z-2| = \frac{1}{2}$.

Centre, radius = $\frac{1}{2}$

$z=1$ lies outside $|z-2| = \frac{1}{2}$

$z=2$ lies inside $|z-2| = \frac{1}{2}$

$$\therefore \int_C \frac{z dz}{(z-1)(z-2)^2} = \int_C \frac{z/z-1}{(z-2)^2} dz$$

Here $f(z) = \frac{z}{z-1}$, $f'(z) = \frac{(z-1)(1) - z(1)}{(z-1)^2} = \frac{z-1-z}{(z-1)^2} = \frac{-1}{(z-1)^2}$

$$\therefore \int_C \frac{f(z)}{(z-2)^2} dz = \frac{2\pi i}{1!} f'(2) = 2\pi i \left[\frac{-1}{(2-1)^2} \right] = -2\pi i$$

8) Evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z|=3$.

Soln:

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

Put $z=1$, $A = -1$

Put $z=2$, $B = 1$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\therefore \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = \int_C \left(\frac{\cos \pi z^2}{z-2} - \frac{\cos \pi z^2}{z-1} \right) dz$$

in the first integral

Here $f(z) = \cos \pi z^2$, $a=2$

$a=2$ lies inside $|z|=3$

$$\therefore \int_{|z|=3} \frac{\cos \pi z^2}{z^2} dz = 2\pi i f(a) = 2\pi i \cos \pi (a^2) = 2\pi i \cos 4\pi = 2\pi i \cos 4\pi$$

In the second integral, $a = 1$

Here $a = 1$ lies inside $|z| = 3$

$$\therefore \int_{|z|=3} \frac{\cos \pi z^2}{z-1} dz = 2\pi i f(a) = 2\pi i \cos \pi a^2 = 2\pi i \cos \pi (1) = 2\pi i \cos \pi$$

$$\therefore \textcircled{1} \text{ becomes } \int_{|z|=3} \frac{\cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i [\cos 4\pi - \cos \pi] = 2\pi i [1 - (-1)] = 4\pi i$$

8) Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is $|z| = 3$

$$\text{Soln: } \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

put $z = 2$, we get $B = 1$, put $z = 1$, we get $A = -1$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\therefore \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{-(\sin \pi z^2 + \cos \pi z^2)}{z-1} dz + \int_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2} dz$$

$$= \int_{|z|=3} \frac{-(\sin \pi z^2 + \cos \pi z^2)}{z-1} dz + \int_{|z|=3} \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2} dz$$

$$= 2\pi i [-\sin \pi (1) - \cos \pi (1) + \sin \pi (4) + \cos \pi (4)] = 2\pi i [0 - (-1) + 0 + 1] = 4\pi i$$

9) Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is $|z+1+i| = 2$

Soln: Given: $|z+1+i| = 2$

i.e. $|z - [-(1+i)]| = 2$ is a circle, whose centre is $-(1+i)$

and radius is 2.

$$z^2 + 2z + 5 = [z - (-1+2i)][z - (-1-2i)]$$

$-1+2i$ i.e. $(-1, 2)$ lies outside C

$-1-2i$ i.e. $(-1, -2)$ lies outside C

$$\int_C \frac{z+4}{z^2+2z+5} dz = \int_C \frac{z+4}{z - (-1-2i)} dz$$

Here $f(z) = \frac{z+4}{z - (-1+2i)}$ is analytic inside C $\therefore$

Hence, by Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a), \text{ we get}$$

$$\begin{aligned} \int_C \frac{z+4}{z^2+2z+5} dz &= 2\pi i f(-1-2i) = 2\pi i \left[\frac{-1-2i+4}{(-1-2i) - (-1+2i)} \right] \\ &= 2\pi i \left[\frac{3-2i}{-1-2i+1-2i} \right] = 2\pi i \left[\frac{3-2i}{-4i} \right] \\ &= -\pi i (3-2i) \end{aligned}$$

10) Evaluate $\int_C \frac{z+1}{z^3-2z^2} dz$, where C is the unit circle $|z| \leq 1$.

Soln:

$$\frac{z+1}{z^3-2z^2} = \frac{z+1}{z^2(z-2)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-2}$$

$$z+1 = A z(z-2) + B(z-2) + C z^2$$

Put $z=0$, $-2B=1 \Rightarrow B = -1/2$

put $z=2$, $A \cdot C = 3 \Rightarrow C = 3/4$

Equating coefficient of z^2 , we get

$$A+C=0 \Rightarrow A = -3/4$$

$$\frac{z+1}{z^3-2z^2} = \frac{-3/4}{z} - \frac{1/2}{z^2} + \frac{3/4}{z-2}$$

$$\therefore \int_C \frac{z+1}{z^3-2z^2} dz = \int_C \left(\frac{-3/4}{z} - \frac{1/2}{z^2} + \frac{3/4}{z-2} \right) dz$$

$$(PH) = -3/4 \int_C \frac{1}{z} dz - 1/2 \int_C \frac{1}{z^2} dz + 3/4 \int_C \frac{1}{z-2} dz$$

$$= -3/4 \cdot 2\pi i f(0) - 1/2 \cdot 2\pi i f'(0) + 0$$

$$= -3/4 \pi i - \pi i \cdot 0 + 0$$

$$= -3/4 \pi i$$

Taylor Series

A function $f(z)$ is analytic inside a circle C with center at a , can be expanded in a series,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \dots$$

$$1 = \frac{1}{(z-a)^0} = \frac{(1)!}{0!} + \frac{f^{(n)}(a)}{n!}(z-a)^n + \dots \text{ to } \infty \rightarrow \textcircled{1}$$

which is convergent at every point in C

Taylor's series about $z=0$ is

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \dots + \dots +$$

$$\frac{f^{(n)}(0)}{n!}z^n + \dots \text{ to } \infty$$

1) Expand e^z in a Taylor's series about $z=0$

Soln: $f(z) = e^z$ Function

$$f(z) = e^z \dots \dots \dots f(0) = e^0 = 1$$

$$f'(z) = e^z \dots \dots \dots f'(0) = e^0 = 1$$

$$f''(z) = e^z \dots \dots \dots f''(0) = e^0 = 1$$

$$f'''(z) = e^z \dots \dots \dots f'''(0) = e^0 = 1$$

Taylor's series about $z=0$ is

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \dots$$

$$= 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots$$

$$= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

2) Expand $\frac{1}{z-2}$ at $z=1$ in a Taylor's series.

Soln: Function

$$f(z) = \frac{1}{z-2} \quad \text{Value at } z=1 \quad f(1) = \frac{1}{1-2} = -1$$

$$f'(z) = \frac{-1}{(z-2)^2} \quad f'(1) = \frac{-1}{(1-2)^2} = -1$$

$$f''(z) = \frac{2}{(z-2)^3} \quad f''(1) = \frac{2}{(1-2)^3} = -2$$

$$f'''(z) = \frac{-6}{(z-2)^4} \quad f'''(1) = \frac{-6}{(1-2)^4} = -6$$

Taylor's series about $z=a$ is

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \dots$$

$$f(z) = -1 + (-1)(z-1) + \frac{-2}{2!}(z-1)^2 + \frac{-6}{3!}(z-1)^3 + \dots$$

$$= -1 - (z-1) - (z-1)^2 - (z-1)^3 - \dots$$

3) Expand $f(z) = \log(1+z)$ as a Taylor's series about $z=0$ if $|z| < 1$.

Soln: Function

$$f(z) = \log(1+z) \quad f(0) = \log 1 = 0$$

$$f'(z) = \frac{1}{1+z} \quad f'(0) = \frac{1}{1+0} = 1$$

$$f''(z) = \frac{-1}{(1+z)^2} \quad f''(0) = \frac{-1}{(1+0)^2} = -1$$

$$f'''(z) = \frac{2}{(1+z)^3} \quad f'''(0) = \frac{2}{(1+0)^3} = 2$$

∴ Taylor's series about $z=0$ is

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \dots$$

$$= 0 + \frac{1}{1!}z + \frac{-1}{2!}z^2 + \frac{2}{3!}z^3 + \dots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots \quad |z| < 1$$

4) Expand $f(z) = \cos z$ about $z = \pi/3$ in Taylor's series

Soln:

Function

Value at $z = \pi/3$

$$f(z) = \cos z$$

$$f(\pi/3) = \cos \pi/3 = 1/2$$

$$f'(z) = -\sin z$$

$$f'(\pi/3) = -\sin \pi/3 = -\sqrt{3}/2$$

$$f''(z) = -\cos z$$

$$f''(\pi/3) = -\cos \pi/3 = -1/2$$

$$f'''(z) = \sin z$$

$$f'''(\pi/3) = \sin \pi/3 = \sqrt{3}/2$$

Taylor's series about $z = a$ is

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \dots$$

$$\begin{aligned} f(z) &= f(\pi/3) + \frac{f'(\pi/3)}{1!}(z-\pi/3) + \frac{f''(\pi/3)}{2!}(z-\pi/3)^2 + \frac{f'''(\pi/3)}{3!}(z-\pi/3)^3 + \dots \\ &= \frac{1}{2} + \frac{(-\sqrt{3}/2)}{1!}(z-\pi/3) + \frac{(-1/2)}{2!}(z-\pi/3)^2 + \frac{\sqrt{3}/2}{3!}(z-\pi/3)^3 + \dots \\ &= \frac{1}{2} - \frac{\sqrt{3}}{2}(z-\pi/3) - \frac{1}{4}(z-\pi/3)^2 + \frac{\sqrt{3}}{12}(z-\pi/3)^3 + \dots \end{aligned}$$

5) Obtain Taylor's series to represent the function $\frac{z^2-1}{(z+2)(z+3)}$

In the region $|z| < 2$.

Soln:

$$\text{Let } f(z) = \frac{z^2-1}{(z+2)(z+3)}$$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{z^2+5z+6}{(z+2)(z+3)} = \frac{(z+2)(z+3) - 5z-7}{(z+2)(z+3)} = 1 + \frac{-5z-7}{(z+2)(z+3)}$$

$$\frac{-5z-7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$-5z-7 = A(z+3) + B(z+2)$$

Put $z = -2$, we get Put $z = -3$, we get

$$10 - 7 = A(1) + B(0) \quad 15 - 7 = A(0) + B(-1)$$

$$A = 3$$

$$8 = B(-1)$$

$$B = -8$$

$$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

Given: $|z| < 2$ & $|z/2| < 1 \therefore |z/3| < 1$

$$\therefore f(z) = 1 + \frac{3}{2\left(\frac{z}{2} + 1\right)} - \frac{8}{3\left(1 + \frac{z}{3}\right)} = 1 + \frac{3}{2\left(1 + \frac{z}{2}\right)} - \frac{8}{3\left(1 + \frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2} \left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3} \left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2} \left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \dots \right] - \frac{8}{3} \left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \dots \right]$$

$$= 1 + \frac{3}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n - \frac{8}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

$$= 1 + \sum_{n=0}^{\infty} (-1)^n \left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}} \right] z^n$$

Laurent's Series

Let C_1, C_2 be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$. Let $f(z)$ be analytic on C_1 and C_2 and in the annular region R between them. Then, for any point z in R , $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$

$$\text{where } a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz + 1 = \frac{1 - R_2^n}{(1-R_2)(1-R_2^n)}$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{1-n}} dz + \frac{1}{1-R_2} = \frac{R_2^n - 1}{(1-R_2)(1-R_2^n)}$$

the integrals being taken anticlockwise.

problems based on Laurent's Series: internal and. 2017 (8)

1) Expand $f(z) = \frac{z+1}{z(z-1)}$ as Laurent's series in powers of z and

State the respective region of validity.

Soln: Given: $f(z) = \frac{1}{z(z-1)}$

$f(z)$ is not analytic at $z=0$ and $z=1$

Now the function $f(z)$ is not analytic at $z=0$ and $z=1$ but it is

analytic in the regions 1) $0 < |z| < 1$
2) $|z| > 1$

Case(i): For all z in the deleted disc $0 < |z| < 1$ that is in the disc $|z| < 1$ from which $z=0$ is deleted. So in this annulus,

$$\begin{aligned} \text{we have } \frac{1}{z(z-1)(1-z)} &= \frac{-1}{z+1} \frac{1}{(1-z)} = \frac{1-z-1}{(1+z)(1-z)} \\ &= \frac{-1}{z} \left[1 + (z+1) + (z+1)^2 + \dots \right] \frac{1}{(1-z)} \\ &= - \left[\frac{1}{z} + 1 + z + z^2 + z^3 + \dots \right] \frac{1}{(1-z)} \end{aligned}$$

The region of validity of this expansion is $0 < |z| < 1$.

Case(ii): In the region $|z| > 1$, we have $|1/z| < 1$.

$$\begin{aligned} \therefore f(z) &= \frac{1}{z} \frac{1}{z-1} = \frac{1}{z} \frac{1}{z(1-1/z)} = \frac{1}{z^2} \left(1 - \frac{1}{z} \right)^{-1} \\ &= \frac{1}{z^2} \left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right] = \frac{(1-z^{-1})^{-1}}{(1+z)(1-z)} \\ &= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots \end{aligned}$$

The region of validity of this expansion is $|z| > 1$.

$$\frac{1}{(1+z)(1-z)} = \frac{1}{(1+z)} \frac{1}{(1-z)} = \frac{1}{(1+z)} \left(1 + z + z^2 + \dots \right)$$

2) Find the Laurent's series of $f(z) = \frac{z^2-1}{z^2+5z+6}$ valid in the region $2 < |z| < 3$

Soln:

$$f(z) = \frac{z^2-1}{z^2+5z+6}$$

$$\frac{z^2+5z+6}{z^2+5z+6} \cdot \frac{z^2-1}{z^2+5z+6} = \frac{1}{-5z-7}$$

$$\therefore \frac{z^2-1}{z^2+5z+6} = 1 + \frac{(-5z-7)}{z^2+5z+6}$$

$$= 1 + \frac{-5z-7}{(z+3)(z+2)}$$

$$\frac{-5z-7}{(z+3)(z+2)} = \frac{A}{z+3} + \frac{B}{z+2} = \frac{A(z+2) + B(z+3)}{(z+3)(z+2)}$$

$$\therefore -5z-7 = A(z+2) + B(z+3)$$

put $z = -2$

$$10-7 = B(1)$$

$$B = 3$$

$$\text{put } z = -3$$

$$15-7 = A(-1)$$

$$A = -8$$

$$\therefore \frac{-5z-7}{(z+3)(z+2)} = \frac{-8}{z+3} + \frac{3}{z+2}$$

$$\therefore 1 + \frac{(-5z-7)}{(z+3)(z+2)} = 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

We have $2 < |z| < 3$

$$2 < |z|, |z| < 3$$

$$\frac{2}{|z|} < 1, |z/3| < 1$$

$$\therefore 1 + \frac{(-5z-7)}{(z+3)(z+2)} = 1 - \frac{8}{3(1+z/3)} + \frac{3}{2(z/2+1)}$$



$$\begin{aligned}
 &= 1 - \frac{8}{3} \left(1 + \frac{z}{3} \right)^{-1} + \frac{2}{z} \left(1 + \frac{3}{z} \right)^{-1} \\
 &= 1 - \frac{8}{3} \left[1 - \frac{z}{3} + \left(\frac{z}{3} \right)^2 - \dots \right] + \frac{2}{z} \left[1 - \frac{3}{z} + \left(\frac{3}{z} \right)^2 - \dots \right] \\
 &= 1 - \frac{8}{3} \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{3^n} + \frac{2}{z} \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{z^n}
 \end{aligned}$$

3) Evaluate $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent's series valid for the regions $|z| > 3$ and $|z| < 3$.

