PIE Tech

POLLACHI INSTITUTE OF ENGINEERING AND TECHNOLOGY

(Approved by **AICTE** and Affiliated to **Anna University**)

sky is the limit

Department of Electronics and Communication Engineering

Regulation 2021

II Year – III Semester

EC3354- SIGNALS AND SYSTEMS

EC 3354 Signals and systems

Unit 1 :- Classification of signals and systems

=> Standard signals Step, Ramp, pulse impulse, real and complex exponential and sinusoides

✓ => Classification of signals
* continuous time signals
* Discrete "

✓ => periodic and aperiodic signals.

=> Deterministic & Random signals.

✓ => Energy and power signals.

✓ => Classification of systems
* CT & DT systems.
* Linear and non linear
* Time variants and Time invariant
* causal and non causal
* Stable and unstable.

Signals :-

A function of one or more independent Variables which contain some information is Called signal.

If signal depends only one variable One dimensional signal

eg :- electric Voltage or current
Such as radio signal, TV signal
telephone signal, computer signal.

* non electric signals such as pressure signal, sound siegnal.

* A signal is defined as a physical quantity that varies with time, space or any other indeependent variable

Systems :-

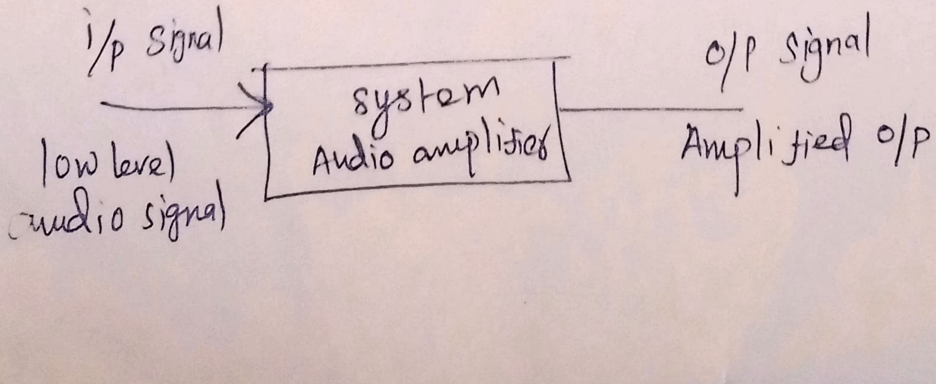
A system is a set of elements or functional block that are connected together and produces an o/p in responses to an i/p signal.

eg :- audio amplifier
attenuator
TV set
Transmitter
receiver.

Two-dimentional signal
eg:- pictures
X-ray images
Sonograms

multidimensional signal,
eg:-
image signal.

Relation b/w signal & system

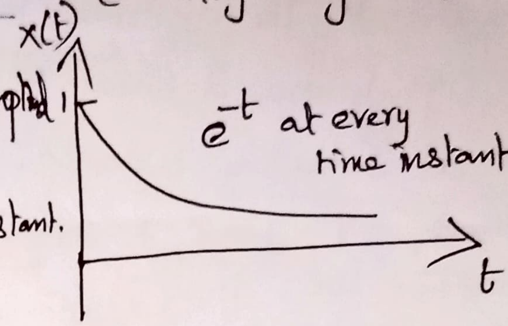


* Every system has one or more i/ps called as excitation

* " " " one or more o/ps called response.

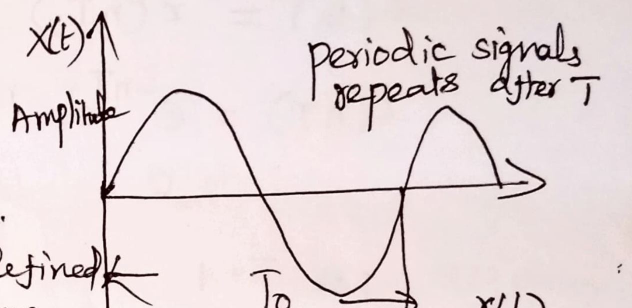
* i/ps and o/ps of the system are always signal

Continuous Time continous amplitude signals :- (analog signals)



* The analog signals are Amplitude continuous in amplitude and defined at every time instant.

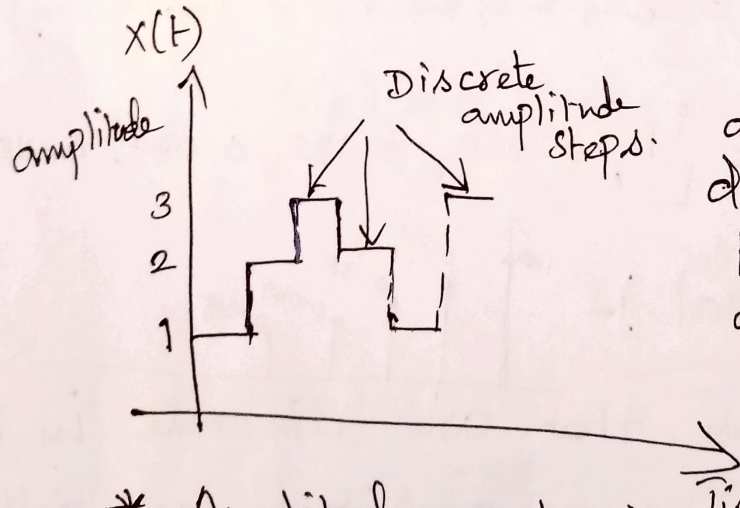
* eg:- ECG signal
  speech signal
  telephone signal
  noise signal etc.



$x(t) = x(t+T_0)$

* The signal that are defined for every instant of time is known as continuous time signal

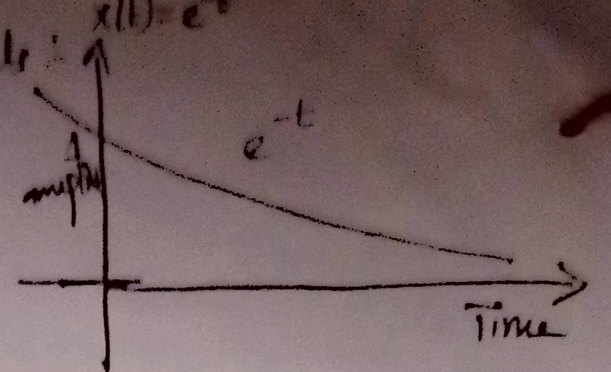
Continuous time, ~~and~~ discrete amplitude signals.



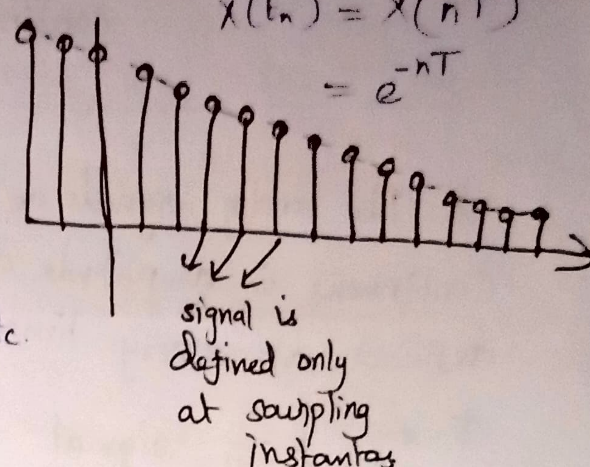
* The signals that are defined at discrete instant of time are known as discrete time singnal denoted as $x(n)$

* Amplitude only in 3 steps but can be defined at any time instants. It is called continuous time discrete amplitude signal.

Discrete Time signals:



* The discrete time signals are obtained by time sampling of continuous time signals.



* The discrete time signals are defined only at sampling instants.

$0, \pm T, \pm 2T, \pm 3T \ldots$ etc.

$$x(t_n) = x(nT)$$

$$x(nT) = e^{-nT} \quad n \geq 0$$
$$= 0, \quad n < 0$$

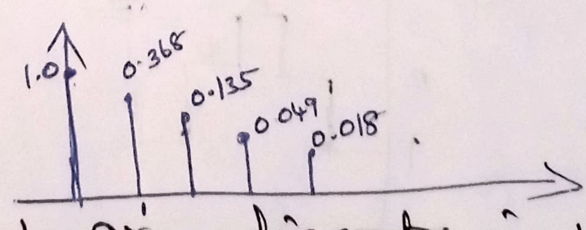
For $T = 1$

$$x(n) = e^{-n} \quad n \geq 0$$
$$0 \quad n < 0$$

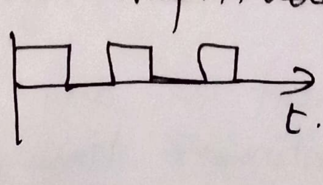
$$x(n) = \{e^0, e^{-1}, e^{-2}, e^{-3}, e^{-4}, \ldots\}$$

$$= \{1, 0.368, 0.135, 0.049, 0.018 \ldots\}$$

Digital signal !-

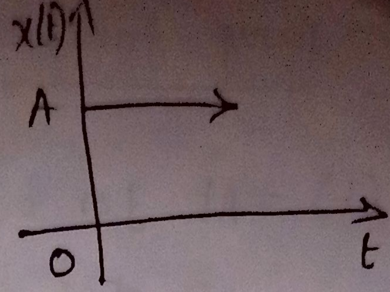


* Signals that are discrete in time and quantized in amplitude are called digital signals.



Continuous Time signals :-

Step signal :-



The step signal can be defined as

$$x(t) = 0 \quad \text{for } t < 0$$
$$= A \quad \text{for } t \geq 0$$

The signal looks like a step it called Step signal.

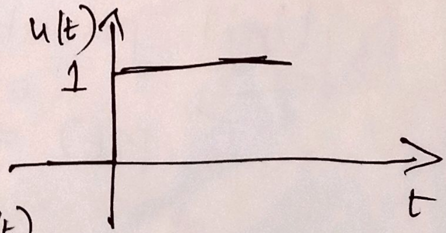
Unit step function u(t) :-

* unit Step fn u(t), it satisfies two conditions.

(i) The amplitude of of unit step fn is always equal to unity.