Soln:

Given: $f(z) = \frac{1}{(z+1)(z+3)} = \frac{-A}{z+1} + \frac{B}{z+3} = \frac{A(z+3) + B(z+1)}{(z+1)(z+3)}$

$\therefore 1 = A(z+3) + B(z+1)$

Put $z = -3$, we get, Put $z = -1$, we get

$$1 = -2B$$

$$B = -\frac{1}{2}$$

$$1 = 2A$$

$$A = \frac{1}{2}$$

$$\therefore \frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

i) Given $|z| > 3$

$(-3) < |z|$

$$\left| \frac{3}{z} \right| < 1 \Rightarrow \left| \frac{3}{z} \right| < 1$$

$$\therefore \textcircled{2} \Rightarrow \frac{1}{(z+1)(z+3)} = \frac{1}{2z(1+\frac{1}{z})} - \frac{1}{2z(1+\frac{3}{z})}$$

$$= \frac{1}{2z} \left(1 + \frac{1}{z} \right)^{-1} - \frac{1}{2z} \left(1 + \frac{3}{z} \right)^{-1}$$

$$= \frac{1}{2z} \left[1 - \left(\frac{1}{z} \right) + \left(\frac{1}{z} \right)^2 - \dots \right] - \frac{1}{2z} \left[1 - \frac{3}{z} + \left(\frac{3}{z} \right)^2 - \dots \right]$$

$$= \frac{1}{2z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z} \right)^n - \frac{1}{2z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{z} \right)^n$$

$$= \frac{1}{2z} \left[\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z} \right)^n - \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{z} \right)^n \right]$$

(ii) Given region is $1 < |z| < 3$

$$\Rightarrow 1 < |z| \text{ and } |z| < 3$$

$$\text{Let } \left| \frac{1}{z} \right| < 1, \left| \frac{z}{3} \right| < 1$$

$$\begin{aligned} \Rightarrow f(z) &= \frac{1}{2} \cdot \frac{1}{z(1+\frac{1}{z})} - \frac{1}{2(3)} \cdot \frac{1}{(1+\frac{z}{3})} \\ &= \frac{1}{2z} \left(1 + \frac{1}{z} \right)^{-1} - \frac{1}{6} \left(1 + \frac{z}{3} \right)^{-1} \\ &= \frac{1}{2z} \left[1 - \frac{1}{z} + \left(\frac{1}{z} \right)^2 - \dots \right] - \frac{1}{6} \left[1 - \frac{z}{3} + \left(\frac{z}{3} \right)^2 - \dots \right] \end{aligned}$$

$$= \frac{1}{2z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z} \right)^n - \frac{1}{6} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3} \right)^n$$

A) Find the Laurent's expansion of $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ Valid in $1 < |z+1| < 3$.

Soln: Given: $f(z) = \frac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z-2 = A(z-2)(z+1) + B(z)(z+1) + C(z)(z-2)$$

Put $z=0$, we get

$$-2 = A(-2)(1)$$

$$A = 1$$

Put $z = -1$, we get

$$-7-2 = C(-1)(-3)$$

$$-9 = 3C$$

$$C = -3$$

Put $z=2$, we get

$$14-2 = B(2)(3)$$

$$12 = 6B$$

$$B = 2$$

$$\therefore \frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Given region is $1 < |z+1| < 3$

$$\text{Let } u = z+1$$

$$z = u-1$$

$$1 < |u| < 3$$

$$\Rightarrow \left| \frac{1}{u} \right| < 1, \left| \frac{u}{3} \right| < 1$$

$$\therefore \Rightarrow f(z) = \left[\frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u} \right] \frac{1}{u}$$

$$\begin{aligned}
 &= \frac{1}{u(1-\frac{1}{u})} + \frac{2}{-3(1-\frac{1}{u})} - \frac{3}{u} \\
 &= \frac{1}{u} (1-\frac{1}{u})^{-1} + \frac{2}{-3} (1-\frac{1}{u})^{-1} - \frac{3}{u} \\
 &= \frac{1}{u} \left[1 + \frac{1}{u} + \left(\frac{1}{u}\right)^2 + \dots \right] - \frac{2}{3} \left[1 + \frac{1}{u} + \left(\frac{1}{u}\right)^2 + \dots \right] - \frac{3}{u} \\
 &= \frac{1}{z+1} \left[1 + \frac{1}{z+1} + \left(\frac{1}{z+1}\right)^2 + \dots \right] - \frac{2}{3} \left[1 + \frac{1}{z+1} + \left(\frac{1}{z+1}\right)^2 + \dots \right] - \frac{3}{z+1} \\
 &= \frac{1}{z+1} \sum_{n=0}^{\infty} \left(\frac{1}{z+1}\right)^n - \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{1}{z+1}\right)^n - \frac{3}{z+1}
 \end{aligned}$$

Singularities (z): point at which a function is not analytic. A point z=a is called a singularity of f(z) if f(z) is not analytic at z=a.

Zero of order m: An analytic function f(z) is said to have a zero of order m if f(z) is expressed as $(z-a)^m \phi(z)$ where $\phi(z)$ is analytic and $\phi(a) \neq 0$.

(i) When m=1, a is called a simple zero.

Problem: 1) Find the zeros of $f(z) = \frac{z^2+1}{1-z^2}$.

Soln: The zeros of f(z) are given by $f(z) = 0$

$$\frac{z^2+1}{1-z^2} = 0 \Rightarrow z^2+1=0$$

$$\Rightarrow z^2 = -1 \Rightarrow z = \pm \sqrt{-1} = \pm i$$

$\therefore$ The zeros of f(z) are $z = \pm i$.

2) Find the zeros of $\frac{z^3-1}{z^3+1}$

Soln: The zeros of f(z) are given by $f(z) = 0$

$$\frac{z^3-1}{z^3+1} = 0 \Rightarrow z^3-1=0 \Rightarrow z^3 = 1$$

$$z = (1)^{1/3} = 1, \omega, \omega^2$$

3) Find the zeros of $f(z) = \sin\left(\frac{1}{z-a}\right)$.

Soln: The zeros of $f(z)$ are given by $f(z) = 0$

$$\sin\left(\frac{1}{z-a}\right) = 0$$

$$\Rightarrow \frac{1}{z-a} = n\pi$$

$$z-a = \frac{1}{n\pi}$$

$$z = a + \frac{1}{n\pi}$$

The zeros are $z = a + \frac{1}{n\pi}$, $n = \pm 1, \pm 2, \dots$

Singular points:

A point $z=a$ at which a function $f(z)$ fails to be analytic is called a singular point (or) singularity of $f(z)$.

(eg) Let $f(z) = \frac{1}{z-2}$. Here $z=2$ is a singular point.

Types of Singularity:

Isolated Singularity:

A point $z=a$ is said to be isolated singularity of $f(z)$ if

i) $f(z)$ is not analytic at $z=a$.

ii) There exists a neighbourhood of $z=a$ containing no other singularity.

Removable singularity:

A point $z=a$ is called a removable singularity of $f(z)$ if

i) $z=a$ is a singular point.

ii) $\lim_{z \rightarrow a} f(z)$ exists.

1) What is the nature of the singularity $z=0$ of the function $f(z) = \frac{\sin z - z}{z^3}$.

Soln: Given: $f(z) = \frac{\sin z - z}{z^3}$

The function $f(z)$ is not defined at $z=0$.

But by L-Hopital's rule

$$\lim_{z \rightarrow 0} \frac{\sin z - z}{z^3} = \lim_{z \rightarrow 0} \frac{\cos z - 1}{3z^2} = \lim_{z \rightarrow 0} \frac{-\sin z}{6z} = \lim_{z \rightarrow 0} \frac{-\cos z}{6} = -\frac{1}{6}$$

Since, the limit exist and is finite, the singularity at $z=0$ is a removable singularity.

2) Find the nature of singularity of $f(z) = \frac{1}{(z-1)^2}$

Soln: Let $f(z) = \frac{1}{(z-1)^2}$

Here $z=1$ is a singular point. There is no other singular point.

In the neighbourhood of $z=1$.

Hence $z=1$ is an isolated singularity.

Poles: $\left[(z-1)^n f(z) \right]_{z=1}$

If we can find a positive integer n such that

it $(z-a)^n f(z) \neq 0$, then $z=a$ is called a pole of order n of $f(z)$.

1) Find the poles of $f(z) = \frac{1}{(z-5)^3(z-4)^2}$

Here $z=5$ is a pole of order 3.

Since $\lim_{z \rightarrow 5} (z-5)^3 f(z) = \lim_{z \rightarrow 5} \frac{1}{(z-4)^2} = \frac{1}{(5-4)^2} = 1 \neq 0$

Similarly $z=4$ is a pole of order 2.

Since $\lim_{z \rightarrow 4} (z-4)^2 f(z) = \lim_{z \rightarrow 4} \frac{1}{(z-5)^3} = \frac{1}{(4-5)^3} = -1 \neq 0$

Essential singularity:

A point $z=a$ is an essential singularity if

- i) $z=a$ is a singular point (eg) identify the type of singularity
- ii) $z=a$ should not be a pole of the function $f(z)$

Meromorphic function:

A function $f(z)$ is called meromorphic function, if it is analytic anywhere except at a finite number of poles. Example: $f(z) = \frac{1}{(z-1)(z-2)}$ is a meromorphic function as it has only two simple poles $z=1$ and $z=2$.

Evaluation of Residues:

- 1) Residue at a simple pole (pole of order 1) is given by

$$[\text{Res } f(z)]_{z=a} = \lim_{z \rightarrow a} (z-a) f(z)$$

- 2) Residue at a pole of order m is given by

$$[\text{Res } f(z)]_{z=a} = \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$$

- 1) Calculate the residue of $f(z) = \frac{1-e^{2z}}{z^3}$

Soln: Given $f(z) = \frac{1-e^{2z}}{z^3}$

Here $z=0$ is a pole of order 3.

$$[\text{Res } f(z)]_{z=0} = \lim_{z \rightarrow 0} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z-0)^3 \frac{1-e^{2z}}{z^3} \right]$$

$$= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} [1-e^{2z}]$$

$$= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d}{dz} [-2e^{2z}] = \frac{1}{2} \lim_{z \rightarrow 0} [-4e^{2z}]$$

$$= \frac{1}{2} \cdot [-4] = -2$$

$$0 + 1 = \frac{1}{e^{(1-2)}} \lim_{A \rightarrow \infty} \frac{1}{s(A-2)^2(1-2)} s(A-2) \lim_{A \rightarrow \infty} s(A-2) \lim_{A \rightarrow \infty} s(A-2)$$

2) Find the residue of $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ at each of the poles.

Soln:

$$[\text{Res } f(z)]_{z=1} = \lim_{z \rightarrow 1} \frac{d}{dz} [(z-1)^2 f(z)] = \lim_{z \rightarrow 1} \frac{d}{dz} \left[\frac{z^2}{(z+2)} \right]$$

$$= \lim_{z \rightarrow 1} \frac{1}{1!} \frac{d}{dz} \left[\frac{z^2}{z+2} \right] = \lim_{z \rightarrow 1} \frac{2z(z+2) - z^2}{(z+2)^2}$$

$$= \lim_{z \rightarrow 1} \frac{z^2 + 4z}{(z+2)^2} = \frac{1+4}{3^2} = \frac{5}{9}$$

$$[\text{Res } f(z)]_{z=-2} = \lim_{z \rightarrow -2} (z+2) f(z) = \lim_{z \rightarrow -2} \frac{z^2}{(z-1)^2}$$

$$= \lim_{z \rightarrow -2} \frac{z^2}{(z-1)^2} = \frac{4}{9}$$

Cauchy's Residue theorem:

If $f(z)$ is analytic at all points inside and on a simple closed curve C , except at a finite number of points $z_1, z_2, \dots, z_n$ within C , then $\int_C f(z) dz = 2\pi i \times \text{sum of the residues of } f(z) \text{ at } z_1, z_2, \dots, z_n$.

1) If C is the circle $|z| = 3$, then evaluate $\int_C \frac{\cos \pi z^2 + i \sin \pi z^2}{(z+1)(z+2)} dz$

Soln:

$$\text{Let } f(z) = \frac{\cos \pi z^2 + i \sin \pi z^2}{(z+1)(z+2)}$$

$$(z+1)(z+2) = 0 \Rightarrow z = -1, -2$$

$z = -2$ lies inside $|z| = 3$

$$[\text{Res } f(z)]_{z=-1} = \lim_{z \rightarrow -1} (z+1) f(z) = \lim_{z \rightarrow -1} \frac{\cos \pi z^2 + i \sin \pi z^2}{(z+2)}$$

$$= \frac{\cos \pi + i \sin \pi}{1} = \frac{-1 + i0}{1} = -1$$

$$\left[\text{Res } f(z) \right]_{z=-2} = \lim_{z \rightarrow -2} (z+2) f(z) = \lim_{z \rightarrow -2} \frac{2(z+2) \cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)}$$

$$= \lim_{z \rightarrow -2} \frac{\cos \pi z^2 + \sin \pi z^2}{z+1} = \frac{\cos 4\pi + \sin 4\pi}{-1} = \frac{1+0}{-1} = -1$$

$$\frac{\cos 4\pi + \sin 4\pi}{-1} = \frac{1+0}{-1} = -1$$

By Cauchy's residue theorem,

$$\int_C f(z) dz = 2\pi i [\text{sum of the residues}]$$

$$= 2\pi i [-1-1] = -4\pi i$$

2) Evaluate $\int_C \frac{dz}{z \sin z}$ where C is $|z|=1$ using Cauchy's

residue theorem.

Soln: Let $f(z) = \frac{1}{z \sin z}$

The singularity of $f(z)$ is given by

$$z \sin z = 0$$

$$z \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right] = 0$$

$$z^2 \left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \right] = 0$$

$z=0$ is a pole of order 2.

$z=0$ lies inside C .

$$\left[\text{Res } f(z) \right]_{z=0} = \lim_{z \rightarrow 0} \frac{1}{1!} \frac{d}{dz} \left(\frac{1}{z \sin z} \right)$$

$$= \lim_{z \rightarrow 0} \frac{1}{1!} \frac{d}{dz} \left(\frac{z^2}{z \sin z} \right) = \lim_{z \rightarrow 0} \frac{d}{dz} \left(\frac{z}{\sin z} \right)$$

$$= \lim_{z \rightarrow 0} \left[\frac{\sin z(1) - z \cos z}{\sin^2 z} \right]$$

$$= \frac{0-0}{0} \Rightarrow \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{z \rightarrow 0} \frac{\cos z - [z(-\sin z) + \cos z(1)]}{2z \sin z \cos z}$$

$$= \frac{0+1}{0} = \infty$$

$$\lim_{z \rightarrow 0} \frac{\cos z + z \sin z - \cos z}{z \sin z \cos z} = \lim_{z \rightarrow 0} \frac{z \sin z}{z \sin z \cos z} = \lim_{z \rightarrow 0} \frac{1}{\cos z} = 1$$

$$\lim_{z \rightarrow 0} \frac{z}{2 \cos z} = \frac{0}{2} = 0$$

By Cauchy's residue theorem, $\int f(z) dz = 2\pi i f(0) = 0$.

Evaluation of real definite integrals by contour integration.

Contour Integration:

The complex integration along the curve used in evaluating the definite integral is called contour integration.

Type 1:

Integrals of the form $\int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta$, where f is a rational function in $\cos \theta$ and $\sin \theta$.

$$\text{Put } z = e^{i\theta}, dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{1}{iz} dz$$

$$\cos \theta = \frac{1}{2} \left[z + \frac{1}{z} \right], \sin \theta = \frac{1}{2i} \left[\frac{z^2 - 1}{z} \right]$$

$$\sin \theta = \frac{1}{2i} \left[\frac{z^2 - 1}{z} \right], \cos \theta = \frac{1}{2} \left[\frac{z^2 + 1}{z} \right]$$

1) Evaluate $\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta}$ by contour integration.

Soln:

$$\text{Let } z = e^{i\theta}, dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{1}{iz} dz$$

$$\cos \theta = \frac{1}{2} \left[\frac{z^2 + 1}{z} \right], \sin \theta = \frac{1}{2i} \left[\frac{z^2 - 1}{z} \right]$$

$$\text{Given: } \int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} = \int_C \frac{1}{2 + \frac{1}{2} \left[\frac{z^2 + 1}{z} \right]} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{2z}{4z + z^2 + 1} \cdot \frac{dz}{iz} = \int_C \frac{2}{z^2 + 4z + 1} dz$$

$$= \frac{2}{i} \int_C \frac{dz}{z^2 + 4z + 1} \rightarrow \text{P}$$

$$z^2 + 4z + 1 = 0$$

$$\Rightarrow z = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm \sqrt{12}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

Let $\alpha = -2 + \sqrt{3} \approx -2 + 1.7321 = -0.2679$ is a simple pole, where lies inside C

$\beta = -2 - \sqrt{3} \approx -2 - 1.7321 = -3.7321$ is a simple pole, where lies outside C .

$$\therefore \int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} = \frac{2}{i} \int_C \frac{dz}{(z-\alpha)(z-\beta)} \rightarrow (2)$$

Here $f(z) = \frac{1}{(z-\alpha)(z-\beta)}$

$$[\text{Res } f(z)]_{z=\alpha} = \lim_{z \rightarrow \alpha} (z-\alpha) \frac{1}{(z-\alpha)(z-\beta)} = \lim_{z \rightarrow \alpha} \frac{1}{(z-\beta)}$$

$$= \frac{1}{\alpha - \beta} = \frac{1}{(-2 + \sqrt{3}) - (-2 - \sqrt{3})} = \frac{1}{2\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

Hence, by Cauchy's residue theorem

$$\int_C f(z) dz = \int_C \frac{dz}{(z-\alpha)(z-\beta)} = 2\pi i [\text{sum of the residues}]$$

$$(2) \Rightarrow \int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} = \frac{2}{i} [2\pi i] \times \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

2) Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4 \cos \theta} d\theta$

Soln: Let $z = e^{i\theta}$, $dz = i e^{i\theta} d\theta \Rightarrow d\theta = \frac{1}{iz} dz$

$$z^2 = (e^{i\theta})^2 = e^{2i\theta} = \cos 2\theta + i \sin 2\theta$$

$$\cos \theta = \frac{1}{2} \left[z + \frac{1}{z} \right] = \frac{1}{2} \left[\frac{z^2 + 1}{z} \right]$$

$$\sin \theta = \frac{1}{2i} \left[z - \frac{1}{z} \right] = \frac{1}{2i} \left[\frac{z^2 - 1}{z} \right]$$

$$\text{Real part of } z^2 = \cos 2\theta$$

Given: $\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta = R \cdot P \int_C \frac{z^2}{5-4\left(\frac{z^2+1}{2z}\right)} \frac{1}{iz} dz$, where C is the unit circle $|z|=1$.

$$= R \cdot P \frac{1}{i} \int_C \frac{z}{5-2\left(\frac{z^2+1}{z}\right)} dz$$

$$= R \cdot P \frac{1}{i} \int_C \frac{z^2}{5z-2z^2-2} dz = R \cdot P \frac{-1}{i} \int_C \frac{z^2}{2z^2-5z+2} dz$$

$$\therefore \text{①} \Rightarrow R \cdot P \frac{-1}{2i} \int_C \frac{z^2}{z^2-5/2z+1} dz \rightarrow \text{①}$$

$$z^2 - 5/2z + 1 = 0$$

$$z = \frac{5/2 \pm \sqrt{(5/2)^2 - 4}}{2} = \frac{5/2 \pm \sqrt{9/4}}{2} = \frac{5/2 \pm 3/2}{2}$$

$$= \frac{1}{4} [5 \pm 3] = \frac{1}{4} (8), \frac{1}{4} (2) = 2, \frac{1}{2}$$

Let $\alpha = 2$ is a simple pole lies outside C

$\beta = 1/2$ is a simple pole lies inside C .

$$\therefore \text{①} \Rightarrow R \cdot P \left(\frac{-1}{2i} \right) \int_C \frac{z^2}{(z-\alpha)(z-\beta)} dz \rightarrow \text{②}$$

Here $f(z) = \frac{z^2}{(z-\alpha)(z-\beta)}$

$$\left[\text{Res } f(z) \right]_{z=\beta} = \lim_{z \rightarrow \beta} (z-\beta) \frac{z^2}{(z-\alpha)(z-\beta)} = \frac{\beta^2}{\beta-\alpha} = \frac{(1/2)^2}{1/2-2} = \frac{1/4}{-3/2} = -\frac{1}{6}$$

Hence by Cauchy's Residue theorem

$$\int_C f(z) dz = 2\pi i \left[\text{sum of the residues} \right] = 2\pi i \left[-\frac{1}{6} \right] = -\frac{\pi i}{3}$$

$$\therefore \text{②} \Rightarrow R \cdot P \frac{-1}{2i} \left(-\frac{\pi i}{3} \right) = R \cdot P \left(\frac{\pi}{6} \right) = \frac{\pi}{6}$$

Type - II Integral of the type $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$

1) Show that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$

Soln:

Consider $\int_C f(z) dz = \int_C \frac{dz}{(z^2+1)^3}$ where C is the upper half of the semi-circle with the boundary diameter $[-R, R]$.

Now, $\int_C f(z) dz = \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz$

The poles of $f(z)$ are the solutions of $(z^2+1)^3 = 0$.

$z = i$ is a pole of order 3 lies inside Γ .

$z = -i$ is a pole of order 3 lies outside Γ .

$\left[\text{Res}(f(z)) \right]_{z=i} = \lim_{z \rightarrow i} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z-i)^3 f(z) \right]$

$= \lim_{z \rightarrow i} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z-i)^3 \frac{1}{(z-i)^3 (z+i)^3} \right]$

$= \lim_{z \rightarrow i} \frac{1}{2} \frac{d^2}{dz^2} \left(\frac{1}{(z+i)^3} \right)$

$= \lim_{z \rightarrow i} \frac{1}{2} \frac{d}{dz} \left[\frac{-3}{(z+i)^4} \right] = \lim_{z \rightarrow i} \frac{1}{2} \frac{12}{(z+i)^5}$

$= \frac{1}{2} \frac{12}{(2i)^5} = \frac{6}{2^5 i^5} = \frac{3}{16i}$

By Cauchy's residue theorem

$\int_C f(z) dz = 2\pi i [\text{sum of the residues}]$

$= 2\pi i \left[\frac{3}{16i} \right] = \frac{3\pi}{8}$

As $R \rightarrow \infty$, $|z| \rightarrow \infty$, $\int_{\Gamma} f(z) dz \rightarrow 0$

$\therefore \int_{-\infty}^{\infty} f(x) dx = \frac{3\pi}{8}$
 $\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$

2) Prove that $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}, a>0, b>0.$

Soln:

Consider $\int_C f(z) dz = \int_C \frac{z^2 dz}{(z^2+a^2)(z^2+b^2)}$

where C is the upper half of the semi-circle Γ with the bounding diameter $[-R, R]$.

By Cauchy's residue theorem, we have

$$\int_C f(z) dz = \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz$$

The poles of $f(z)$ are the soln of $(z^2+a^2)(z^2+b^2)=0$

$$z^2+a^2=0$$

$$z^2+b^2=0$$

$$z = \pm ai$$

$$z = \pm bi$$

$z = ai$ is a simple pole lies inside Γ

$z = bi$ is a simple pole lies inside Γ

$z = -ai$ " " outside Γ

$$z = -bi$$

Res $f(z)$ at $z = ai$ = $\lim_{z \rightarrow ai} (z - ai) f(z)$

$$= \lim_{z \rightarrow ai} (z - ai) \frac{z^2}{(z - ai)(z + ai)(z^2 + b^2)}$$

$$= \lim_{z \rightarrow ai} \frac{z^2}{(z + ai)(z^2 + b^2)} = \frac{a^2}{(2ai)(-a^2 + b^2)} = \frac{-a^2}{2ai(-a^2 + b^2)}$$

Res $f(z)$ at $z = bi$ = $\lim_{z \rightarrow bi} (z - bi) f(z)$

$$= \lim_{z \rightarrow bi} (z - bi) \frac{z^2}{(z^2 + a^2)(z - bi)(z + bi)}$$

$$= \lim_{z \rightarrow bi} \frac{z^2}{(z^2 + a^2)(z + bi)} = \frac{b^2}{(b^2 + a^2)(2bi)}$$

$$= \frac{-b^2}{(2bi)(-b^2+a^2)} \cdot \frac{-b}{(2i)(a^2-b^2)} \quad \text{Soln:}$$

Hence, by Cauchy's residue theorem

$$\int_C f(z) dz = 2\pi i [\text{sum of the residues}]$$

$$= 2\pi i \left[\frac{a}{2i(a^2-b^2)} + \frac{-b}{2i(a^2-b^2)} \right]$$

$$= \frac{2\pi i}{2i(a^2-b^2)} (a-b) = \frac{\pi}{(a^2-b^2)} (a-b) = \frac{\pi}{a+b}$$

$$\text{e) } \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz = \frac{\pi}{a+b}$$

If $R \rightarrow \infty$, then $\int_{\Gamma} f(z) dz \rightarrow 0$

$$\text{e) } \int_{-\infty}^{\infty} f(x) dx = \frac{\pi}{a+b}$$

$$\text{e) } \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$$

Type - III

Integrals of the form $\int_{-\infty}^{\infty} f(x) \cos mx dx$ or $\int_{-\infty}^{\infty} f(x) \sin mx dx$ where $f(x) \rightarrow 0$ as $x \rightarrow \infty$.

1) Evaluate $\int_0^{\infty} \frac{\cos ax}{(x^2+1)(x^2+4)} dx$, $a > 0$

Soln:

$$2 \int_0^{\infty} \frac{\cos ax}{x^2+1} dx = \int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+1)(x^2+4)} dx$$

$$\int_0^{\infty} \frac{\cos ax}{(x^2+1)(x^2+4)} dx = \frac{1}{3} \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} dx - \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+4} dx$$

To find: $\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} dx$

Consider $\int_C f(z) dz = \int_{\Gamma} \frac{\cos az}{(z^2+1)(z^2+4)} dz$

$$= R-P \int_C \frac{e^{iaz}}{(z^2+1)(z^2+4)} dz$$

where c is the upper half of the semi-circle Γ with the bounding diameter $[-R, R]$.

By Cauchy's Residue theorem, we have

$$\int_c f(z) dz = \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz$$

The poles of $f(z)$ are the roots of $z^2 + 1 = 0$
 $z = \pm i$

$z = i$ is a simple pole lies inside Γ

$z = -i$ is a simple pole lies outside Γ

$$\therefore \left[\text{Res. } f(z) \right]_{z=i} = \lim_{z \rightarrow i} (z-i) f(z)$$

$$= \lim_{z \rightarrow i} (z-i) \frac{e^{iaz}}{z^2+1} = 0/0$$

$$= \lim_{z \rightarrow i} \frac{(z-i) a e^{iaz} + e^{iaz}}{2z}$$

$$= \frac{0 + e^{-a}}{2i} = \frac{e^{-a}}{2i}$$

$$\begin{aligned} \text{Hence, } \int_c f(z) dz &= R \cdot p \cdot 2\pi i [\text{sum of the residues}] \\ &= R \cdot p \cdot 2\pi i \left[\frac{e^{-a}}{2i} \right] \cdot 1 \\ &= R \cdot p \cdot \pi e^{-a} = \pi e^{-a} \cdot \pi R \end{aligned}$$

$$\text{ie) } \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz = \pi e^{-a}$$

If $R \rightarrow \infty$, then $\int_{\Gamma} f(z) dz \rightarrow 0$

$$\therefore \int_{-\infty}^{\infty} f(x) dx = \pi e^{-a} \quad \text{ie) } \int_0^{\infty} \frac{\cos ax}{x^2+1} dx = \frac{\pi}{2} e^{-a}$$

2) Show that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

$$\text{Soln: } 2 \int_0^{\infty} \frac{\sin x}{x} dx = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx \Rightarrow \int_0^{\infty} \frac{\sin x}{x} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx \rightarrow \text{D}$$

To find $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$

Consider, $\int_C f(z) dz = \int_C \frac{\sin z}{z} dz = \frac{1}{i} \cdot P \int_C \frac{e^{iz}}{z} dz$

where C is the upper half of the semi-circle Γ with the bounding diameter $[-R, R]$.

a) $\int_C f(z) dz = \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz$

The poles of $f(z)$ are the solns of $z = 0$

$z = 0$ is a simple pole lies on the real axis inside Γ .