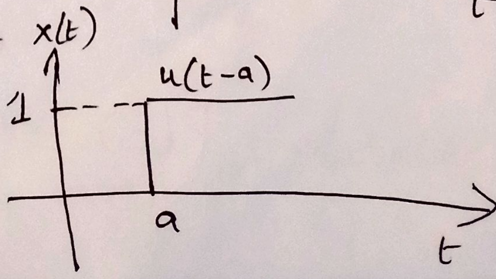
(ii) u(t) is zero, whenever argument (t) inside brackets (-ve) and unity when argument (t) inside brackets is greater than equal to zero (ie +ve values of t).

$$u(t) = 0 \quad \text{for } t < 0$$
$$1 \quad \text{for } t \geq 0$$



Time shifted unit step function :-

$$u(t-a) = 0 \quad \text{for } t < a$$
$$= 1 \quad \text{for } t \geq 0$$



Unit ramp function :-

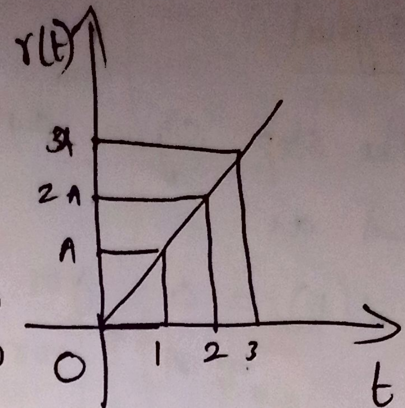
$r(t) = At$ for $t \geq 0$

$\quad\quad 0$ for $t < 0$

$A = 1$

$r(t) = t$ for $t \geq 0$

$\quad\quad 0$ for $t < 0$



* If ramp signal has unit magnitude $A=1$ the ramp signal is said to be unit ramp Signal.

* The ramp signal can be obtained by applying unit step fn to an integrator.

$$r(t) = \int u(t)\,dt = \int 1\,dt = t$$

In otherwords

Unit step fn can be obtained by differentiating the Unit ramp function.

$$u(t) = \frac{d}{dt} r(t)$$

Unit parabolic function :-

$p(t) = \frac{t^2}{2}$ for $t \geq 0$

$\quad\quad 0$ fo $t < 0$

$$p(t) = \frac{t^2}{2} u(t)$$

* The unit parabolic function can be obtained by integrating the ramp fn.

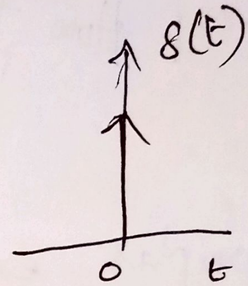
$$p(t) = \int r(t)\, dt = \int t\, dt = t^2/2$$

$$r(t) = \frac{d}{dt} p(t)$$

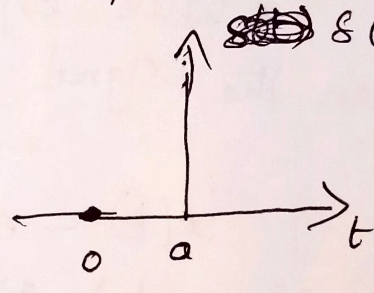
Impulse function :-

Impulse fn occupies an important place in signal analysis. It is defined as

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

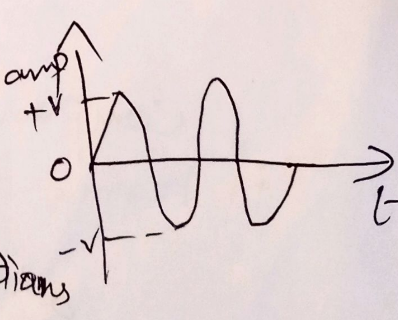


* impulse fn has zero amplitude every where except at $t=0$



$\Longrightarrow$ Delayed unit impulse function.

Sinusoidal signal :-



$$x(t) = A \sin(\omega t + \theta)$$

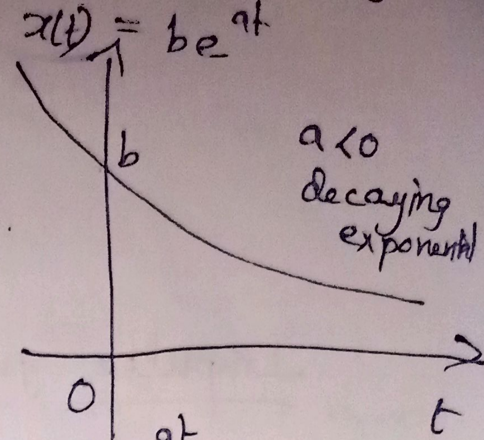
ω - frequency in radians/sec

A - Amplitude.

θ = phase angle in radians.

Real exponential signal :-

* It is exponentially growing or decaying signal.

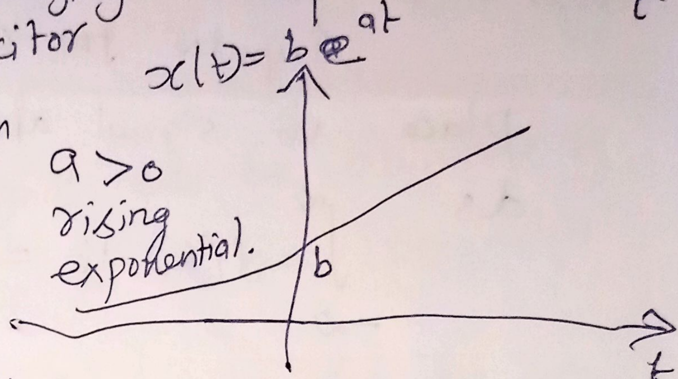


$$x(t) = be^{at}$$

b & a are real.

uses :-

* charging and discharging of a capacitor
* ct flow through an inductor.



Complex exponential :-

* when exponent is purely imaginary then the signal is said to complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

Sinusoidal signal is given by

$$x(t) = \cos(\omega_0 t + \phi)$$

Complex exponential can be written in terms of sinusoidal signal

$$e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t$$

Similarly sinusoidal signal can be written ~~as~~ interms of complex exponential.

$$\cos(\omega_0 t + \phi) = \frac{e^{j(\omega_0 t + \phi)} + e^{-j(\omega_0 t + \phi)}}{2}$$

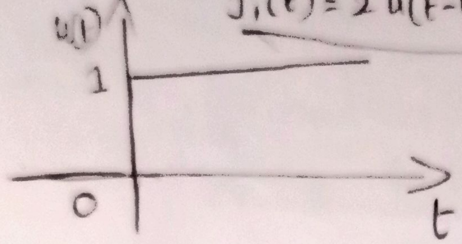
① Draw the wave forms represented by following step functions.

(i) $f_1(t) = 2\,u(t-1)$

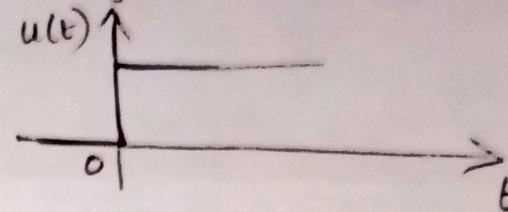
(ii) $f_2(t) = -2\,u(t-2)$

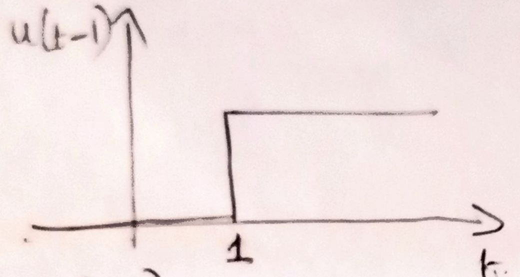
(iii) $f_1(t) + f_2(t)$

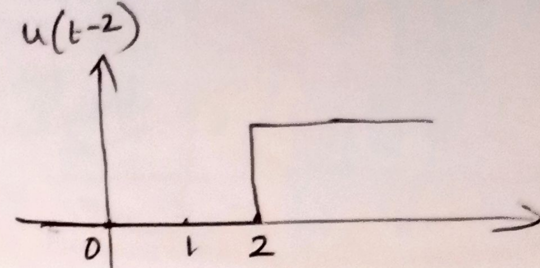
$f_1(t) = 2\,u(t-1)$

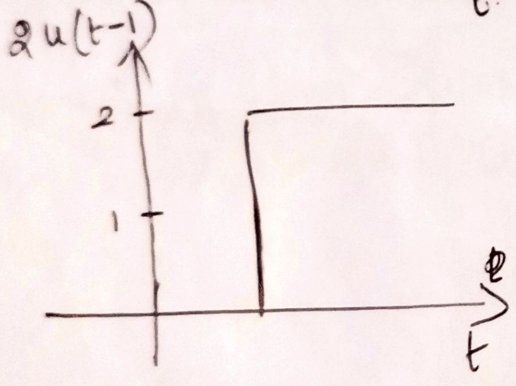


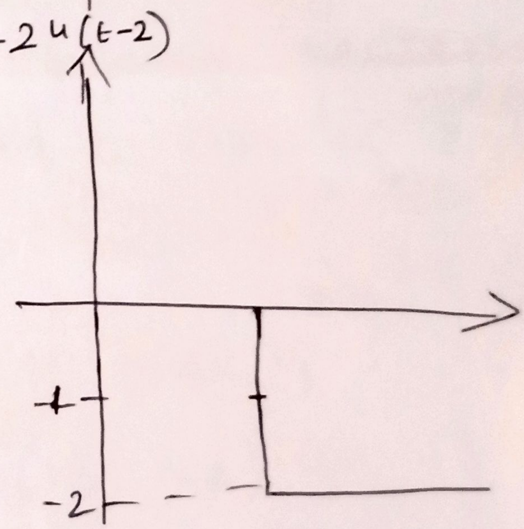
$f_2(t) = -2\,u(t-2)$





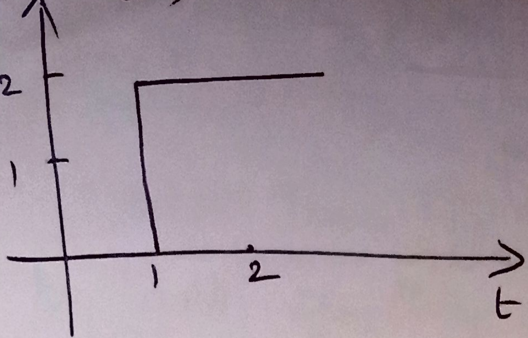




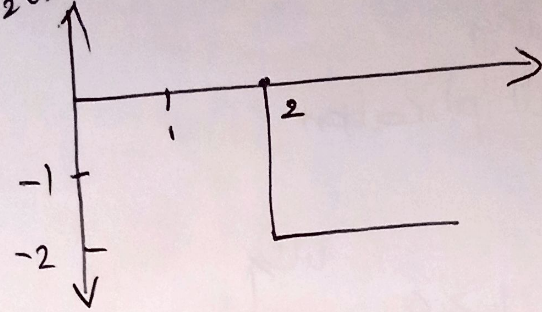


$f_1(t) + f_2(t)$

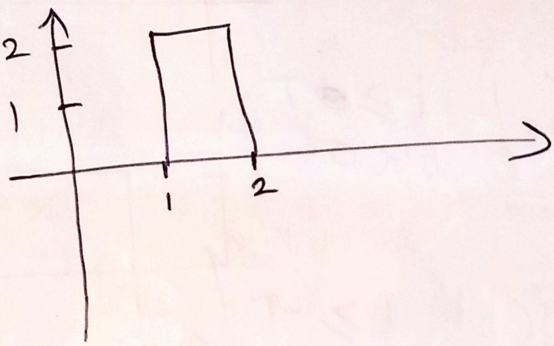
$f_1(t) = 2u(t-1)$



$f_2(t) = -2u(t-2)$



$f(t) = f_1(t) + f_2(t)$

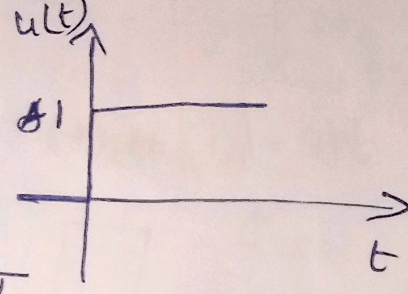


Basic properties of signals :-

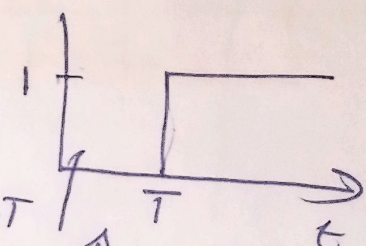
1. Time shifting
2. Amplitude Scaling
3. Time reversal
4. Time Scaling
5. Addion & multiplication