$\therefore [Res f(z)]_{z=0} = \lim_{z \rightarrow 0} (z-0) f(z) = \lim_{z \rightarrow 0} (z) \frac{e^{iz}}{z} = \lim_{z \rightarrow 0} e^{iz}$

Hence by Cauchy residue theorem $\int_C \frac{e^{iz}}{z} dz = 2\pi i \cdot [Res f(z)]_{z=0} = 2\pi i \cdot 1 = 2\pi i$

$\int_C f(z) dz = \int_{-R}^R f(x) dx + \int_{\Gamma} f(z) dz$

$\int_C \frac{e^{iz}}{z} dz =$

$\int_C f(z) dz = \frac{1}{i} \cdot \text{sum of the residues}$

$= \frac{1}{i} \cdot P [2\pi i] \left[\frac{1}{1} \right] = 2\pi$

$= 2\pi \cdot 1 = 2\pi$

$\therefore \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2\pi$

$0 \leftarrow \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$

$\therefore \frac{1}{i} \cdot \frac{2\pi i}{1} = 2\pi$

$\therefore \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2\pi$

$\therefore \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2\pi$

Differential equations.

Higher order linear differential equations with constant coefficients.

The general form of the linear differential equation of second order is $\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$.

Differential operators.

The symbol, $Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dx^2}$, $\frac{1}{D}$ stands for the

operation of integration.

$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$ can be written as

$$D^2y + PDy + Qy = R \text{ (or) } (D^2 + PD + Q)y = R.$$

The complete soln is $y = \text{Complementary function} + \text{particular Integral}$.

To find the complementary functions:

Roots of A.E

C.F

1) Roots are real & distinct

$$Ae^{m_1x} + Be^{m_2x}.$$

$$m_1, m_2 (m_1 \neq m_2)$$

2) Roots are real and equal

$$(Ax+B)e^{mx} \text{ (or) } (A+Bx)e^{mx}.$$

$$m_1 = m_2 = m \text{ (say)}$$

3) Roots are imaginary

$$e^{\alpha x} (A \cos \beta x + B \sin \beta x).$$

$$\alpha \pm i\beta$$

To find the particular integral:

$$P \cdot I = \frac{1}{f(D)} X$$

1) $X = e^{ax}$

$$P \cdot I = \frac{1}{f(D)} e^{ax} = e^{ax} \frac{1}{f(a)}, f(a) \neq 0$$

$$= \alpha e^{ax} \frac{1}{f'(a)}, f(a) = 0, f'(a) \neq 0$$

$$= \alpha^2 e^{ax} \frac{1}{f''(a)}, f(a) = 0, f'(a) = 0, f''(a) \neq 0$$

2) $X = x^n$

$$P \cdot I = \frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$$

Expand $[f(D)]^{-1}$ and then operate.

3) $X = \sin ax$ (or) $\cos ax$

$$P \cdot I = \frac{1}{f(D)} [\cos ax \text{ (or) } \sin ax]$$

4) $X = e^{ax} \phi(x)$

Replace D^2 by $-a^2$

$$P \cdot I = \frac{1}{f(D)} e^{ax} \phi(x) = e^{ax} \frac{1}{f(D+a)} \phi(x)$$

Result:

i) $\frac{1}{D-a} \phi(x) = e^{ax} \int e^{-ax} \phi(x) dx$ (ii) $\frac{1}{D+a} \phi(x) = e^{-ax} \int e^{ax} \phi(x) dx$

Problems based on R.H.S of the given differential equation is zero.

Example:

1) Solve $(D^2 - 5D + 6)y = 0$

The auxiliary equation is $m^2 - 5m + 6 = 0$

$$m = 2, m = 3$$

$$\therefore C.F = Ae^{2x} + Be^{3x}$$

$\therefore$ The general solution is $y = C.F + P.I$ $y = Ae^{2x} + Be^{3x}$

2) Solve $(D^2 + 6D + 9)y = 0$

The auxiliary equation is $m^2 + 6m + 9 = 0 \Rightarrow m = -3, m = -3$

$$\therefore y = C.F = (Ax + B)e^{-3x}$$

3) Solve $(D^2 + 2D + 1)y = 0$.

The auxiliary eqn is $m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$

$\therefore y = (Ax + B)e^{-x}$

4) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$

Soln: Given: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$

i.e. $(D^2 - 6D + 13)y = 0$

The auxiliary eqn is $m^2 - 6m + 13 = 0$

$\Rightarrow m = 3 \pm 2i$

The solution is $e^{3x} [A \cos(2x) + B \sin(2x)]$

5) Solve $(D^2 + 1)y = 0$, given $y(0) = 0, y'(0) = 1$

Soln: Given $(D^2 + 1)y = 0$

The auxiliary eqn is $m^2 + 1 = 0 \Rightarrow m^2 = -1, m = \pm i$

$y = A \cos x + B \sin x$

$y(x) = A \cos x + B \sin x \rightarrow (1)$

Given: $y(0) = 0 \Rightarrow y(0) = A = 0$

$y'(x) = -A \sin x + B \cos x$

Given: $y'(0) = 1 \Rightarrow y'(0) = B = 1$

$\therefore (1) \Rightarrow y(x) = \sin x$

6) Solve $(D^2 + D + 1)y = 0$

7) Solve $(D^3 - 3D^2 + 3D - 1)y = 0$

The solution is $y = e^x [A + Bx + Cx^2]$

8) Solve $(D^3 + D^2 + 4D + 4)y = 0$

The auxiliary eqn is $m^3 + m^2 + 4m + 4 = 0$

$m = -1, \pm 2i$

$y = Ae^{-x} + e^{0x} [B \cos 2x + C \sin 2x]$

$= Ae^{-x} + B \cos 2x + C \sin 2x$

problems based on $p.I = \frac{1}{f(D)} e^{ax} \Rightarrow$ Replace D by a . 37

1) Solve $(D^2 - 4D + 13)y = e^{2x}$

The auxiliary eqn is $m^2 - 4m + 13 = 0$

$$m = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$\therefore C.F = e^{2x} (A \cos 3x + B \sin 3x)$$

$$p.I = \frac{1}{D^2 - 4D + 13} e^{2x} = \frac{1}{4 - 8 + 13} e^{2x} \quad [\because \text{Replace } D \text{ by } 2]$$

$$= \frac{1}{9} e^{2x}$$

$$\therefore y = C.F + p.I = e^{2x} (A \cos 3x + B \sin 3x) + \frac{1}{9} e^{2x}$$

2) Find the p.I of $(D^2 + 4D + 8)y = e^{2x}$

$$p.I = \frac{1}{D^2 + 4D + 8} e^{2x} = \frac{1}{4 + 8 + 8} e^{2x}$$

$$= \frac{1}{20} e^{2x}$$

3) Solve $y'' + 7y' - 8y = e^{2x}$

$$\text{Ans: } y = e^{2x} + B e^{-8x} + \frac{1}{10} e^{2x}$$

4) Solve $(D-2)^2 y = e^{2x}$

Soln: The auxiliary eqn is $(m-2)^2 = 0$ i.e. $m = 2, 2$

$$\therefore C.F = e^{2x} (A x + B)$$

$$p.I = \frac{1}{(D-2)^2} e^{2x} = \frac{1}{0} e^{2x}$$

$$= x \frac{1}{2(D-2)} e^{2x} = \frac{x}{2} \left[\frac{1}{2-2} \right] e^{2x} = \frac{x}{0} e^{2x}$$

$$= \frac{x}{2} \left[\frac{1}{1} \right] e^{2x} = \frac{x^2}{2} e^{2x}$$

$$\therefore y = C.F + p.I = e^{2x} (A x + B) + \frac{x^2}{2} e^{2x}$$

5) Find the particular integral of $(D^2 - 4D + 4)y = 2^x$. St

Soln: Given: $(D^2 - 4D + 4)y = 2^x$

$$(D-2)^2 y = 2^x = e^{\log 2^x} = e^{x \log 2} = e^{(\log 2)x}$$

$$P.I. = \frac{1}{(D-2)^2} e^{(\log 2)x} = \frac{1}{(\log 2 - 2)^2} e^{(\log 2)x} \quad [\text{Replace } D \text{ by } \log 2]$$

6) Solve $(D^2 + 7D + 12)y = 14e^{-3x}$

Soln: Given: $(D^2 + 7D + 12)y = 14e^{-3x}$

The auxiliary eqn is $m^2 + 7m + 12 = 0$

$$\Rightarrow m = -3, m = -4$$

$$\therefore C.F. = Ae^{-3x} + Be^{-4x}$$

$$P.I. = \frac{1}{D^2 + 7D + 12} 14e^{-3x}$$

$$= 14 \frac{1}{9 - 21 + 12} e^{-3x}$$

$$= \frac{1}{0} e^{-3x}$$

$$= 14x \frac{1}{2D+7} e^{-3x} = 14x \frac{1}{0 - 6 + 7} e^{-3x} = 14xe^{-3x}$$

$$y = C.F. + P.I. = Ae^{-3x} + Be^{-4x} + 14xe^{-3x}$$

7) Solve $(D^2 - 4)y = e^{2x} + e^{-4x}$

The auxiliary eqn is $m^2 - 4 = 0 \Rightarrow (m+2)(m-2) = 0$

$$m = 2, m = -2$$

$$\therefore C.F. = Ae^{2x} + Be^{-2x}$$

$$P.I. = \frac{1}{D^2 - 4} e^{2x}$$

$$= \frac{1}{4 - 4} e^{2x} = \frac{1}{0} e^{2x}$$

$$= x \frac{1}{2D - 4} e^{2x}$$

$$= x \frac{1}{0} e^{2x}$$

$$= x^2 \frac{1}{2} e^{2x}$$

$$= \frac{x^2}{2} e^{2x}$$

$$P.I. = \frac{1}{D^2 - 4} e^{-2x}$$

$$= \frac{1}{4 - 4} e^{-2x} = \frac{1}{0} e^{-2x}$$

$$= x \frac{1}{2D - 4} e^{-2x}$$

$$= x \frac{1}{-8} e^{-2x}$$

$$= -\frac{x}{8} e^{-2x}$$

$\therefore$ The general soln is

$$y = Ae^{2x} + Be^{-2x} + \frac{x^2}{2} e^{2x} - \frac{x}{8} e^{-2x}$$



problems based on $p.I = \frac{1}{f(D)} \sin ax$ or $\frac{1}{f(D)} \cos ax$

$\Rightarrow$ Replace D^2 by $-a^2$.

1) Find the p.I of $(D^2 + 1)y = \sin x$

Soln: Given: $(D^2 + 1)y = \sin x$

$$p.I = \frac{1}{D^2 + 1} \sin x$$

$$= \frac{1}{-1 + 1} \sin x \quad (\text{Replace } D^2 \text{ by } -1^2)$$

$$= x \frac{1}{2D} \sin x = \frac{x}{2} \frac{1}{D} \sin x$$

$$p.I = \frac{x}{2} (-\cos x) = -\frac{x \cos x}{2}$$

2) Find the p.I of $\frac{d^3 y}{dx^3} + 4 \frac{dy}{dx} = \sin 2x$

Soln: $(D^3 + 4D)y = \sin 2x$

$$p.I = \frac{1}{D^3 + 4D} \sin 2x$$

$$= \frac{1}{D(D^2 + 4)} \sin 2x$$

$$= \frac{1}{D(-4 + 4)} \sin 2x \quad [\text{Replace } D^2 \text{ by } -2^2 = -4]$$

$$= x \frac{1}{3D^2 + 4} \sin 2x$$

$$= x \frac{1}{3(-4) + 4} \sin 2x$$

$$= x \frac{1}{-12 + 4} \sin 2x$$

$$= -\frac{x}{8} \sin 2x$$

$$p.I = -\frac{x}{8} \sin 2x$$

$$3) (D^2 - 3D + 2)y = \cos 3x \quad 66$$

Soln:

The A.E is $m^2 - 3m + 2 = 0$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

$$\therefore \text{C.F} = Ae^x + Be^{2x}$$

$$P.I = \frac{1}{D^2 - 3D + 2} \cos 3x$$

$$= \frac{(1 - \frac{1}{D} \frac{d}{dx} \cos 3x)}{-9 - 3D + 2} \cos 3x$$

$$= \frac{1}{-7 - 3D} \cos 3x$$

$$= -\frac{1}{3D + 7} \cos 3x$$

$$= -\frac{(3D - 7)}{(3D + 7)(3D - 7)} \cos 3x$$

$$= \frac{7 - 3D}{9D^2 - 49} \cos 3x$$

$$= \frac{7 - 3D}{9(-9) - 49} \cos 3x$$

$$= \frac{7 \cos 3x + 9 \sin 3x}{-81 - 49}$$

$$= -\frac{1}{130} [7 \cos 3x + 9 \sin 3x]$$

$$y = \text{C.F} + P.I$$

$$= Ae^x + Be^{2x} - \frac{1}{130} [7 \cos 3x + 9 \sin 3x]$$

$$4) \text{ Solve: } (D^2 + 3D + 2)y = \sin 3x \cos 2x$$

Soln:

The A.E is $m^2 + 3m + 2 = 0$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

$$\therefore \text{C.F} = Ae^{-x} + Be^{-2x}$$

$$P.I = \frac{1}{D^2 + 3D + 2} [\sin 3x \cos 2x]$$

$$= \frac{1}{D^2+3D+2} \left[\frac{\sin 5x + \sin x}{2} \right] \quad (\because \sin(A+B) + \sin(A-B) = 2 \sin A \cos B)$$

$$= \frac{1}{2} [P \cdot I_1 + P \cdot I_2]$$

$$P \cdot I_1 = \frac{1}{D^2+3D+2} \sin 5x$$

$$= \frac{1}{-25+3D+2} \sin 5x$$

$$= \frac{1}{3D-23} \sin 5x$$

$$= \frac{3D+23}{(3D-23)(3D+23)} \sin 5x$$

$$= \frac{3D+23}{9D^2-529} \sin 5x$$

$$= \frac{3D+23}{9(-25)-529} \sin 5x$$

$$= \frac{3D+23}{-754} \sin 5x$$

$$= \frac{3D \sin 5x + 23 \sin 5x}{-754}$$

$$= -\left(\frac{15 \cos 5x + 23 \sin 5x}{754} \right)$$

$$P \cdot I_2 = \frac{1}{D^2+3D+2} \sin x$$

$$= \frac{1}{-1+3D+2} \sin x$$

$$= \frac{1}{3D+1} \sin x$$

$$= \frac{3D-1}{9D^2-1} \sin x$$

$$= \frac{3D-1}{-10} \sin x$$

$$= -\frac{1}{10} [3D \sin x - \sin x]$$

$$= -\frac{1}{10} [3 \cos x - \sin x]$$

$$y = A e^{-x} + B e^{-2x} - \frac{1}{754} \cdot \frac{1}{2} (15 \cos 5x + 23 \sin 5x) - \frac{1}{20} (3 \cos x - \sin x).$$

Solve: $(D^2 - 4D + 4)y = e^{2x} + \cos 2x$

Soln: The A.E is $m^2 - 4m + 4 = 0$

$$(m-2)(m-2) = 0$$

$$m = 2, 2.$$

$$\therefore \text{C.F} = (Ax+B)e^{2x}$$

$$P.I_1 = \frac{1}{D^2 - 4D + 4} e^{2x}$$

$$= \frac{1}{4 - 8 + 4} e^{2x}$$

$$= \frac{1}{0} e^{2x}$$

$$= x \frac{1}{2D-4} e^{2x}$$

$$= x \frac{1}{4-4} e^{2x}$$

$$= x \frac{1}{0} e^{2x}$$

$$= x^2 \frac{1}{2} e^{2x}$$

$$P.I_1 = \frac{x^2}{2} e^{2x}$$

$$P.I_2 = \frac{1}{D^2 - 4D + 4} \cos 2x$$

$$= \frac{1}{-2^2 - 4D + 4} \cos 2x$$

$$= \frac{-1}{4D} \cos 2x$$

$$= \frac{-1}{4} \left[\frac{1}{D} \cos 2x \right]$$

$$= \frac{-1}{4} \frac{\sin 2x}{2} = \frac{-\sin 2x}{8}$$

$$\therefore y = \text{C.F} + P.I_1 + P.I_2$$

$$= (Ax+B)e^{2x} + \frac{x^2}{2} e^{2x} - \frac{1}{8} \sin 2x$$

$$\left[\frac{x \sin 2x - x \cos 2x}{2} \right] \frac{1}{2} =$$

$$\left[\frac{x \sin 2x - x \cos 2x}{2} \right] \frac{1}{2} =$$



2) Solve: $(D^3 + 2D^2 + D)y = e^{2x} + \sin x$.

Solution:

A.E. & $m^3 + 2m^2 + m = 0$
 $m(m^2 + 2m + 1) = 0$

$m = 0, m^2 + 2m + 1 = 0$

$\Rightarrow (m+1)(m+1) = 0$

$\Rightarrow m = -1, -1$

$\therefore$ C.F. & $A + (Bx + C)e^{-x}$

P.T.₁ = $\frac{1}{D^3 + 2D^2 + D} e^{2x}$

= $\frac{1}{2^3 + 2(2)^2 + 2} e^{2x}$

= $\frac{1}{8 + 8 + 2} e^{2x} = \frac{e^{2x}}{18}$

P.T.₂ = $\frac{1}{D^3 + 2D^2 + D} \sin x$

= $\frac{1}{-1 \cdot D + 2(-1) + D} \sin x$

= $\frac{1}{-2} \sin x$

$y = \text{C.F.} + \text{P.T.}_1 + \text{P.T.}_2$
 $= A + (Bx + C)e^{-x} + \frac{e^{2x}}{18} - \frac{1}{2} \sin x$

3) Solve: $(D^2 + 4)y = e^x + \sin 2x + \cos 2x$ no boxed method

Solution:

A.E. & $m^2 + 4 = 0$

$m^2 = -4 \Rightarrow m = \pm 2i$

$m = \pm 2i$

$\therefore$ C.F. = $A \cos 2x + B \sin 2x$

P.T.₁ = $\frac{1}{D^2 + 4} e^x$
 $= \frac{1}{(1)^2 + 4} e^x$
 $= \frac{e^x}{5}$

$$P \cdot T_2 = \frac{1}{D^2+4} \sin 2x$$

$$= \frac{1}{-4+4} \sin 2x$$

$$= \frac{x}{2D} \sin 2x$$

$$= \frac{x}{2} \left(\frac{-\cos 2x}{2} \right)$$

$$= \frac{-x \cos 2x}{4}$$

$$P \cdot T_3 = \frac{1}{D^2+4} \cos 2x$$

$$= \frac{+1}{-4+4} \cos 2x$$

$$= \frac{x}{2D} \cos 2x$$

$$= \frac{x}{2} \left(\frac{\sin 2x}{2} \right)$$

$$= \frac{x \sin 2x}{4}$$

$$y = C \cdot F + P \cdot T_1 + P \cdot T_2 + P \cdot T_3$$

$$= A \cos 2x + B \sin 2x + \frac{x^2}{5} + \frac{x \cos 2x}{4} + \frac{x \sin 2x}{4}$$

Problems based on R.H.S = $x^n e^{ax} + \dots$

$$\text{Note: } (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$1) \text{ Solve } \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = x^2 + 3$$

Soln:

$$\text{Given: } \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = x^2 + 3$$

$$(D^2 - 5D + 6)y = x^2 + 3$$

The A.E is $m^2 - 5m + 16 = 0$

$$(m-3)(m-2) = 0$$

$$m = 3, 2$$

$$C.F = Ae^{2x} + Be^{3x}$$

$$P.I = \frac{1}{D^2 - 5D + 16} (x^2 + 3)$$

$$= \frac{1}{6 \left[1 + \frac{D^2 - 5D}{6} \right]} (x^2 + 3)$$

$$= \frac{1}{6} \left[1 + \frac{D^2 - 5D}{6} \right]^{-1} (x^2 + 3)$$

$$= \frac{1}{6} \left[1 - \left(\frac{D^2 - 5D}{6} \right) + \left(\frac{D^2 - 5D}{6} \right)^2 \right] (x^2 + 3)$$

$$= \frac{1}{6} \left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{25}{36} D^2 \right] (x^2 + 3)$$

$$= \frac{1}{6} \left[1 + \frac{5D}{6} + \frac{19}{36} D^2 \right] (x^2 + 3)$$

$$= \frac{1}{6} \left[x^2 + 3 + \frac{5}{6} (2x) + \frac{19}{36} (2) \right]$$

$$= \frac{1}{6} \left[x^2 + 3 + \frac{5}{3} x + \frac{19}{18} \right]$$

$$P.I = \frac{1}{108} [18x^2 + 30x + 73]$$

$$\therefore y = C.F + P.I = Ae^{2x} + Be^{3x} + \frac{1}{108} [18x^2 + 30x + 73]$$

2) Find the Particular Integral of $(D^2 + 2)y = x^2$

$$P.I = \frac{1}{D^2 + 2} x^2$$

$$= \frac{1}{2 \left[1 + \frac{D^2}{2} \right]} x^2$$

$$= \frac{1}{2} \left[1 + \frac{D^2}{2} \right]^{-1} x^2$$

$$= \frac{1}{2} \left[1 - \frac{D^2}{2} + \dots \right] x^2$$

$$= \frac{1}{2} [x^2 - 1]$$

3) Solve: $(D^2 + 5D + 4)y = x^2 + 7x + 9$

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Soln:

The A.E is $m^2 + 5m + 4 = 0$

$$(m+4)(m+1) = 0$$

$$m = -4, -1.$$

$$C.F = Ae^{-4x} + Be^{-x}$$

$$P.T = \frac{1}{D^2 + 5D + 4} (x^2 + 7x + 9)$$

$$= \frac{1}{4 \left[1 + \left(\frac{D^2 + 5D}{4} \right) \right]} (x^2 + 7x + 9)$$

$$= \frac{1}{4} \left[1 + \left(\frac{D^2 + 5D}{4} \right) \right]^{-1} (x^2 + 7x + 9)$$

$$= \frac{1}{4} \left[1 - \left(\frac{D^2 + 5D}{4} \right) + \left(\frac{D^2 + 5D}{4} \right)^2 - \dots \right] (x^2 + 7x + 9)$$

$$= \frac{1}{4} \left[1 - \frac{D^2}{4} - \frac{5D}{4} + \frac{D^4}{16} + \frac{25D^2}{16} + \frac{10D^3}{16} - \dots \right] (x^2 + 7x + 9)$$

$$= \frac{1}{4} \left[x^2 + 7x + 9 - \frac{D^2}{4} (x^2 + 7x + 9) - \frac{5D}{4} (x^2 + 7x + 9) + \frac{25D^2}{16} (x^2 + 7x + 9) \right]$$

$$= \frac{1}{4} \left[x^2 + 7x + 9 - \frac{1}{2} - \frac{10x}{4} - \frac{35}{4} + \frac{50}{16} \right]$$

$$= \frac{1}{4} \left[x^2 + \frac{9x}{2} + \frac{23}{8} \right]$$

$$= \frac{1}{32} [8x^2 + 36x + 23]$$

$$y = C.F + P.T$$

$$= Ae^{-4x} + Be^{-x} + \frac{1}{32} (8x^2 + 36x + 23)$$

1) Solve $(D^2 + 5D + 4)y = 4e^{-x} + x$.

Soln:

The A.E is $m^2 + 5m + 4 = 0$

$$(m+4)(m+1) = 0$$

$$m = -4, -1$$

$$\therefore \text{C.F} = Ae^{-4x} + Be^{-x}$$

$$P.I_1 = \frac{1}{D^2 + 5D + 4} 4e^{-x}$$

$$= \frac{1}{(-1)^2 + 5(-1) + 4} 4e^{-x}$$

$$= \frac{1}{1 - 5 + 4} 4e^{-x} = \frac{1}{0} 4e^{-x}$$

$$= x \cdot \frac{1}{2D + 5} 4e^{-x}$$

$$= x \cdot \frac{1}{2(-1) + 5} 4e^{-x}$$

$$= x \cdot \frac{1}{3} 4e^{-x}$$

$$= \frac{4}{3} x e^{-x}$$

$$P.I_2 = \frac{1}{D^2 + 5D + 4} x = \frac{1}{4} \frac{x}{1 + \left(\frac{D^2 + 5D}{4}\right)}$$

$$= \frac{1}{4} \left[1 + \left(\frac{D^2 + 5D}{4}\right) \right]^{-1} x$$

$$= \frac{1}{4} \left[1 - \left(\frac{D^2 + 5D}{4}\right) + \left(\frac{D^2 + 5D}{4}\right)^2 - \dots \right] x$$

$$= \frac{1}{4} \left[x - \frac{D^2}{4} x - \frac{5D}{4} x + \dots \right]$$

$$= \frac{1}{4} \left[x - 0 - \frac{5}{4} x \right]$$

$$= \frac{1}{4} \left[x - \frac{5x}{4} \right] = \frac{x}{4} - \frac{5}{16}$$

$$\therefore y = Ae^{-4x} + Be^{-x} + \frac{4}{3} x e^{-x} + \frac{x}{4} - \frac{5}{16}$$



problems based on P.H.S. = e^{ax} X.

$$P.I = \frac{1}{f(D)} e^{ax} x = e^{ax} \frac{1}{f(D+a)} x.$$

1) Find the P.I of $(D^2 - 4D)y = x e^x$

$$\begin{aligned} \text{Soln: } P.I &= \frac{1}{(D^2 - 4D)} x e^x \\ &= e^x \frac{1}{(D+1)^2 - 4(D+1)} x \quad (\text{Replace } D \text{ by } D+1) \\ &= e^x \frac{1}{D^2 + 2D + 1 - 4D - 4} x \\ &= e^x \frac{1}{D^2 - 2D - 3} x \\ &= e^x \frac{1}{-3 \left[1 - \left(\frac{D^2 - 2D}{3} \right) \right]} x \\ &= \frac{e^x}{-3} \left[1 - \left(\frac{D^2 - 2D}{3} \right) \right]^{-1} x \\ &= \frac{e^x}{-3} \left[1 + \left(\frac{D^2 - 2D}{3} \right) + \left(\frac{D^2 - 2D}{3} \right)^2 + \dots \right] x \\ &= \frac{e^x}{-3} \left[x - \frac{2}{3} \right]. \end{aligned}$$

2) Find the particular integral of $(D^2 - 4D + 3)y = e^{2x} \cos 2x$.

$$\begin{aligned} \text{Soln: } P.I &= \frac{1}{D^2 - 4D + 3} e^{2x} \cos 2x \\ &= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 3} \cos 2x \\ &= e^{2x} \frac{1}{D^2 + 4D + 4 - 4D - 8 + 3} \cos 2x \\ &= e^{2x} \frac{1}{D^2 - 1} \cos 2x \\ &= \frac{e^{2x}}{-4} \cos 2x \left[1 + \frac{D^2}{4} + \frac{D^4}{16} + \dots \right] \\ &= \frac{-e^{2x}}{5} \cos 2x. \end{aligned}$$

3) Solve $\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4y = e^{-x} \sin 2x$