Time shifting :-

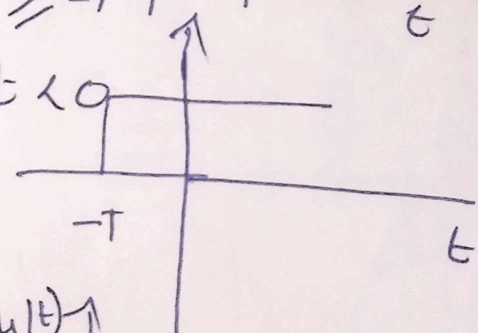
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$u(t-T) = \begin{cases} 1 & t \geq T \\ 0 & t < 0 \end{cases}$$

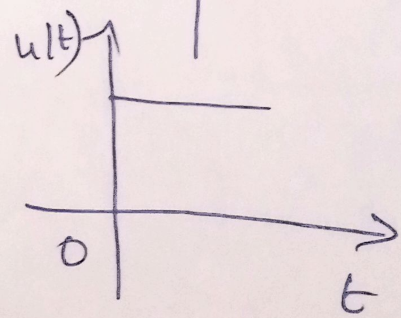


$$u(t+T) = \begin{cases} 1 & t \geq -T \\ 0 & t < 0 \end{cases}$$

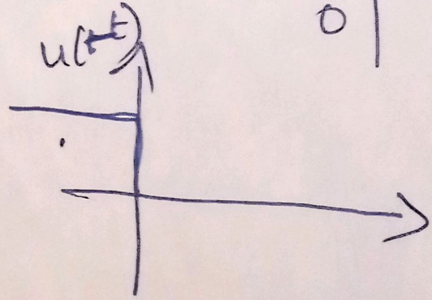


Time reversal :-

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

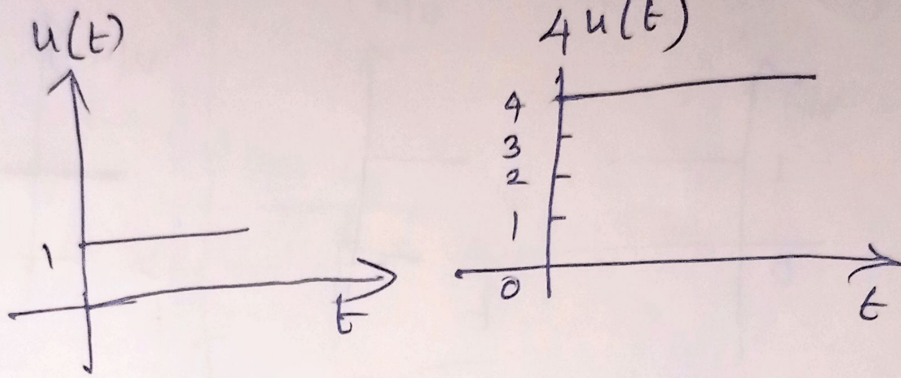


$u(-t) =$

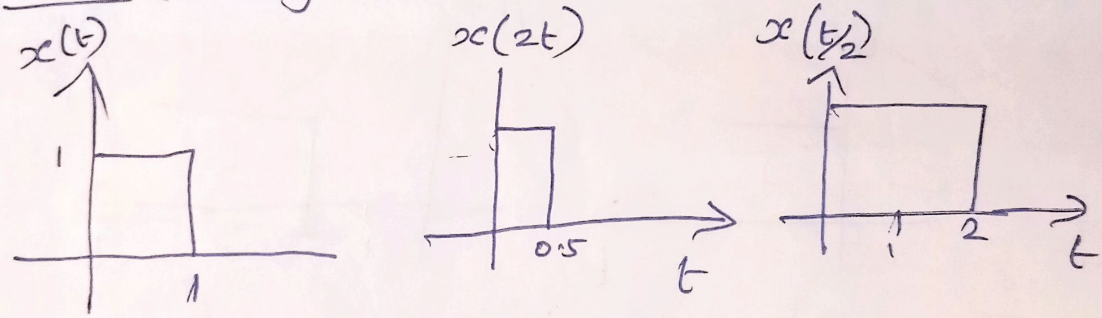


Amplitude scaling :-

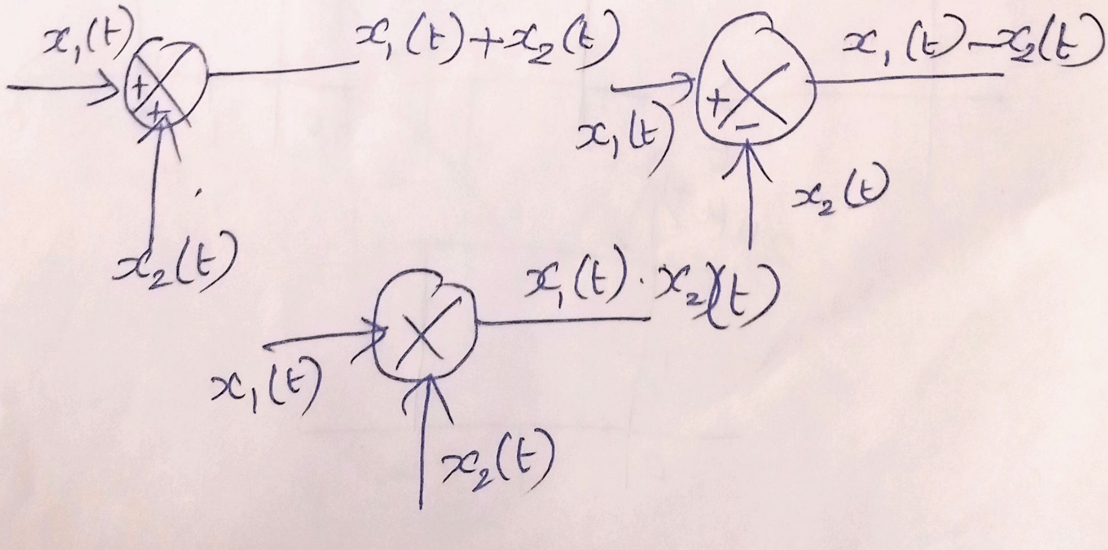
~~u(t)~~ The signal $x(t)$ is multiplied by gain of amplifier is said to be amplitude scaling



Time scaling :-



Signal addition and multiplication



Draw the signals

(a) $u(-t+1)$

b) $u(t-2)$

c) $2u(-t+2)$

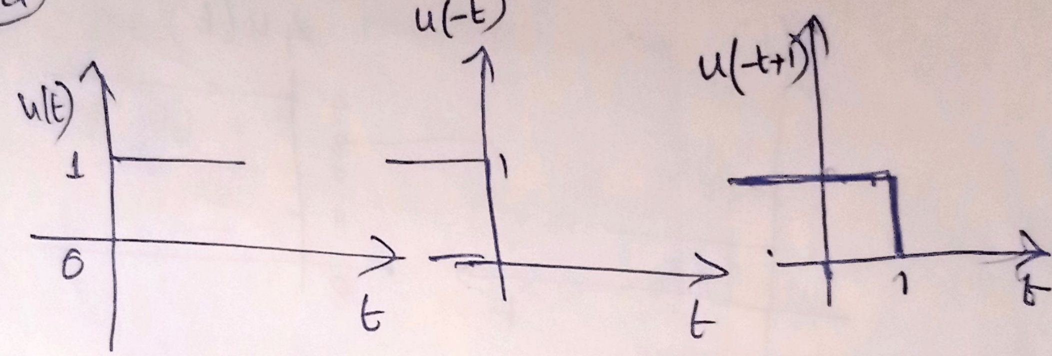
d) $-3u(t-2)$

(e) $r(-t+5)$

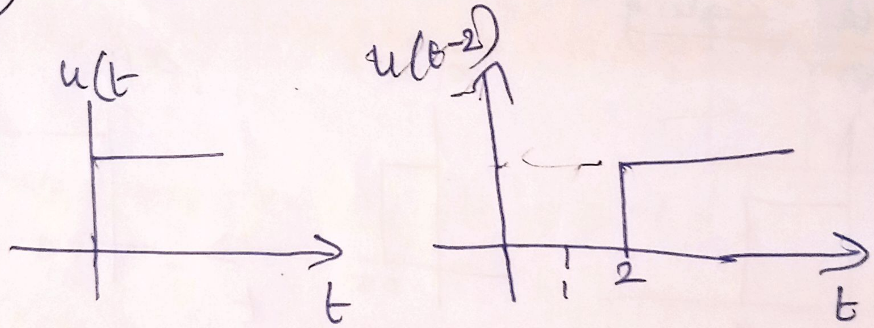
(f) $-3r(t)$

(g) $3r(t-1)$

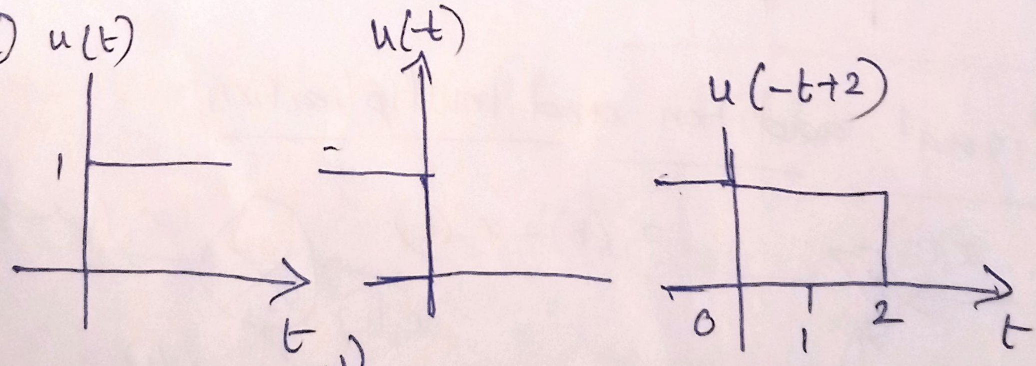
(a)