Soln:

$$\begin{aligned} A.E \text{ is } m^2 + 4m + 4 &= 0 \\ (m+2)(m+2) &= 0 \\ m &= -2, -2 \end{aligned}$$

$$\therefore C.F = (Ax+B)e^{-2x}$$

$$P.I = \frac{1}{D^2+4D+4} e^{-x} \sin 2x$$

$$= e^{-x} \frac{1}{D^2+4D+4} \sin 2x$$

$$= e^{-x} \frac{1}{(D-1)^2+4(D-1)+4} \sin 2x \quad [\text{Replacing } D \text{ by } D-1]$$

$$= e^{-x} \frac{1}{-4+2D+4} \sin 2x \quad [\text{Replacing } D^2 \text{ by } -2^2]$$

$$= e^{-x} \frac{1}{2D-3} \sin 2x$$

$$= e^{-x} \frac{(2D+3)}{(2D-3)(2D+3)} \sin 2x$$

$$= e^{-x} \frac{(2D+3) \sin 2x}{4D^2-9}$$

$$= e^{-x} \frac{2D \sin 2x + 3 \sin 2x}{4(-4)-9}$$

$$= e^{-x} \frac{(4 \cos 2x + 3 \sin 2x)}{-25}$$

$$y = C.F + P.I$$

$$= (Ax+B)e^{-2x} - \frac{1}{25} e^{-x} (4 \cos 2x + 3 \sin 2x)$$

$$\text{Solve } (D^2-4D+13)y = e^{2x} \sin 3x \cdot \frac{1}{5}$$

Soln:

$$\text{The A.E is } m^2-4m+13=0$$

$$m = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$C.F = e^{2x} (C_1 \cos 3x + C_2 \sin 3x)$$

$$P.I = \frac{1}{D^2-4D+13} e^{2x} \sin 3x$$

$$= e^{2x} \frac{1}{(D+2)^2-4(D+2)+13} \sin 3x$$

$$= e^{2x} \frac{1}{D^2+4D+4-4D-8+13} \sin 3x$$

$$= e^{2x} \frac{1}{D^2 + 9} \sin 3x$$

$$= e^{2x} \frac{1}{-9 + 9} \sin 3x$$

$$= e^{2x} \frac{x}{2D} \sin 3x$$

$$= \left[\frac{-x e^{2x}}{2} \right] \left[\frac{\cos 3x}{3} \right]$$

$$= \left[\frac{-x e^{2x} \cos 3x}{6} \right]$$

$$y = C.F + P.I = e^{2x} (C_1 \cos 3x + C_2 \sin 3x) - \frac{x e^{2x} \cos 3x}{6}$$

5) Solve $(D^2 - 4D + 4)y = x^3 e^{2x}$.

Soln:

The A.E is $m^2 - 4m + 4 = 0$

$$m = 2, 2$$

$$C.F = (Ax + B)e^{2x}$$

$$P.I = \frac{1}{D^2 - 4D + 4} x^3 \cdot e^{2x}$$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^3$$

$$= e^{2x} \frac{1}{D^2 + 4D + 4 - 4D - 8 + 4} x^3$$

$$= e^{2x} \frac{1}{D^2} x^3$$

$$= e^{2x} \cdot \frac{x^5}{20}$$

$$y = C.F + P.I = (Ax + B)e^{2x} + \frac{e^{2x} \cdot x^5}{20}$$

6) Solve: $(D^2 - 2D + 1)y = x^2 e^{3x}$.

Ans: $(A + B)e^x + \frac{e^{3x}}{4} \left[x^2 - 2x + \frac{3}{2} \right]$

7) Solve: $(D^2 - 5D + 4)y = e^x \cos 2x$

Ans: $Ae^{2x} + Be^{3x} - \frac{e^x}{20} [\cos 2x + 3 \sin 2x]$

based on $f(x) = x^n \sin ax$ or $x^n \cos ax$.

$$I = \frac{1}{f(D)} x^n \sin ax \text{ (or)} x^n \cos ax$$

$$\frac{1}{f(D)} x v = x \frac{1}{f(D)} v - \frac{f'(D)}{[f(D)]^2} v + \frac{f''(D)}{[f(D)]^3} v$$

Solve $(D^2 - 2D + 1)y = x e^x \sin x$.

soln:

Given: $(D^2 - 2D + 1)y = x e^x \sin x$.

auxiliary eqn is $m^2 - 2m + 1 = 0, m = 1, 1$

C.F = $(Ax + B)e^x$.

$I = \frac{1}{D^2 - 2D + 1} x e^x \sin x$

$$= e^x \left[\frac{1}{(D+1)^2 - 2(D+1) + 1} \right] x \sin x = e^x \left[\frac{1}{D^2 + 2D + 1 - 2D - 2 + 1} \right] x \sin x$$

$$= e^x \left[\frac{1}{D^2} \right] x \sin x = e^x \left[x \frac{1}{D^2} \sin x \right] - e^x \frac{2D}{(D^2)^2} \sin x$$

$$= e^x x \left[\frac{\sin x}{-1} \right] - e^x \frac{2D}{(-1)^2} \sin x = -x e^x \sin x - 2e^x \cos x$$

$$= -e^x [x \sin x + 2 \cos x]$$

$y = C.F + P.I = (Ax + B)e^x - e^x (x \sin x + 2 \cos x)$.

Solve $(D^2 + 4)y = x \sin x$.

soln:

Given: $(D^2 + 4)y = x \sin x$

A.E is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

C.F = $A \cos 2x + B \sin 2x$.

$I = \frac{1}{D^2 + 4} x \sin x = x \frac{1}{D^2 + 4} \sin x - \frac{2D}{(D^2 + 4)^2} \sin x$

$$= x \frac{1}{(-1 + 4)} \sin x - \frac{2D}{(-1 + 4)^2} \sin x = \frac{x}{3} \sin x - \frac{2}{9} \cos x$$

$y = C.F + P.I = A \cos 2x + B \sin 2x + \frac{x}{3} \sin x - \frac{2}{9} \cos x$

3) Solve $(D^2 + 4)y = x^2 \cos 2x$

soln: The A.E is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

C.F = $A \cos 2x + B \sin 2x$.

$$p \cdot I = \frac{1}{D^2 + 4} R \cdot p \cdot x^2 e^{i2x}$$

$$= R \cdot p \cdot x^2 e^{i2x} \cdot \frac{1}{(D^2 + 4)^2 + 4} x^2$$

$$= R \cdot p \cdot x^2 e^{i2x} \cdot \frac{1}{D^2 + 41D} x^2$$

$$= R \cdot p \cdot x^2 e^{i2x} \cdot \frac{1}{41D(D^2 + 41)} x^2$$

$$= R \cdot p \cdot x^2 e^{i2x} \cdot \frac{1}{41D} \left(1 + \frac{D^2}{41} \right) x^2$$

$$= R \cdot p \cdot x^2 e^{i2x} \cdot \frac{1}{41D} \left(1 + \frac{D^2}{41} \right) x^2$$

$$= R \cdot p \cdot x^2 e^{i2x} \cdot \frac{1}{41D} \left(1 - \frac{D^2}{41} + \frac{D^2}{(-16)} \right) x^2$$

$$= R \cdot p \cdot x^2 e^{i2x} \cdot \frac{1}{41D} \left(x^2 - \frac{2x}{41} - \frac{1}{8} \right)$$

$$= R \cdot p \cdot x^2 (\cos 2x + i \sin 2x) \left(\frac{1}{41} \right) \left(\frac{x^3}{3} - \frac{1}{21} x^2 + \frac{x}{8} \right)$$

$$= R \cdot p \cdot x^2 (\cos 2x + i \sin 2x) \left(\frac{1}{41} \right) \left(\frac{x^3}{3} + \frac{i x^2}{4} + \frac{x}{8} \right) \quad (\because i^2 = -1)$$

$$= R \cdot p \cdot x^2 (\cos 2x + i \sin 2x) \left(\frac{-i x^3}{12} + \frac{x^2}{16} + \frac{i x}{32} \right)$$

$$\therefore Y = C \cdot F + p \cdot I$$

$$= C_1 \cos 2x + C_2 \sin 2x + \frac{x^2 \cos 2x}{16} + \frac{x^3 \sin 2x}{32} + \frac{x \sin 2x}{32}$$

Method of Variation of parameters

Consider $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X$ [where X is a function of x]

The complementary function of (1) $\rightarrow$ (2)

$$C \cdot F = C_1 f_1 + C_2 f_2 \rightarrow$$

where C_1, C_2 are constants and f_1 and f_2 are functions of x .

$$\text{Then } p \cdot I = p f_1 + Q f_2 \rightarrow$$

$$P = - \int \frac{f_2 x}{f_1 f_2' - f_1' f_2} dx \rightarrow (4)$$

$$Q = \int \frac{f_1 x}{f_1 f_2' - f_1' f_2} dx \rightarrow (5)$$

$$\therefore y = c_1 f_1 + c_2 f_2 + p \cdot I$$

problems based on method of variation of parameters.

1) Use the method of variation of parameters to solve $y' + y = 4e^x$, given that $y(0) = 2$.

Soln: Given: $y' + y = 4e^x \rightarrow (1)$

$$(D+1)y = 4e^x$$

$$A.E \text{ is } m+1=0 \text{ i.e. } m=-1$$

$$\therefore y = Ae^{-x} \rightarrow (2)$$

$$\text{Now } y' = -Ae^{-x} + A'e^{-x} \rightarrow (3)$$

Sub (3) in (1), we get

$$-Ae^{-x} + A'e^{-x} + Ae^{-x} = 4e^x$$

$$A'e^{-x} = 4e^x$$

$$A' = 4e^{2x}$$

$$A = \int 4e^{2x} dx = \frac{A \cdot e^{2x}}{2} + c$$

$$A = 2e^{2x} + c$$

$$\therefore \text{Using (2), } y = (2e^{2x} + c)e^{-x}$$

$$y = ce^{-x} + 2e^x$$

$$\text{When } x=0, y=2, 2 = c+2 \Rightarrow c=0$$

$$\therefore y = 2e^x$$

2) Solve the equation $(D^2 + a^2)y = \tan ax$ by the method of variation of parameters.

$$\text{Soln: } A.E \text{ is } m^2 + a^2 = 0$$

$$m^2 = -a^2$$

$$m = \pm ia$$

$$C.F = C_1 \cos ax + C_2 \sin ax$$

$$P \cdot I = P f_1 + Q f_2$$

$$f_1 = \cos ax \quad f_2 = \sin ax$$

$$f_1' = -a \sin ax \quad f_2' = a \cos ax$$

$$f_1 f_2' - f_1' f_2 = a \cos^2 ax + a \sin^2 ax = a$$

$$P = - \int \frac{f_2 x}{f_1 f_2' - f_1' f_2} dx = - \int \frac{\sin ax \cdot \tan ax}{a} dx$$

$$= - \frac{1}{a} \int \frac{\sin^2 ax}{\cos ax} dx = - \frac{1}{a} \int \left(\frac{1 - \cos^2 ax}{\cos ax} \right) dx$$

$$= - \frac{1}{a} \left[\int \sec ax dx - \int \cos ax dx \right]$$

$$= - \frac{1}{a} \left[\frac{1}{a} (\log(\sec ax + \tan ax)) - \frac{\sin ax}{a} \right]$$

$$Q = + \int \frac{f_1 x}{f_1 f_2' - f_1' f_2} dx = \int \frac{\cos ax \cdot \tan ax}{a} dx = \frac{1}{a} \int \sin ax dx$$

$$= - \frac{1}{a^2} \cos ax$$

$$P \cdot I = P f_1 + Q f_2 = \frac{1}{a^2} \left[\frac{\sin ax}{\cos ax} - \log(\sec ax + \tan ax) \right] \cos ax - \frac{1}{a^2} \cos ax \sin ax$$

$$Y = C \cdot F + P \cdot I = (C_1 \cos ax + C_2 \sin ax) + \frac{1}{a^2} \left[\frac{\sin ax}{\cos ax} - \log(\sec ax + \tan ax) \right] \cos ax - \frac{1}{a^2} \cos ax \sin ax$$

2) Solve $(D^2 + 4)y = \sec 2x$ by the method of variation of parameters.

Soln: Given: $(D^2 + 4)y = \sec 2x$

The A.E. is $m^2 + 4 = 0$

$$m = \pm 2i$$

$$\therefore C \cdot F = C_1 \cos 2x + C_2 \sin 2x \quad P \cdot I = P f_1 + Q f_2$$

$$f_1 = \cos 2x \quad f_2 = \sin 2x$$

$$f_1' = -2 \sin 2x \quad f_2' = 2 \cos 2x$$

$$f_1 f_2' - f_1' f_2 = 2 \cos^2 2x + 2 \sin^2 2x = 2$$

$$P = - \int \frac{f_2 x}{f_1 f_2' - f_1' f_2} dx = - \int \frac{\sin 2x \sec 2x}{2} dx = - \frac{1}{2} \int \tan 2x dx$$

$$= -\frac{1}{2} \left[\frac{-\log(\cos 2x)}{2} \right] = \frac{1}{4} \log(\cos 2x) \quad 75$$

$$Q = \int \frac{f_1 x}{f_1 f_2' - f_1' f_2} dx = \int \frac{\cos 2x \sec 2x}{2} dx$$

$$= \frac{1}{2} \int \cos 2x \frac{1}{\cos 2x} dx = \frac{1}{2} \int dx = \frac{1}{2} x$$

$$P \cdot I = P f_1 + Q f_2 = \frac{1}{4} \log[\cos 2x] \cos 2x + \frac{1}{2} x \sin 2x$$

$$y = C \cdot F + P \cdot I = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{4} \log(\cos 2x) \cos 2x + \frac{1}{2} x \sin 2x$$

3) Apply the method of variation of parameters to solve $(D^2 + 4) = \cot 2x$.