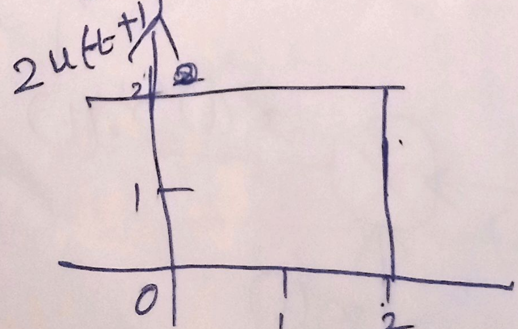


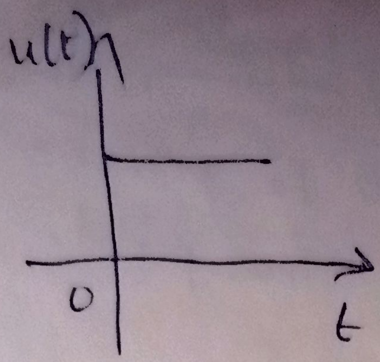
(b)

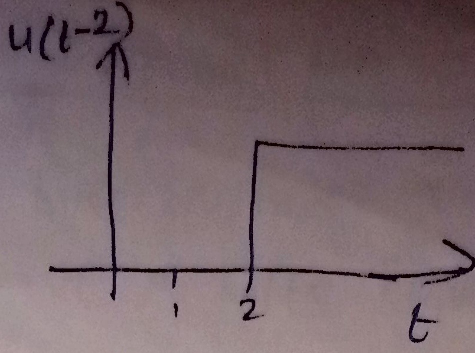


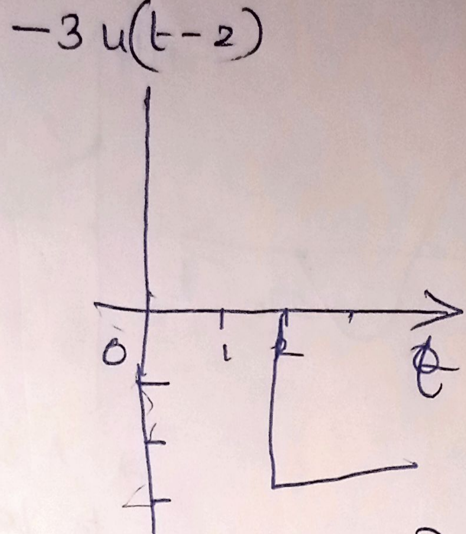
(c)



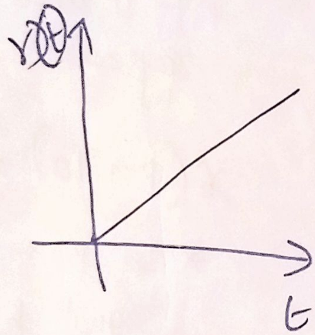


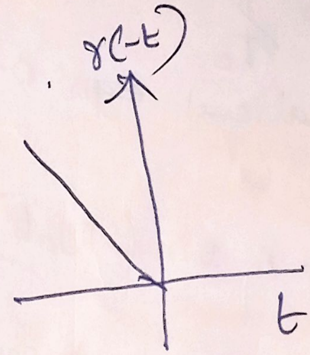


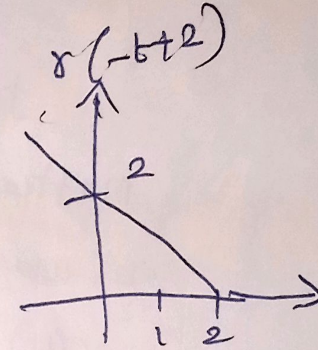




(e)

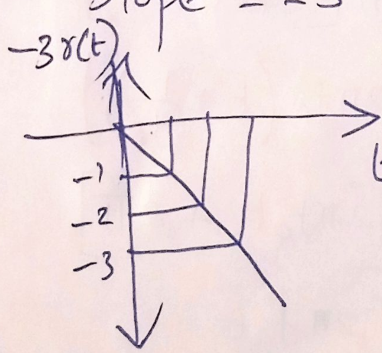






(f)

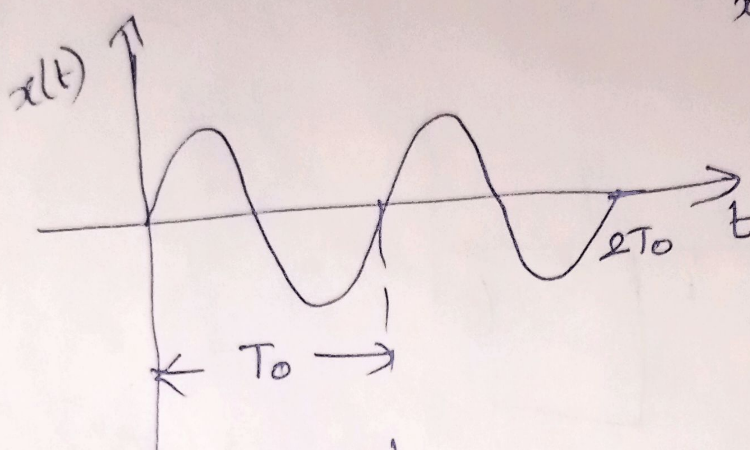
slope = −3



Periodic and Non periodic signals.

* A Signal is said to be periodic if it repeats after a fixed time period.

$$x(t) = x(t + T_0)$$



$f_0 = 1/T_0$

mathematical eqn of sin wave is

$$x(t) = A \sin(2\pi f_0 t)$$

$f_0 = \frac{1}{T_0}$ = freq of the signal

$$x(t+T_0) = A \sin[2\pi f_0 (t+T_0)]$$

$$= A \sin\left[2\pi f_0 \left(t + \frac{1}{f_0}\right)\right]$$

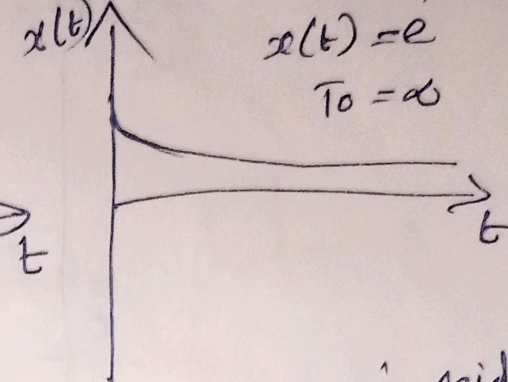
$$= A \sin[2\pi f_0 t + 2\pi]$$

$$= A \sin 2\pi f_0 t$$

$$= x(t)$$

$x(t) = e^{-at}$

$T_0 = \infty$



* A signal is said to be non peridic if it does not repeat.

$$x(t+T_0) = e^{-a(t+T_0)}$$

* In non periodic signal have period T_0 is equal to ∞

$$x(t+T_0) = e^{-a(t+\infty)}$$

$$= e^{-at} \cdot e^{\infty}$$

$$= 0$$

which is not equal to $x(t)$

Periodicity of $x_1(t) + x_2(t)$

$x(t) = x_1(t) + x_2(t)$

$x_1(t)$ will be periodic

$x_1(t) = x_1(t+T_1) = x(t+2T_1) = \dots$

$x_1(t) = x_1(t+mT_1)$ m is an integer

Similarly $x_2(t)$ will be periodic

$x_2(t) = x_2(t+T_2) = x(t+2T_2) = \dots$

$x_2(t) = x_2(t+nT_2)$

n is an integer

$x(t) = x_1(t) + x_2(t)$

$x(t)$ will be periodic

$mT_1 = nT_2 = T_0$

T_0 - period of $x(t)$

$$\frac{T_1}{T_2} = \frac{n}{m}$$ ie ratio of 2 integers

This is the condition for peridicity.

Problem what is the peridicity of the signal

$x(t) = \sin 100\pi t + \cos 150\pi t$

$x(t) = \sin 100\pi t + \cos 150\pi t$

compare

$2\pi f_1 t = 100\pi t$

$\Rightarrow f_1 = 50, \quad T_1 = \frac{1}{f_1} = \frac{1}{50}$

$2\pi f_2 t = 150\pi t$

$\Rightarrow f_2 = 75, \quad T_2 = \frac{1}{f_2} = \frac{1}{75}$

$\frac{T_1}{T_2} = \frac{1/50}{1/75} = \frac{3}{2}$ ie rational

$\therefore$ signal is periodic

The fundamental period

$T = 2T_1 = 3T_2$

$T = 2 \times \frac{1}{50} = 3 \times \frac{1}{75} = \frac{1}{25}$

$$\boxed{T = \frac{1}{25}}$$

② Find whether the signal $x(t) = 2\cos(10t+1) - \sin(4t-1)$ is periodic or not.

$\omega_1 = 10 \quad T_1 = \frac{2\pi}{\omega_1}$

$\omega_2 = 4 \quad = \frac{2\pi}{10}$

$\boxed{T_1 = \pi/5}$

$2\pi f_1 = \omega_1$

$f_1 = \frac{\omega_1}{2\pi}$

$T_1 = \frac{2\pi}{\omega_1}$

$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{4} = \pi/2$

$\boxed{T_2 = \pi/2}$

$T_1/T_2 = \frac{\pi/5}{\pi/2} = \frac{\cancel{\pi}}{5} \times \frac{2}{\cancel{\pi}} = 2/5$

which is rational hence signal is periodic.

③ Find the fundamental period T of the continuous Time signal $x(t) = 20\cos\left(10\pi t + \frac{\pi}{6}\right)$

$\omega = 10\pi$

$T = \frac{2\pi}{\omega} = \frac{2\cancel{\pi}}{10\cancel{\pi}} = 0.2$ Sec.

① Classify whether the signal is periodic or aper...
If periodic identify the period.

(a) $f(t) = \cos((2\pi/3)t) + 3\sin((2\pi/4)t)$

(b) $x(t) = 2\cos(3\pi t + (\pi/3)) - \sin(2\pi t)$

(a)

$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{2\pi/3} = 3$ $\omega_1 = 2\pi/3$ $\omega_2 = 2\pi/4$

$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{2\pi/4} = 4$

$\frac{T_1}{T_2} = \frac{3}{4}$ which is rational number

∴ hence signal is periodic

$T = 4T_1 = 3T_2 = 4 \times 3 = 3 \times 4 = 12$ sec.

(b)

$\omega_1 = 3\pi$ $\omega_2 = 2\pi$

$T_1 = \frac{2\pi}{\omega_1}$ $T_2 = \frac{2\pi}{\omega_2}$

$T_1 = \frac{2\pi}{3\pi} = 2/3$ $T_2 = \frac{2\pi}{2\pi} = 1$

$\frac{T_1}{T_2} = \frac{\frac{2}{3}}{1} = 2/3$ which is rational

$T = 3T_1 = 2T_2 = 3 \times 2/3 = 2$ sec.

Power ...

Energy and power of the signals :-

* Before that first define energy and power concepts.

* Average value

$$\langle x(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T_0/2}^{T_0/2} x(t)\,dt$$

$\langle\ \rangle$ - is the time average operator.

* Average value is also called as time mean or dc value of the signal.

* If the signal is periodic.

Average value of periodic signal $\Big\}$: $\langle x(t)\rangle = \dfrac{1}{T_0}\displaystyle\int_{-T_0/2}^{T_0/2} x(t)\,dt$

* Average normalised power :-

* If the signal $x(t)$ is the voltage across the resistor R

Instaneous power $= \dfrac{x^2(t)}{R}$ → voltage

* If the signal $x(t)$ is current signal thro' resistor R.

$$\left.\begin{array}{r}\text{instantaneous}\\ \text{power}\end{array}\right\} = x^2(t)\,R$$

take $R = 1$

<u>Average normalized power</u>

$$P = \langle x^2(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,dt$$

This eqn can be written as

$$P = \langle |x^2(t)| \rangle = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{T_0/2} |x(t)|^2\,dt.$$

* <u>Rms value</u>

Rms value of periodic signal is

$$X_{rms} = \left[\frac{1}{T_0} \int_0^{T_0} x^2(t)\,dt\right]^{1/2}$$

$$X_{rms} = [P]^{1/2} = \sqrt{P}$$

$$P = X^2_{rms}$$

Power signals:-

* If the signal $x(t)$ is said to be power signal if and only if the normalized power P is finite and non zero

Total normalized energy:-

* $$\left.\begin{array}{r}\text{Total normalized}\\ \text{energy}\end{array}\right\} = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

if $x(t)$ is real signal — if $x(t)$ is complex signal.

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} x^2(t)\,dt = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

* For energy signal,

* total normalized energy is finite and non zero.

Power of the energy signal :-

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \cdot \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \cdot E$$

$$= 0$$

The power of the energy signal is zero over infinite time

Energy of power signal :-

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)\, dt$$

$$= \lim_{T\to\infty} \left[T \cdot \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt \right]$$

$$= \lim_{T\to\infty} T \cdot \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)\, dt$$

$$= \lim_{T\to\infty} T \cdot P$$

$$= \infty$$

The energy of the power signal is infinite over infinite time.

Power signal :	Energy Signal
*. The normalized ~~average~~ power is finite and non zero.	* Total normalized energy is finite and non zero.
* $P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt$	$E = \lim_{T \to \infty} \int_{-T}^{T} x^2(t)\, dt$
* periodic signal are power signals	non periodic signals are energy signal
* Energy of the power signal is infinite over infinite Time.	power of the energy signal is zero over infinite time.

$$p(t) = v(t)\, i(t)$$

$$= v(t) \cdot \frac{v(t)}{R} = \frac{v^2(t)}{R}$$

$$p(t) = i^2(t) R$$

$$E = \lim_{T \to \infty} \int_{-T}^{T} i^2(t)\, dt \qquad R = 1$$

If signal is $x(t)$ The total energy is

$$\boxed{E = \lim_{T \to \infty} \int_{-T}^{T} x^2(t)\, dt.}$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)\, dt \text{ watts}$$

If signal is $x(t)$, The total power is

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\, dt$$

For discrete signal $x(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

Energy signal ~~x(t)~~

* A signal $x(t)$ is called an energy signal if the energy satisfies $0 < E < \infty$ and $P = 0$

power signal :-

* A signal $x(t)$ is called an power signal if the power satisfies $0 < P < \infty$ & $E = \infty$

① Determine the power and Rms value of the following signals.

(i) $x_1(t) = 5 \cos(50t + \pi/3)$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[5 \cos(50t + \pi/3)\right]^2 dt$$

$$= \lim_{T \to \infty} \frac{5^2}{2T} \int_{-T}^{T} \left[\frac{1 + \cos 2(50t + \pi/3)}{2}\right] dt$$

$$= \lim_{T \to \infty} \frac{5^2}{2 \times 2T} \left[\int_{-T}^{T} dt + \underbrace{\int_{-T}^{T} \cos 2(50t + \pi/3)\, dt}_{\downarrow\, 0}\right]$$

$$= \lim_{T \to \infty} \frac{5^2}{4T} \left[t\right]_{-T}^{T}$$

$$= \lim_{T\to\infty} \frac{5\cancel{0}^2}{4T}\left[T-(-T)\right]$$

$$= \lim_{T\to\infty} \frac{5\cancel{0}^2}{2\cancel{4}\cancel{T}} \cancel{2T}$$

$$= \frac{5\cancel{0}^2}{2}$$

$$= \cancel{25\,w}\ 12.5 \text{ watts}$$

Rms value of signal

$$X_{rms} = \sqrt{P}$$

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$= \sqrt{12.5}$$

$$=$$

(ii) $x_2(t) = 10\sin\left(50t + \frac{\pi}{4}\right) + 16\cos\left(100t + \frac{\pi}{3}\right)$

$$\text{power} = \frac{10^2}{2} + \frac{16^2}{2}$$

$$= 178 \text{ watts}$$

$$X_{rms} = \sqrt{178} = 13.341$$

(iii) $x_3(t) = 10\cos 5t \cos 10t$

$$= \frac{10}{2}\left[\cos 15t + \cos 5t\right]$$

$$= 5\left[\cos 15t + \cos 5t\right]$$

$$= 5\cos 15t + 5\cos 5t$$

power of the signal is

$$= \frac{5^2}{2} + \frac{5^2}{2}$$

$$= 25\,W.$$

$|e^{j\alpha t}| = 1$

$\cos^2 \Omega_0 t$

$= \frac{1+\cos 2x}{2}$

Rms value $= \sqrt{25} = 5.$

(iv) $x_4(t)\ e^{j\alpha t} \cdot \cos \Omega_0 t.$

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(e^{j\alpha t} \cos \Omega_0 t\right)^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{j2\alpha t} \cdot \cos^2 \Omega_0 t\ dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 1 \cdot \left(\frac{1+\cos 2\Omega_0 t}{2}\right) dt$$

$$= \lim_{T\to\infty} \frac{1}{4T} \left[\int_{-T}^{T} dt + \underbrace{\int_{-T}^{T} \frac{\cos 2\Omega_0 t}{2} dt}_{0}\right]$$

$$= \lim_{T\to\infty} \frac{1}{4T} \left[T - (-T)\right]$$

$$= \lim_{T\to\infty} \frac{1}{4\cancel{T}} 2\cancel{T} = \frac{1}{2}$$

$$x_s(t) = A e^{j\Omega_0 t}$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} A^2 e^{j2\Omega_0 t} \, dt$$

$$= \lim_{T \to \infty} \frac{A^2}{2T} \int_{-T}^{T} dt$$

$$= \frac{A^2}{\cancel{2T}} \cancel{2T}$$

$$= A^2$$

$$RMS = \sqrt{A^2} = A.$$

(2) Determine the value of power and energy for each of the signals.

(i) $x_1(n) = e^{j(\pi n/2 + \pi/8)}$

(ii) $x_2(n) = \left(\frac{1}{2}\right)^n u(n)$

(i) $x_1(n) = e^{j(\pi n/2 + \pi/8)}$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_1(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j\left[\pi n/2 + \pi/4\right]} \right|^2$$

$$e^{j\theta} = 1$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} (2N+1) = 1$$

$\therefore$ power is finite and nozero, energy of the signal will be infinite.

(ii) $x_2(n) = \left(\frac{1}{2}\right)^n u(n)$

$$x(n) = \begin{cases} \left(\frac{1}{2}\right)^n u(n) & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$$

$$1 + A + A^2 + A^3 + \cdots \frac{1}{1-A} \quad \text{if } |A| < 1$$

$$A = \frac{1}{2}$$

$$E = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

$$\boxed{E = 2}$$

3 Check wheter the signal ~~x(n)~~ = ~~Re~~

$$x(n) = R_e\{e^{jn\pi/12}\} + I_m(e^{jn\pi/18})$$

is periodic or not, it is periodic means Determine the period.

$$x(n) = R_e\{e^{jn\pi/12}\} + I_m(e^{jn\pi/18})$$

$$= R_e\left(\cos n\pi/12 + j\sin n\pi/12\right) + I_m\left(\cos n\pi/18 + j\sin n\pi/18\right)$$

$$x(n) = \cos\left(n\pi/12\right) + \sin\left(n\pi/18\right)$$

compare

$$x(n) = \cos(2\pi f_1 n) + \sin(2\pi f_2 n)$$

$$2\pi f_1 = \pi/12 \qquad 2\pi f_2 = \pi/18$$

$$f_1 = \frac{1}{24} = \frac{k}{N_1} \qquad f_2 = \frac{1}{36} = \frac{k}{N_2}$$

$$N_1 = 24 \qquad N_2 = 36$$

$$\frac{N_1}{N_2} = \frac{24}{36} = \frac{2}{3}$$ which is rational

$\therefore x(n)$ is periodic.

Fundamental period

$$3N_1 = 2N_2 = 2\times 36 = 3\times 24$$

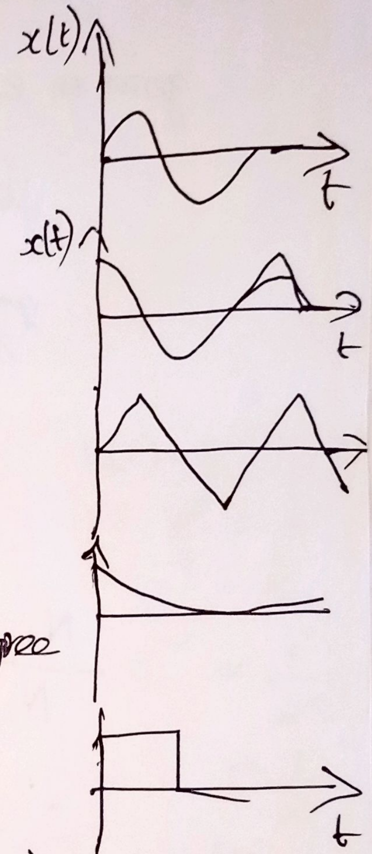
$$= 72 \text{ samples.}$$

Deterministic & Random signals.

* A deterministic signal is a signal which there is no uncertainty w.r.t its Value at any time. It can be completely represented by mathematical eqn at any time.