Soln: A.E. is $m^2 + 4 = 0$

$$m = \pm 2i$$

$$\therefore C \cdot F = C_1 \cos 2x + C_2 \sin 2x$$

$$P \cdot I = P f_1 + Q f_2$$

$$f_1 = \cos 2x \quad f_2 = \sin 2x$$

$$f_1' = -2 \sin 2x \quad f_2' = 2 \cos 2x$$

$$f_1 f_2' - f_1' f_2 = 2 \cos^2 2x + 2 \sin^2 2x = 2$$

$$P = - \int \frac{f_2 x}{f_1 f_2' - f_1' f_2} dx = - \int \frac{\sin 2x \cdot \cot 2x}{2} dx = - \frac{1}{2} \int \cos 2x dx$$

$$= -\frac{1}{2} \frac{\sin 2x}{2} = -\frac{\sin 2x}{4}$$

$$Q = \int \frac{f_1 x}{f_1 f_2' - f_1' f_2} dx = \int \frac{\cos 2x \cot 2x}{2} dx = \frac{1}{2} \int \frac{\cos^2 2x}{\sin 2x} dx$$

$$= \frac{1}{2} \int \frac{1 - \sin^2 2x}{\sin 2x} dx = \frac{1}{2} \left[\int \frac{1}{\sin 2x} dx - \int \sin 2x dx \right]$$

$$= \frac{1}{2} \left[\int \operatorname{cosec} 2x dx + \frac{\cos 2x}{2} \right] = \frac{1}{2} \left[\frac{1}{2} \log(\operatorname{cosec} 2x - \cot 2x) + \frac{\cos 2x}{2} \right]$$

$$P \cdot I = P f_1 + Q f_2 = -\frac{1}{4} \sin 2x \cos 2x + \left[\frac{1}{4} \log(\operatorname{cosec} 2x - \cot 2x) + \frac{\cos 2x}{4} \right] \sin 2x$$

$$\therefore y = C \cdot F + P \cdot I$$

$$= C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4} \sin 2x \cos 2x + \left[\frac{1}{4} \log(\operatorname{cosec} 2x - \cot 2x) + \frac{\cos 2x}{4} \right] \sin 2x$$

4) Solve $\frac{d^2y}{dx^2} + 4y = 4 \tan 2x$ using method of variation of parameters. 76

Ans: $y = C_1 \cos 2x + C_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$

5) Solve $y'' + y = \sec x$ by the method of variation of parameters.

Ans: $y = C_1 \cos x + C_2 \sin x + \log(\cos x) \cdot \cos x + x \cdot \sin x$

Homogeneous equations of Euler's type.

An equation of the form $x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = f(x) \rightarrow (1)$

where $a_1, a_2, \dots, a_n$ are constants and $f(x)$ is a function of x is called a linear differential equation with variable coefficients.

put $x = e^z$ ($z = \log x$).

$$D' \equiv d/dz \quad x \frac{dy}{dx} = D'y \rightarrow (2)$$

$$x^2 \frac{d^2 y}{dx^2} = D'(D'-1)y \rightarrow (3)$$

$$x^3 \frac{d^3 y}{dx^3} = D'(D'-1)(D'-2)y \rightarrow (4)$$

$$x^4 \frac{d^4 y}{dx^4} = D'(D'-1)(D'-2)(D'-3)y \rightarrow (5)$$

1) Transform the equation $x^2 y'' + \alpha y' = \alpha$ into a linear differential equation with constant coefficients.

Soln: $[x^2 D^2 + \alpha D]y = \alpha \rightarrow (1)$

Put $x = e^z$ or $z = \log x$

$$xD = D' \rightarrow (2)$$

$$x^2 D^2 = D'(D'-1), D' = d/dz$$

Sub (2) in (1), we get

$$[D'(D'-1) + D']y = e^z$$

$$D'^2 y = e^z$$

$$\frac{d^2 y}{dz^2} = e^z \text{ where } e^z = x$$

which is a linear differential equation with constant coefficients.

2) Reduce the equation $(x^2 D^2 + xD + 1)y = \log x$ into an ordinary differential eqn with constant coefficients.

Soln: Given: $(x^2 D^2 + xD + 1)y = \log x \rightarrow ①$

Put $x = e^z$ or $z = \log x$

$$xD = D'$$

$$x^2 D^2 = D'(D'-1), D' = d/dz \rightarrow ②$$

Sub ② in ①, we get $[D'(D'-1) + D' + 1]y = z$
 $[D'^2 + 1]y = z$

$$\frac{d^2 y}{dz^2} + y = z, \text{ where } z = \log x$$

3) Solve: $(x^2 D^2 - xD - 2)y = x^2 \log x$

Soln: Given $(x^2 D^2 - xD - 2)y = x^2 \log x \rightarrow ①$

Put $x = e^z$ or $z = \log x$

$$xD = D'$$

$$x^2 D^2 = D'(D'-1)$$

Sub ② in ①, we get $[D'(D'-1) - D' - 2]y = e^{2z} \cdot z$
 $[D'^2 - 2D' - 2]y = e^{2z} \cdot z$

The A.E is $m^2 - 2m - 2 = 0$

$$m = \frac{2 \pm \sqrt{4 + 8}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

$$\therefore \text{C.F} = Ae^{(1+\sqrt{3})z} + Be^{(1-\sqrt{3})z}$$

$$P.I = \frac{1}{D^2 - 2D - 2} z e^{2z}$$

$$= e^{2z} \frac{1}{(D'+2)^2 - 2(D'+2) - 2} z$$

$$= e^{2z} \frac{1}{D'^2 + 4D' + 4 - 2D' - 4 - 2} z$$

$$= e^{2z} \frac{1}{D'^2 + 2D' - 2} z$$

$$= e^{2z} \frac{1}{-2 \left[1 - \left(\frac{D'^2 + 2D'}{2} \right) \right]} z$$

$$\begin{aligned}
 &= \frac{e^{2z}}{-2} \left[1 - \left(\frac{D'^2}{2} + \frac{2D'}{2} \right) \right] z \\
 &= \frac{e^{2z}}{-2} \left[1 + \left(\frac{D'^2}{2} + D' \right) - \dots \right] z \\
 &= \frac{e^{2z}}{-2} [z + D'z] \\
 &= \frac{e^{2z}}{-2} [z + 1]
 \end{aligned}$$

$$\begin{aligned}
 \therefore y &= C.F + P.I \\
 &= A e^{(1+\sqrt{3}) \log \alpha} + B e^{(1-\sqrt{3}) \log \alpha} - \frac{e^{2 \log \alpha}}{2} (\log \alpha + 1) \\
 &= A e^{(1+\sqrt{3}) \log \alpha} + B e^{(1-\sqrt{3}) \log \alpha} - \frac{\alpha^2 (\log \alpha + 1)}{2} \\
 &= A \alpha^{(1+\sqrt{3})} + B \alpha^{(1-\sqrt{3})} - \frac{\alpha^2 (\log \alpha + 1)}{2}
 \end{aligned}$$

2) Solve: $x^2 y'' - 4xy' + 6y = x^2 + \log x$

Ans: $Ax^3 + Bx^2 + \frac{1}{6}(\log x + \frac{5}{6}) - x^2 \log x$

3) Solve $(x^2 D^2 - xD + 1)y = x^2$

Soln:

Given: $(x^2 D^2 - xD + 1)y = x^2 \rightarrow (1)$

Put $x = e^z$ or $z = \log x$

$x D = D'$ $\rightarrow (2)$

$x^2 D^2 = D'(D'-1) \rightarrow (3)$

Sub (2) & (3) in (1), $[D'(D'-1) - D' + 1]y = e^{2z}$

$[D'^2 - 2D' + 1]y = e^{2z} \rightarrow (4)$

The A.E is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$\therefore C.F = (Ax + B)e^z$

$$p \cdot I = \frac{1}{D'^2 - 2D' + 1} e^{2z}$$

$$= \frac{1}{A - A + 1} e^{2z}$$

$$p \cdot I = e^{2z}$$

∴ The general soln is $y = C.F + p \cdot I$

$$y = (Az + B)e^z + e^{2z}$$

$$= (A \log \alpha + B)x + \alpha^2$$

④ Solve $(x^2 D^2 + 3D + 1)y = \frac{\sin(\log \alpha)}{x^2}$

Soln:

put $x = e^z$ or $z = \log \alpha \rightarrow$ ①

$$xD = D'$$

$$x^2 D^2 = D'(D' - 1), D' \equiv d/dz$$

Sub ② in ①, we get $[D'(D' - 1) + 3D' + 1]y = \frac{\sin z}{e^{2z}}$

$$[D'^2 + 2D' + 1]y = e^{-2z} \sin z$$

A.E is $m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$

∴ C.F = $(Az + B)e^{-z}$

$$p \cdot I = \frac{1}{D'^2 + 2D' + 1} e^{-2z} \sin z$$

$$= \frac{1}{(D' - 2)^2 + 2(D' - 2) + 1} e^{-2z} \sin z$$

$$= e^{-2z} \frac{1}{D'^2 + 4 - 4D' + 2D' - A + 1} \sin z$$

$$= e^{-2z} \frac{1}{D'^2 - 2D' + 1} \sin z = e^{-2z} \frac{1}{-1 - 2D' + 1} \sin z = -e^{-2z} \frac{1}{2D'} \sin z$$

$$= -\frac{e^{-2z}}{2} (-\cos z) = \frac{e^{-2z} \cos z}{2}$$

∴ $y = C.F + p \cdot I = (Az + B)e^{-z} + \frac{e^{-2z} \cos z}{2}$

$$= (A \log \alpha + B)e^{-\log \alpha} + \frac{1}{2} e^{-2 \log \alpha} \cos(\log \alpha)$$

$$= (A \log \alpha + B)x^{-1} + \frac{1}{2} x^{-2} \cos(\log \alpha)$$

$$= \frac{1}{x} (A \log \alpha + B) + \frac{1}{2x^2} \cos(\log \alpha)$$

4) Solve $(x^2 D^2 - xD + 1)y = \left[\frac{\log x}{x}\right]^2$

Soln:

Given: $(x^2 D^2 - xD + 1)y = \left[\frac{\log x}{x}\right]^2$

Put $x = e^z$, $z = \log x$

$x D = D'$

$x^2 D^2 = D'(D' - 1)$

$\therefore [D'(D' - 1) - D' + 1]y = \left(\frac{z}{e^z}\right)^2$

$[D'^2 - D' - D' + 1]y = z^2 e^{-2z}$

$[D'^2 - 2D' + 1]y = z^2 e^{-2z}$

The A.E is $m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$

$\therefore C.F = (Ax + B)e^z$

P.I = $\frac{1}{D'^2 - 2D' + 1} z^2 e^{-2z}$

$= e^{-2z} \frac{1}{(D' - 2)^2 - 2(D' - 2) + 1} z^2$

$= e^{-2z} \frac{1}{D'^2 - 4D' + 4 - 2D' + 4 + 1} z^2$

$= e^{-2z} \frac{1}{D'^2 - 6D' + 9} z^2$

$= e^{-2z} \frac{1}{9 \left[\frac{D'^2}{9} - \frac{2D'}{3} + 1 \right]} z^2$

$= \frac{e^{-2z}}{9} \left[1 + \left(\frac{D'^2}{9} - \frac{2D'}{3} \right) \right]^{-1} z^2$

$= \frac{e^{-2z}}{9} \left[1 - \left(\frac{D'^2}{9} - \frac{2D'}{3} \right) + \left(\frac{D'^2}{9} - \frac{2D'}{3} \right)^2 - \dots \right] z^2$

$(\text{oper}) = \frac{e^{-2z}}{9} \left[z^2 - \frac{2}{9} z^2 + \frac{4z}{3} + \frac{1}{9} (2) \right]$

$= \frac{e^{-2z}}{9} \left[z^2 + \frac{4z}{3} + \frac{2}{9} \right]$

$\therefore (x \log x) = \frac{1}{x} \left[z^2 + \frac{4z}{3} + \frac{2}{9} \right]$

Homogeneous equations of Legendre's type

The equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + K_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + K_n y = f(x) \rightarrow (1)$$

can be reduced to homogeneous linear differential equation by putting $ax+b = e^z$ or $z = \log(ax+b)$

Now $\frac{dy}{dx} (ax+b) = aD'y \rightarrow (2)$

$$(ax+b)^2 \frac{d^2 y}{dx^2} = a^2 D'(D'-1)y \rightarrow (3)$$

$$(ax+b)^3 \frac{d^3 y}{dx^3} = a^3 D'(D'-1)(D'-2)y \rightarrow (4)$$

Example:

- 1) Transform the equation $(2x+3)y'' - 2(2x+3)y' + 2y = 6x$ into a linear differential equation with constant coefficients.

Soln:

The given equation can be written as

$$[(2x+3)^2 D^2 - 2(2x+3)D + 2]y = 6x \rightarrow (1)$$

Put $2x+3 = e^z$ or $z = \log(2x+3)$

$$(2x+3)D = 2D'$$

$$(2x+3)^2 D^2 = 2^2 (D')^2 (D'-1) \quad D' = \frac{d}{dz} \rightarrow (2)$$

Sub (2) in (1), we get $[4D'(D'-1) - 2 \cdot 2D' + 2]y = 6 \left[\frac{e^z - 3}{2} \right]$

$$[4D'^2 - 4D' - 4D' + 2]y = 3(e^z - 3)$$

$$(4D'^2 - 8D' + 2)y = 3(e^z - 3), \quad z = \log(2x+3)$$

2) $(x+2)^2 \frac{d^2 y}{dx^2} - (x+2) \frac{dy}{dx} + y = 3x+4$

Soln: Put $x+2 = e^z$ or $z = \log(x+2)$

$$\therefore (x+2)D = D', \quad (x+2)^2 D^2 = D'(D'-1), \quad D' = \frac{d}{dz}$$

$$[D'(D'-1) - D' + 1]y = 3(e^z - 2) + 4$$

$$[D'^2 - 2D' + 1]y = 3e^z - 2$$

3) Solve $(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y = 2x \sin \log(1+x)$.