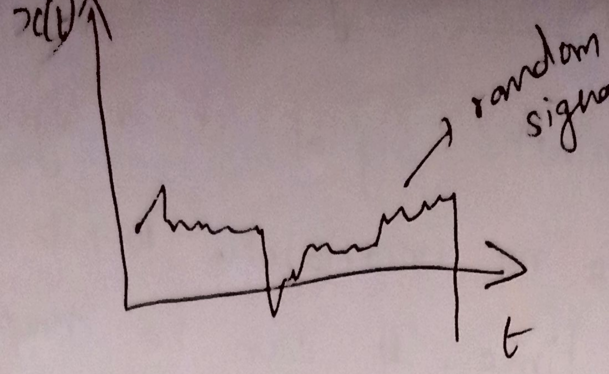
eg:- * Sine wave
* Cosine wave
* triangular wave
* exponential
* pulse etc.

$x(t) = 100 \sin 50t$.



Random signal:-

* A random signal has degree of uncertainity befor it ~~actu~~ actually occurs.
* It can not be defined by mathematical expressions.
* The value of random signal is not predefined
* it can not be calculated ~~from~~ from previous Value of the signal.



Assignment - 1

1 (i) $e^{j(2\pi/3)n} + e^{j(3\pi/4)n}$

general eqn $e^{j\omega n}$

$\omega = 2\pi f_1$

$2\pi f_1 = 2\pi/3$

$f_1 = \frac{1}{3} = \frac{k}{N_1}$

$N_1 = 3$

$2\pi f_2 = 3\pi/4$

$f_2 = \frac{3}{8} = \frac{k}{N_2}$

$N_2 = 8$

$\frac{N_1}{N_2} = \frac{3}{8}$ is rational

∴ It is periodic

$8N_1 = 3N_2 = 8 \times 3 = 3 \times 8 = 24$ samples.

(ii) $12 \cos(20n)$

$2\pi f_1 = 20$

$f_1 = \frac{20}{2\pi} = \frac{10}{\pi}$

$f_1 = \frac{k}{N_1}$ $N_1 = \pi$ which is not rational number

∴ it is a aperiodic.

Find which of the following signals are energy signals, power signals, neither energy or nor power signals.

(i) $x_1(t) = e^{-3t} u(t)$

(ii) $x_2(t) = e^{j(2t + \pi/4}$

(iii) $x_3(t) = \cos t$

(iv) $x_1(n) = (1/3)^n u(n)$

(v) $x_2(n) = e^{j(\pi/2 \, n + \pi/8)}$

(vi) $x_3(n) = \cos(\pi/4 \, n)$

(i) $x_1(t) = e^{-3t} u(t)$

$u(t) = 1$

$e^0 = 1$

$e^{\infty} = \infty$

$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x_1(t)|^2 \, dt$$

$$= \lim_{T \to \infty} \int_{-T}^{T} (e^{-3t})^2 \, dt$$

$$= \lim_{T \to \infty} \int_{-T}^{T} e^{-6t} \, dt = \lim_{T \to \infty} \left[\frac{e^{-6t}}{-6} \right]_{-T}^{T}$$

$$= \lim_{T \to \infty} -\frac{1}{6} e^{-6t} = -\frac{1}{6} \left[e^{-\infty} - e^{0} \right]$$

$$= \cancel{\lim_{T \to \infty}} -\frac{1}{6} [0 - \infty] = -\frac{1}{6} [0 - 1] = \frac{1}{6}$$

The paer of the signal

$$P = \lim_{T \to \alpha} \frac{1}{2T} \int_{-T}^{T} |x_1(t)|^2 dt$$

$$= \lim_{T \to \alpha} \frac{1}{2T} \int_{0}^{T} e^{-6t} dt$$

$$= \lim_{T \to \alpha} \frac{1}{2T} \left(-\frac{1}{6}\right) \left[e^{-6t}\right]_0^T$$

$$= \lim_{T \to \alpha} \frac{1}{-12T} \left[e^{-6T} - e^{-0}\right]$$

$$= \lim_{T \to \alpha} \frac{1}{-12(\alpha)} \left[0 - 1\right]$$

$$= \lim_{T \to \alpha} \frac{1}{\alpha}$$

$$= 0$$

$\therefore$ The energy of the signal is finite & power is zero $\therefore$

$x_1(t)$ is an energy signal.

(ii) $x_2(t) = e^{j(2t+\pi/4)}$

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x_2(t)|^2 \, dt \qquad e^{j(2t+\pi/4)} = 1$$

$$= \lim_{T \to \infty} \int_{-T}^{T} dt$$

$$= \lim_{T \to \infty} [t]_{-T}^{T}$$

$$= \lim_{T \to \infty} T-(-T)$$

$$= \lim_{T \to \infty} 2T$$

$$\boxed{E = \infty}$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x_2(t)|^2 \, dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{j(2t+\pi/4)} \cdot dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} [t]_{-T}^{T}$$

$$= \lim_{T \to \infty} \frac{1}{2T} [T-(-T)]$$

$$= \lim_{T \to \infty} \frac{1}{2T} [2T] = 1$$

The power of the signal is finite and Energy of the signal is ∞

$\therefore x_2(t)$ is a power signal.

(iii) $x_3(t) = \cos(t)$

$$E = \lim_{T \to \infty} \int_{-T}^{T} \cos^2 t \, dt$$

$$= \frac{1}{2} \lim_{T \to \infty} \int_{-T}^{T} (1 + \cos 2t) \, dt$$

$$= \frac{1}{2} \left[\lim_{T \to \infty} \left[\int_{-T}^{T} dt \right] + \underbrace{\lim_{T \to \infty} \int_{-T}^{T} \cos 2t \, dt}_{0} \right]$$

$$= \frac{1}{\cancel{2}} \lim_{T \to \infty} [\cancel{2}T]$$

$$= \infty$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \cos^2 t \, dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \frac{1}{2} \left[\int_{-T}^{T} dt + \int_{-T}^{T} \cos 2t \, dt \right]$$

$$= \lim_{T \to \infty} \frac{1}{4\cancel{T}} [2\cancel{T}] = \frac{1}{2}$$

Energy of the signal is ∞ & power is finite $\therefore$ It is power signal

(iv) $x_1(n) = \left(\frac{1}{3}\right)^n u(n)$

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \left(\left(\frac{1}{3}\right)^n\right)^2$$

$$= \lim_{N\to\infty} \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n$$

$$E = \frac{1}{1-\frac{1}{9}} = \frac{1}{\frac{9-1}{9}} = \frac{9}{8}$$

$$\boxed{E = \frac{9}{8}}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n$$

$$= \lim_{N\to\infty} \frac{1}{2\infty+1} \times \frac{9}{8}$$

$$= 0$$

power is zero & Energy is finite value

$\therefore$ It is an energy signal

(v) $x_2(n) = e^{j(\pi n/2 + \pi/8)}$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (x(n))^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|e^{j(\pi n/2 + \pi/8)}\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \lim_{N\to\infty} \frac{1}{\cancel{2N+1}} \cancel{(2N+1)}$$

$$= 1$$

power is finite value means energy is ∞

$\therefore$ It is power signal.

(v) $x_3(n) = \cos\left(\frac{\pi}{4}n\right)$

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \sum_{n=-N}^{N} \cos^2\left(\frac{\pi}{4}n\right)$$

$$= \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1+\cos 2\left(\frac{\pi}{4}n\right)}{2}$$

$$= \frac{1}{2} \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1+\cos\left(\frac{\pi}{2}n\right)\right)$$

$$= \frac{1}{2} \lim_{N\to\infty} \sum_{n=-N}^{N} 1+\cos\left(\frac{\pi}{2}N\right)$$

$$= \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos^2 \frac{\pi}{4} n$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1+\cos 2\left(\frac{\pi}{4}\right)n}{2}$$

$$= \frac{1}{2} \lim_{N\to\infty} \frac{1}{2N+1} \left[\sum_{n=-N}^{N} 1 + \sum_{n=-N}^{N} \cos 2\left(\frac{\pi}{4}\right)n\right]$$

$$= \frac{1}{2} \lim_{N\to\infty} \frac{1}{2N+1} \left[\sum_{n=-N}^{N} 1 + \sum_{n=-N}^{N} \cos\left(\frac{\pi}{2}n\right)\right]$$

$$= \frac{1}{2} \lim_{N\to\infty} \frac{1}{\cancel{2N+1}} (\cancel{2N+1}) + 0$$

$$P = \frac{1}{2}$$

Power is finite and energy is ∞

$\therefore$ It is power signal.

Determine the energy and power of the following signals.

(i) $x(t) = t\,u(t)$ u(t) = unit step

(ii) $x(n) = 2e^{j3\pi n}$

(i) $$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt$$

$$= \lim_{T\to\infty} \int_{0}^{T} t^2\, dt$$

$$= \lim_{T\to\infty} \left[\frac{t^3}{3} \right]_0^T$$

$$= \lim_{T\to\infty} \left[\frac{T^3}{3} \right]$$

$$= \infty.$$

(ii) $$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} t^2\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[\frac{t^3}{3} \right]_0^T$$

$$= \lim_{T\to\infty} \frac{1}{2\cancel{T}} \left[\frac{T^{\cancel{3}\,2}}{3} \right]$$

$$= \lim_{T\to\infty} \frac{T^2}{6} = \infty.$$

Energy and power of the signal are ∞

$\therefore$ signal is neither energy nor power signal!

(ii) $x(n) = 2e^{j3\pi n}$.

$$E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \sum_{n=-N}^{N} 4e^{j6\pi n}$$

$$= 4 \lim_{N \to \infty} \sum_{n=-N}^{N} 1$$

$$= 4 \lim_{N \to \infty} 2N+1$$

$$= \infty.$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |e^{j3\pi n}|^2$$

$$= 4 \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= 4 \lim_{N \to \infty} \frac{1}{2\cancel{N}+1} \, 2\cancel{N}+1$$

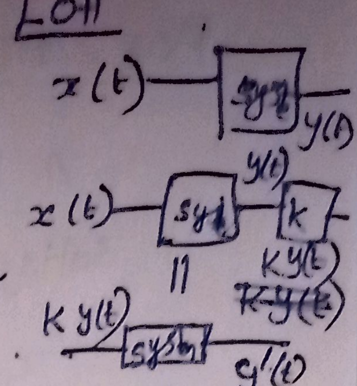
$$= 4$$

power is finite

Enery is ∞

$\therefore$ The signal is power signal.

CT systems :-



CT systems classified as

* Linear & non linear.
* Time invariant & time variant
* casual & non casual
* Stable & un stable.

Linear & non Linear systems :-

For CT :-

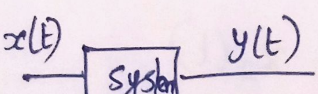
A system said to be Linear if Superposition theorem applies to that system.
↳ Law of additivity
↳ Law of Homogeneous

$y_1(t) = f[x_1(t)]$ ie $x_1(t)$ is excitation
$y_1(t)$ is response

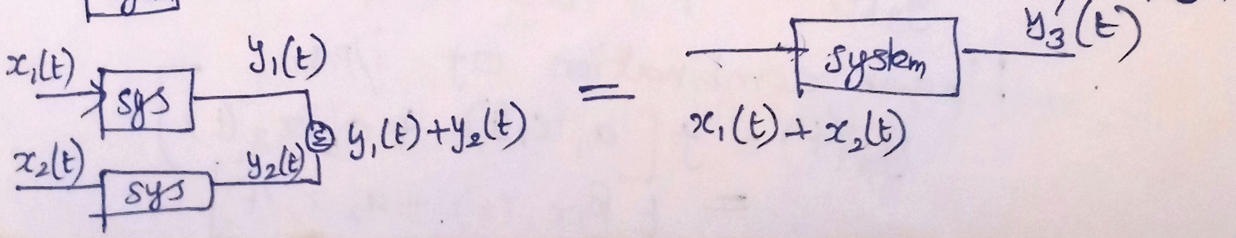
$y_2(t) = f[x_2(t)]$ ie $x_2(t)$ is excitation
$y_2(t)$ is response

For linear system

$$f[a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)$$



a_1 & a_2 are constants.



For DT systems

A system is said to be Linear if it satisfies the super position principle

$x_1(n)$ & $x_2(n)$ be the 2 i/p sequences

$$T\{a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)]$$

a_1, a_2 - are constants.

① Check whether the following systems are Linear or non linear

(i) $y(t) = t\,x(t)$

(ii) $y(t) = x^2(t)$

(iii) $\dfrac{d\,y(t)}{dt} + 3t\,y(t) = t^2 x(t)$

$$y_1(t) = f(x_1(t)) = t\,x_1(t)$$
$$y_2(t) = f(x_2(t)) = t\,x_2(t)$$

Hence Linear combination of o/ps

$$y_3(t) = a_1 y_1(t) + a_2 y_2(t)$$
$$y_3(t) = a_1\, t\, x_1(t) + a_2\, t\, x_2(t)$$

Linear combination of i/ps.

$$y_3'(t) = f[a_1 x_1(t) + a_2 x_2(t)]$$
$$= t\,[a_1 x_1(t) + a_2 x_2(t)]$$

$y_3'(t) = a_1 t x_1(t) + a_2 t x_2(t)$

$y_3(t) = y_3'(t)$

$\therefore$ It is Linear system.

(ii) $y(t) = x^2(t)$

The o/p of the system to two i/ps become

$y_1(t) = f[x_1(t)] = x_1^2(t)$

$y_2(t) = f[x_2(t)] = x_2^2(t)$

Linear combination of o/p are

$y_3(t) = a_1 y_1(t) + a_2 y_2(t)$

$= a_1 x_1^2(t) + a_2 x_2^2(t)$

$y_3'(t) = f[a_1 x_1(t) + a_2 x_2(t)]$

$= [a_1 x_1(t) + a_2 x_2(t)]^2$

$= a_1^2 x_1^2(t) + a_2^2 x_2^2(t) + 2a_1 a_2 x_1(t) x_2(t)$

$y_3(t) \neq y_3'(t)$

$\therefore$ This is non linear system.

(iii) $\frac{dy(t)}{dt} + 3t\,y(t) = t^2 x(t)$

The two i/ps are

$$\frac{dy_1(t)}{dt} + 3t\,y_1(t) = t^2 x_1(t)$$

$$\frac{dy_2(t)}{dt} + 3t\,y_2(t) = t^2 x_2(t)$$

Step 2 :- multiplying 1^{st} eqn by a_1 & 2^{nd} eqn by a_2 and adding them

$$a_1\left[\frac{dy_1(t)}{dt} + 3t\,y_1(t)\right] + a_2\left[\frac{dy_2(t)}{dt} + 3t\,y_2(t)\right]$$
$$= a_1 t^2 x_1(t) + a_2 t^2 x_2(t)$$

$$\frac{d}{dt}\left[a_1 y_1(t) + a_2 y_2(t)\right] + 3t\left[a_1 y_1(t) + a_2 y_2(t)\right]$$
$$= t^2\left[a_1 x_1(t) + a_2 x_2(t)\right]$$

Step 3 :- Let us express the system eqn for linear combination of 2 i/ps.

~~$a_1 y_1(t) + a_2 y_2(t)$~~

$$\therefore \frac{d}{dt}\left[a_1 y_1(t) + a_2 y_2(t)\right] + 3t\left[a_1 y_1(t) + a_2 y_2(t)\right]$$
$$= t^2\left[a_1 x_1(t) + a_2 x_2(t)\right]$$

$\because$ The above 2 eqns are similar

$\therefore$ The system is Linear.

Determine whether the following systems

(i) $y(n) = x(n^2)$

(ii) $y(n) = x^2(n)$

(iii) $y(n) = 2x(n) + \frac{1}{x(n-1)}$

are linear (or) non linear.

$$y(n) = x(n^2)$$

$$y_1(n) = x_1(n^2)$$

$$y_2(n) = x_2(n^2)$$

The response of the system to the linear combination of $x_1(n)$ & $x_2(n)$ will be

$$y_3(n) = T[a_1 x_1(n^2) + a_2 x_2(n^2)]$$

Since the linear systems satisfy additive property

$$y_3'(n) = T[a_1 x_1(n^2)] + T[a_2 x_2(n^2)]$$

$y(t) = x(2t)$

Eg: $y_1(t) = x_1(2t)$

$y_2(t) = x_2(2t)$

$a y_1(t) + b y_2(t) = a x_1(2t) + b x_2(2t)$ — ①

$y'(t) = a y_1(t) + b y_2(t)$

$y'(t) = a x_1(2t) + b x_2(2t)$ — ②

eqn ① & eqn ② are equal

∴ It is linear system.

$x(t) = \frac{1 + \cos 2(2\pi t)}{2}$

$= \frac{1}{2} + \frac{\cos 4\pi t}{2}$

$2\pi f_1 = \frac{4\pi}{2}$

$f_1 = \frac{2\pi}{2\pi} = 1$

$T = \frac{1}{1} = 1/1$ (ra

$T = 1$ sec

① Determine whether the following systems are Linear or not

$$y(n) = 2x(n) + \frac{1}{x(n-1)}$$

$$y_1(n) = 2x_1(n) + \frac{1}{x_1(n-1)}$$

$$y_2(n) = 2x_2(n) + \frac{1}{x_2(n-1)}$$

$$y_3(n) = ay_1(n) + by_2(n) = 2ax_1(n) + \frac{a}{x_1(n-1)} + 2bx_2(n) + \frac{b}{x_2(n-1)}$$

$$y_3'(n) = T\left[ax_1(n) + bx_2(n)\right]$$

$$= 2\left[ax_1(n) + bx_2(n)\right] + \frac{1}{ax_1(n-1) + bx_2(n-1)}$$

$$y_3(n) \neq y_3'(n)$$

$\therefore$ The system is non Linear.

② $y(n) = n\,x(n)$

$$y_1(n) = n\,x_1(n)$$

$$y_2(n) = n\,x_2(n)$$

The o/p due to weighted sum of i/p is

$$y_3(n) = a n x_1(n) + b \cdot n x_2(n)$$

The weighted sum of o/p is

$y_3'(n) = a\,n x_1(n) + b\,\cancel{n} n x_2(n)$

$= a n x_1(n) + b n x_2(n)$

$y_3(n) = y_3'(n)$

$\therefore$ It is Linear.

(3) check the following for Linearity

$y(n) = x(n) + n x(n+1)$

$y_1(n) = x_1(n) + n x_1(n+1)$

$y_2(n) = x_2(n) + n x_2(n+1)$

The o/p due to weighted Sum of two inputs

$y_3(n) = a_1 x_1(n) + a_1 n x_1(n+1) + a_2 x_2(n)$

$+ a_2 n x_2(n+1)$

The weighted sum of o/p is

$y_3'(n) = a_1 x_1(n) + a_2 x_2(n) + a_1 n x_1(n+1)$

$+ a_2 n x_2(n+1)$

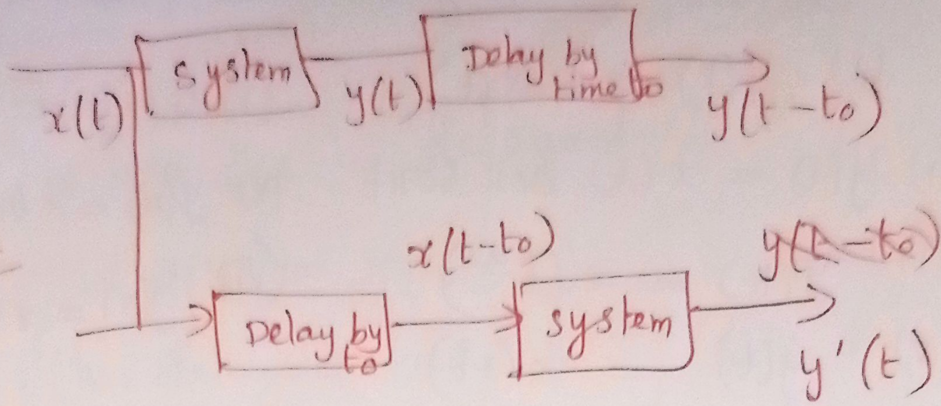
$\therefore y_3 = y_3'$

$\therefore$ It is ~~Line~~ Linear

Time invariant and time variant systems

step 1

step 2



1st case

$$y'(t) = y(t-t_0)$$

Time invariant

$$y'(t) \neq y(t-t_0)$$

Time varient.

Time invariant system is a system in which any delay provided in the input must be reflected in outputs.

eg ① $y(t) = x(2t)$

Step 1

$$y(t-t_0) = x(2(t-t_0)) = x(2t-2t_0)$$

Step 2

$$x(t-t_0) = x(2t-t_0)$$

Both eqns are not same

$\therefore$ It is Time variant.

① For each of the following systems, determine whether or not the system is time invariant.

(i) $y(t) = t\,x(t)$ (v) $y(n) = x(2n)$

(ii) $y(t) = x(t)\cos 50\pi t$ (vi) $y(n) = x(n) + n\,x(n-1)$

(iii) $y(t) = x(t^2)$ (vii) $y(n) = x^2(n$

(iv) $y(t) = x(-t)$

(i) $y(t) = t\,x(t)$

The o/p due to delayed i/p is

$$y(t-t_0) = t\,x(t-t_0) \quad — ①$$

If the o/p is delayed by t_0

$$y(t-t_0) = (t-t_0)\,x(t-t_0) — ②$$

$$\text{eqn } ① \neq \text{eqn } ②$$

∴ The system is time variant.