Solution:

Given: $[(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y] = 2x \sin \log(1+x)$

$[(1+x)^2 D^2 + (1+x) D + 1] y = 2x \sin \log(1+x)$

Put $1+x = e^z$ or $z = \log(1+x)$

$(x+1)D = D'$, $(x+1)^2 D^2 = D'(D'-1)$

$[D'(D'-1) + D' + 1] y = 2 \sin z$

$[D'^2 - D' + D' + 1] y = 2 \sin z$

$[D'^2 + 1] y = 2 \sin z$

The A.E is $m^2 + 1 = 0$

$m^2 = -1$

$\therefore C.F = A \cos z + B \sin z$

$P.I = \frac{1}{D'^2 + 1} 2 \sin z$

$= 2 \frac{1}{-1+1} \sin z = 2 \frac{1}{0} \sin z$

$= 2x \frac{1}{-1+1} \sin z$

$y = C.F + P.I = A \cos z + B \sin z + 2x \cos z$

A) solve $[(x+2)^2 D^2 + 3(3x+2)D - 36] y = 2x \cos z$

Soln:

put $3x+2 = e^z$ or $z = \log(3x+2) \rightarrow ①$

$(3x+2)D = 3D'$, $(3x+2)^2 D^2 = 3^2 D'(D'-1) \rightarrow ②$

Sub ② in ①, we get

$[3^2 D'(D'-1) - 36] y = 2x \cos z$

$$[9D'(D'-1) + 3 \cdot 3D' - 36]y = 3\left(\frac{e^x - 2}{3}\right)^2 + 4\left(\frac{e^x - 2}{3}\right) + 1 \quad 83$$

$$[9D'^2 - 9D' + 9D - 36]y = \frac{e^{2x} + 4 - 4e^x}{e^{2x} + 4 - 4e^x} + 4(e^{2x} - 2) + 1$$

$$[9D^2 - 36]y = \frac{e^{2x}}{3} + 4e^x - 4e^x + 4e^x - 8 + 1$$

$$\frac{e}{3} + \frac{4}{3} - \frac{4e^2}{3} + \frac{4e^2}{3} - \frac{8}{3} + 1$$

$$= \frac{2x}{3} - \frac{1}{3}$$

$$[D^2 - A]y = \frac{1}{27}(e^{2x} - 1)$$

$$A \cdot E \text{ \& } m^2 - A = 0 \Rightarrow m^2 = A \rightarrow 0 = y'(t) + y(t)$$

$$m = \frac{2}{1}$$

$$\therefore C.F = A_0^{2Z} + B_0^{0}$$

$$p\text{-}T = \frac{1}{D^2 - 4} \frac{1}{27} \left(\frac{e^2 - 1}{e^2 + 1} \right)^2 \left[\bar{A} - (e^2 + 1) \left(\frac{e^2 - 1}{e^2 + 1} \right) \right]$$

$$= \frac{1}{27} \left[\frac{e^{2z}}{A-A} \right] - \frac{1}{27} \left[\frac{e^{2z}}{0.4A} \right] + \frac{1}{27} \left[\frac{e^{2z}}{0.4A} \right] - \frac{1}{27} \left[\frac{e^{2z}}{0.4A} \right]$$

$$= \frac{1}{27} \left[\frac{z_e^{2x}}{2D'} + \frac{1}{27} - \frac{1}{27} - \frac{1}{27} + \frac{1}{27} \right]$$

$$= \frac{1}{27} \left[\frac{x}{2} \frac{e^{2x}}{2} + \frac{1}{4} \right]$$

$$= \frac{1}{108} [2x + 1]$$

$$y = C \cdot F + p \cdot I = A e^{2x} + B e^{-2x} + \frac{1}{108} [2e^{2x} + 1], \text{ where } z = \log(3x+2).$$

3) Solve $(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \cos \log(1+x)$

Ans: $y = c_1 \cos z + c_2 \sin z$, where $z = \log(1+x)$.

$$1 + m(\sin \alpha) = 1 + m \sin \alpha \quad \text{and} \quad A \cdot E = A \cos \alpha + B \sin \alpha = 1 \cdot \cos \alpha + 1 \cdot \sin \alpha = 1$$

$$f_0 \cdot \frac{1}{x} = f_0 \cdot \frac{1}{1-f} = \frac{f_0}{1-f}$$

$$1 = \frac{1}{1.65} = 0.609 = 60.9\%$$

System of simultaneous linear differential equations with constant coefficients.

1) Eliminate x and find the equation in y from

$$\frac{dx}{dt} + 2x - 2y = t, \quad \frac{dy}{dt} + 2x + y = 0.$$

Soln:

The given eqn can be written as:

$$(D+2)x - 2y = t \rightarrow (1)$$

$$2x + (D+1)y = 0 \rightarrow (2)$$

$$(1) \times 2 \Rightarrow 2(D+2)x - 4y = 2t$$

$$(2) \times D+2 \Rightarrow 2(D+2)x + (D+1)(D+2)y = 0$$

$$(1) - (2) \Rightarrow [-(D+1)(D+2) - 4]y = 2t$$

$$[-(D^2 + 3D + 2) - 4]y = 2t$$

$$(D^2 + 3D + 6)y = -2t$$

$$(D^2 + 3D + 6)y = -2t$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 6y = -2t$$

$$(2) \text{ Solve: } \frac{dx}{dt} + y = e^t, \quad x - \frac{dy}{dt} = t.$$

Soln: (2) put $x = e^{mt}$

$$\text{Given: } \frac{dx}{dt} + y = e^t, \quad x - \frac{dy}{dt} = t$$

$$\text{ie) } Dx + y = e^t \rightarrow (1) \quad x - Dy = t \rightarrow (2)$$

$$(2) \times D \Rightarrow Dx - D^2y = 1 \rightarrow (3)$$

$$(1) - (3) \Rightarrow (D^2 + 1)y = e^t - 1 = e^t - e^{0t}$$

The A.E is $m^2 + 1 = 0$ ie) $m = \pm i$

$\therefore$ C.F = $A \cos t + B \sin t$

$$P.I_1 = \frac{1}{D^2 + 1} e^t = \frac{1}{1+1} e^t = \frac{1}{2} e^t$$

$$P.I_2 = \frac{1}{D^2 + 1} e^{0t} = \frac{1}{1} e^{0t} = 1.$$

$$y = C \cdot F + P \cdot I, \quad + P \cdot I_2 = A \cos t + B \sin t + \frac{1}{2} e^t - 1$$

$$Dy = -A \sin t + B \cos t + \frac{1}{2} e^t$$

$$\textcircled{2} \Rightarrow x = Dy + t = -A \sin t + B \cos t + \frac{1}{2} e^t + t$$

hence the desired equations are $x = -A \sin t + B \cos t + \frac{1}{2} e^t + t$

$$y = A \cos t + B \sin t + \frac{1}{2} e^t - 1$$

$$1) \text{ Solve } \frac{dx}{dt} - y = t, \quad \frac{dy}{dt} + x = t^2$$

Soln:

The given eqn can be written as $Dx - y = t \rightarrow \textcircled{1}$

$$(Dy + x = t^2) \rightarrow \textcircled{2}$$

$$\textcircled{1} \times D \Rightarrow D^2 x - Dy = 1$$

$$x + Dy = t^2$$

$$D^2 x + x = 1 + t^2$$

$$(D^2 + 1)x = 1 + t^2$$

$$A.E. \text{ is } m^2 + 1 = 0 \Rightarrow m^2 = -1$$

$$m = \pm i$$

$$\therefore C.F. = A \cos t + B \sin t$$

$$P.I. = \frac{1}{D^2 + 1} (1 + t^2)$$

$$= \frac{1}{D^2 + 1} (1) + \frac{1}{D^2 + 1} (t^2)$$

$$= \frac{1}{D^2 + 1} e^{0t} + (1 + D^2)^{-1} (t^2)$$

$$= \frac{1}{1} + (1 - D^2 + D^4 - \dots) (t^2)$$

$$\therefore x = C.F. + P.I. = A \cos t + B \sin t + t^2 - 1$$

$$\textcircled{1} \Rightarrow y = Dx - t = D[A \cos t + B \sin t + t^2 - 1] - t$$

$$= -A \sin t + B \cos t + 2t - t$$

$$= -A \sin t + B \cos t + t$$

$$4) \text{ Solve } \frac{dx}{dt} + y = \sin t, \quad x + \frac{dy}{dt} = \cos t \text{ given that } x = 2 \text{ and } y = 0 \text{ at } t = 0$$

Soln:

$$\text{Given } Dx + y = \sin t \rightarrow \textcircled{1}$$

$$x + Dy = \cos t \rightarrow \textcircled{2}$$



$$Dx + y = \sin t$$

$$(2) \Rightarrow \frac{Dx + D^2y}{(1) \quad (2)} = \frac{-\sin t}{(1)}$$

$$-D^2y + y = 2\sin t$$

$$D^2y - y = -2\sin t$$

$$(D^2 - 1)y = -2\sin t$$

$$A.E \text{ is } m^2 - 1 = 0 \Rightarrow m^2 = 1$$

$$m = \pm 1$$

$$\therefore C.F = C_1 e^t + C_2 e^{-t}$$

$$P.I = \frac{1}{D^2 - 1} (-2\sin t) = \frac{1}{-1 - 1} (-2\sin t)$$

$$= \sin t$$

$$\therefore y = C_1 e^t + C_2 e^{-t} + \sin t \rightarrow (5)$$

$$\text{Now } Dy = C_1 e^t - C_2 e^{-t} + \cos t \rightarrow (6)$$

Sub (5) & (6) in (2), we get $x = -Dy + \cos t$

$$x = -C_1 e^t + C_2 e^{-t} - \cos t + \cos t = -C_1 e^t + C_2 e^{-t}$$

$$y = C_1 e^t + C_2 e^{-t} + \sin t \rightarrow (8)$$

Given $x(0) = 2$ and $y(0) = 0$

$$\therefore (7) \Rightarrow x(0) = -C_1 + C_2 = 2 \rightarrow (9)$$

$$y(0) = C_1 + C_2 = 0 \rightarrow (10)$$

$$\therefore C_1 = -C_2 \therefore C_2 = 2 \rightarrow (11)$$

Sub (11) in (7) & (8), we get

$$x(t) = e^t + e^{-t}, y(t) = -e^t + e^{-t} + \sin t$$

Method of undetermined coefficients.

Example 1: Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$.

Solution:

Given $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x} \rightarrow (1)$

A.E is $m^2 + 2m + 4 = 0$

$$m = \frac{-2 \pm \sqrt{4 - 16}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i$$

$\therefore C.F = e^{-x} [c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x] \rightarrow (1)$

Let P.I = $y = a_1x^2 + a_2x + a_3 + a_4e^{-x} \rightarrow (2)$

Differentiating (2) with respect to x , we get

$$\frac{dy}{dx} = 2a_1x + a_2 - a_4e^{-x} \rightarrow (3)$$

$$\frac{d^2y}{dx^2} = 2a_1 + a_4e^{-x} \rightarrow (4)$$

Sub (3), (4) in (1), we get

$$2a_1 + a_4e^{-x} + 2[2a_1x + a_2 - a_4e^{-x}] + 4[a_1x^2 + a_2x + a_3 + a_4e^{-x}] = 2x^2 + 3e^{-x}$$

$$2a_1 + a_4e^{-x} + 4a_1x + 2a_2 - 2a_4e^{-x} + 4a_1x^2 + 4a_2x + 4a_3 + 4a_4e^{-x} = 2x^2 + 3e^{-x}$$

$$4a_1x^2 + (4a_1 + 4a_2)x + 2a_1 + 2a_2 + 4a_3 + 3a_4e^{-x} = 2x^2 + 3e^{-x}$$

$$4a_1x^2 + 4x(a_1 + a_2) + 2(a_1 + a_2 + 2a_3) + 3a_4e^{-x} = 2x^2 + 3e^{-x}$$

Equating coefficients of like terms, we get

$$4a_1 = 2, 4a_1 + 4a_2 = 0, 2(a_1 + a_2 + 2a_3) = 0$$

$$a_1 = \frac{1}{2}$$

$$4a_1 = -4a_2 \Rightarrow \frac{1}{2} = -\frac{1}{2} + 2a_3 = 0$$

$$a_1 = -a_2$$

$$a_2 = -\frac{1}{2}$$

$$a_3 = 0$$

$$3a_4 = 3$$

$$a_4 = 1$$



Substituting the values of a_1, a_2, a_3 and a_4 in (2), we get

$$p \cdot I = \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$$

$$y = C \cdot F + p \cdot I$$

$$= e^{-x} [C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x] + \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$$

(2) Solve $(D^2 - 2D - 3)y = 5 \cos 2x$.

Solution:

Given: $(D^2 - 2D - 3)y = 5 \cos 2x \rightarrow (A)$

A.E. is $m^2 - 2m - 3 = 0$

$$(m-3)(m+1) = 0$$

$$m = 3, -1$$

$\therefore C.F = C_1 e^{3x} + C_2 e^{-x} \rightarrow (1)$

Let $p \cdot I$ be $y = a_1 \cos 2x + a_2 \sin 2x \rightarrow (2)$

Diff (2) w. r. to x , we get

$$\frac{dy}{dx} = -2a_1 \sin 2x + 2a_2 \cos 2x \rightarrow (3)$$

$$\frac{d^2y}{dx^2} = -4a_1 \cos 2x - 4a_2 \sin 2x \rightarrow (4)$$

Sub (2), (3) and (4) in (A), we get

$$-4a_1 \cos 2x - 4a_2 \sin 2x = 2[-2a_1 \sin 2x + 2a_2 \cos 2x]$$

$$-3[a_1 \cos 2x + a_2 \sin 2x] = 5 \cos 2x$$

$$-4a_1 \cos 2x - 4a_2 \sin 2x + 4a_1 \sin 2x - 4a_2 \cos 2x - 3a_1 \cos 2x - 3a_2 \sin 2x = 5 \cos 2x$$

$$\cos 2x (-7a_1 - 4a_2) + \sin 2x (-4a_1 - 7a_2) = 5 \cos 2x$$

Equating coefficients of like terms, we get

$$-7a_1 - 4a_2 = 5 \rightarrow (5)$$

$$4a_1 - 7a_2 = 0 \rightarrow (6)$$

$$4a_1 = 7a_2$$

$$a_1 = \frac{7a_2}{4} \rightarrow (7)$$



Substituting $a_1 = \frac{7a_2}{4}$ in (5) we get

$$-7\left(\frac{7a_2}{4}\right) - 4a_2 = 5$$

$$\frac{-49a_2}{4} - 4a_2 = 5$$

$$\frac{-65a_2}{4} = 5$$

$$a_2 = \frac{-20}{65} = -\frac{4}{13}$$

Sub $a_2 = -\frac{4}{13}$ in (7), we get

$$a_1 = \frac{7}{4} \left(-\frac{4}{13}\right) = -\frac{7}{13}$$

Sub a_1, a_2 in (2), we get

$$p \cdot I = -\frac{7}{13} \cos 2x - \frac{4}{13} \sin 2x$$

$$y = C \cdot F + p \cdot I = C_1 e^{3x} + C_2 e^{-x} - \frac{7}{13} \cos 2x - \frac{4}{13} \sin 2x$$

Transform the equation $xy'' + y' + 1 = 0$ into linear equation with constant coefficient. 90

Soln: $xy'' + y' + 1 = 0$

$$x^2 y'' + xy' + x = 0$$

$$x^2 D^2 y + x D y + x = 0$$

$$(x^2 D^2 + x D) y = -x$$

Put $x^2 D^2 = D'(D'-1)$, $x D = D'$, $xc = e^z$, $z = \log x$

$$(D'(D'-1) + D') y = -e^z$$

$$D'^2 y = -e^z$$

$$\frac{d^2 y}{dz^2} = -e^z, \text{ where } e^z = x$$