(ii) $y(t) = x(t)\cos(50\pi t)$

The i/p is delayed by T sec, the o/p is

$$y(t,T) = T[x(t-T)] = x(t-T_0)\cos(50\pi t)$$

The o/p is delayed by T sec.

$$y(t-T) = x(t-T)\cos[50\pi(t-T)]$$

$$y(t,T) \neq y(t-T)$$

$\therefore$ It is time variant.

(iii) $y(t) = x(t^2)$

The o/p due to delayed i/p is

$$y(t,T) = x(t^2 - T)$$

The delayed output

$$y(t-T) = x\left[(t-T)^2\right]$$

$$y(t,T) \neq y(t-T)$$

$\therefore$ The system is time variant.

(iv) $y(t) = x(-t)$

The o/p due to delaye i/p is

$$y(t,T) = y(-t-T)$$

The delayed o/p is

$$y(t-T) = y(-(t-T))$$

$$= y(-t+T)$$

$$y(t,T) \neq y(t-T)$$

$\therefore$ It is time variant.

(v) $y(t) = e^{x(t)}$

The output due ~~to~~ delayed input is

$$y(t,T) = e^{x(t-T)}$$

The delayed output is

$$y(t-T) = e^{x(t-T)}$$

$$y(t,T) = e^{x(t-T)}\; y(t-T)$$

$\therefore$ The system is time invariant

(vi) $y(n) = x(2n)$

If the o/p due to delayed input is

$$y(n,k) = x(2n-k)$$

The o/p delayed by k units of time

$$y(n-k) = x(2(n-k))$$

$$= x(2n-2k)$$

$$y(n,k) \neq y(n-k)$$

$\therefore$ It is Time variant.

(vii) $y(n) = x(n) + n\,x(n-1)$

The o/p due to delayed input is

$y(n,k) = x(n-k) + n\,x(n-k-1)$

The delayed output is

$y(n-k) = x(n-k) + (n-k)\,x(n-k-1)$

$y(n,k) \neq y(n-k)$

∴ The system is time variant.

(viii) $y(n) = x^2(n-1)$

The output due to delayed input

$y(n,k) = x^2(n-k-1)$

The delayed o/p is

$y(n-k) = x^2(n-k-1)$

∴ $y(n,k) = y(n-k)$

∴ The system is time invariant.

① For the given signals, determine whether it is periodic (or) non periodic signal & if periodic find the fundamental period.

(a) $x(t) = 5\cos(200\pi t)$

(b) $x(n) = 12\sin(25\pi n)$

(c) $x(n) = 9\cos(25n)$

(d) $x(t) = \cos^2(2\pi t)$

(e) $x(t) = 4\cos(\pi/100\, t) + 2\cos\left(\frac{2\pi}{180} t\right)$

(a) $x(t) = 5\cos(200\pi t)$

$$w_0 = 200\pi$$

$$2\pi f_0 = 200\pi$$

$$f_0 = \frac{200\cancel{\pi}}{2\cancel{\pi}} = 100$$

$$f_0 = 100$$

$$T_0 = 1/100$$ (rational number) so. it is periodic.

$$T_0 = 0.01$$

$$\boxed{T_0 = 10ms}$$

(b) $x(n) = 12\sin(25\pi n)$

$$W_0 = 25\pi$$

$$2\pi f_0 = 25\pi$$

$$f_0 = \frac{25\pi}{2\pi} = 25/2$$ ie k/N ratio of integers

$\therefore$ $x(n)$ is periodic signal

$N = 2$ samples.

(3) $x(n) = 9 \cos(25n)$

$\omega_0 = 25$

$2\pi f_0 = 25$

$f_0 = \frac{25}{2\pi} \neq \frac{k}{N}$ is not rational number

$\therefore$ It is non periodic signal

(4) $x(t) = \cos^2(2\pi t)$

$(1+\cos 2\theta) = 2\cos^2\theta$

~~$x(t) = 2\cos(1+$~~

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$x(t) = \frac{1+\cos 2(2\pi t)}{2}$

$= \frac{1}{2} + \frac{\cos 4\pi t}{2}$

$= 0.5 + 0.5 \cos 4\pi t$

$\downarrow$

does not have freq. component

$\omega_0 = 4\pi$

$2\pi f_0 = 4\pi$

$f_0 = 4\pi / 2\pi = 2$ Hz.

$T_0 = 1/2$ it is a rational number.

∴ It is periodic

$$\boxed{T_0 = 0.5 \text{ sec}}$$

(5) $x(t) = 4\cos\left(\frac{\pi}{100}t\right) + 2\cos\left(\frac{2\pi}{180}t\right)$

$\omega_1 = \pi/100$

$2\pi f_1 = \pi/100$

$f_1 = \frac{\cancel{\pi}}{2\cancel{\pi} \times 100}$

$f_1 = \frac{1}{200}$

$T_1 = 200$

$\omega_2 = \frac{2\pi}{180}$

$2\pi f_2 = \frac{2\pi}{180}$

$f_2 = \frac{\cancel{2\pi}}{\cancel{2\pi} \times 180}$

$= \frac{1}{180}$

$T_2 = 180$

$\frac{T_1}{T_2} = \frac{\cancel{200}^{10}}{\cancel{180}_{9}} = 10/9$ It is a rational number.

∴ It is a periodic signal.

$\frac{T_1}{T_2} = 10/9$

$T_1 9 = 10 T_2$

$T = 9 \times 200$

$$\boxed{T = 1800 \text{ sec}}$$

(2) Determine fundamental period of the given signal.

(a) $x(t) = \sin^2(4\pi t)$

(b) $x(t) = \sin 6\pi t + \cos 5\pi t$.

(a) $x(t) = \sin^2 4\pi t$

$x(t) = \frac{1 - \cos 2(4\pi t)}{2}$ $\quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$= \frac{1}{2} - \frac{\cos 8\pi t}{2}$

$= \underbrace{0.5}_{\text{DC component}} - 0.5 \cos 8\pi t$

$w_0 = 8\pi$

$2\pi f_0 = 8\pi$

$f_0 = \frac{\cancel{8\pi}^{4}}{\cancel{2\pi}} = 4$

$T_0 = 1/4$ it is rational number

∴ It is periodic signal

$\boxed{T_0 = 0.25 \text{ sec}}$

(b) $x(t) = \sin 6\pi t + \cos 5\pi t$

$w_1 = 6\pi$

$2\pi f_1 = 6\pi$

$f_1 = 6\cancel{\pi}/2\cancel{\pi} = 3$ $\quad T_1 = 1/3$

$w_2 = 5\pi$

$2\pi f_2 = 5\pi$

$f_2 = 5\cancel{\pi}/2\cancel{\pi} = 5/2$

$T_2 = 2/5$

$T_1/T_2 = \frac{1/3}{5/2}$

$= \frac{1}{3} \times \frac{5}{2}$

$\frac{T_1}{T_2} = \frac{5}{6}$ (It is a rational number)

$\therefore$ It is a periodic

$T = T_1 \cdot 6$

$= \frac{1}{3} \times 6$

$= \frac{6}{3}$

$\boxed{T = 2 \text{ sec.}}$

$\boxed{T_0 = 2 \text{ sec}}$

$f = K/N \rightarrow$ rational number

It is periodic

$f = K/N \Rightarrow$ irrational no...

It is non periodic

(d) Find the periodicity of the signal.

(i) $x(t) = \sin^2(400\pi t)$

(ii) $x(t) = \cos(2t) + \sin(3t)$

(iii) $x(t) = \sin(4\pi t) + \sin 5t$.

(i) $x(n) = \sin(3n)$

(ii) $x(n) = \cos(0.3\pi n + \pi/4)$

(iii) $x(n) = \sin\left(\frac{7\pi}{37} n\right)$

(i) $x(n) = \sin 3n$

$\omega_0 = 3$

$2\pi f_0 = 3$

$f_0 = 3/2\pi = \frac{K}{N}$ is not rational number

$\therefore$ It is non periodic signal.

(ii) $x(n) = \cos(0.3\pi n + \pi/4)$

$\omega_0 = 0.3\pi$

$2\pi f_0 = 0.3\pi$

$f_0 = \frac{0.3\cancel{\pi}}{2\cancel{\pi}} = \frac{0.3}{2} = \frac{K}{N} (or) \frac{3}{20}$

It is a rational no..

$\therefore$ It is Periodic

N = 20 samples/cycle.

(iii) $x(n) = \sin\left(\frac{7\pi}{37} n\right)$

$\omega_0 = 7\pi/37$

$$2\pi f_0 = \frac{7\pi}{37}$$

$$f_0 = \frac{7\cancel{\pi}}{37 \times 2\cancel{\pi}} = \frac{7}{74} = k/N$$ It is a rational numb

$\therefore$ It is periodic

$N = 74$ samples.

$$H\{ax_1(t) + bx_2(t)\} = aH\{x_1(t)\} + bH\{x_2(t)\}$$

① $y(t) = x(t-t_0)$

$$H\{ax_1(t) + bx_2(t)\} = ax_1(t-t_0) + bx_2(t-t_0) \quad \text{— ①}$$

$$aH[x_1(t)] \cancel{+ bH[x_2(t)]} = ax_1(t-t_0)$$

$$bH[x_2(t)] = bx_2(t-t_0)$$

$$aH[x_1(t)] + bH[x_2(t)] = ax_1(t-t_0) + bx_2(t-t_0) \quad \text{— ②}$$

$$① = ②$$

$\therefore$ It is Linear.

② $y(n) = x(n) + u(n+1)$

$$H[ax_1(n) + bx_2(n)] = ax_1(n) + bx_2(n) + u(n+1) \quad \text{— ①}$$

$$aH\{x_1(n)\} = ax_1(n) + u(n+1)$$

$$bH\{x_2(n)\} = bx_2(n) + u(n+1)$$

$$aH\{x_1(n)\} + bH[x_2(n)] = ax_1(n) + u(n+1) + bx_2(n) + u(n+1) \quad \text{— ②}$$

$$① \neq ②$$

system is non linear.

① $y(t) = x(t) + \frac{1}{x(t-4)}$

Casual $\Rightarrow$ o/p depends on present
(or) past values of the i/p.

non casual $\Rightarrow$ o/p depends on future value
of the i/p.

$x(n)$ $\downarrow$ present

$\left.\begin{matrix} x(n-1) \\ x(n-2) \\ x(n-3) \\ \vdots \end{matrix}\right\}$ past values.

$\left.\begin{matrix} x(n+1) \\ x(n+2) \\ \vdots \end{matrix}\right\}$ Future value.

① $y(t) = \underbrace{x(t)}_{\text{present}} + \frac{1}{x(t-4)} \Rightarrow$ past

$\therefore$ It is casual system.

② $y(n) = \underbrace{x(n)}_{\text{present}} + 3\,\underbrace{x(n+4)}_{\text{Future}}$

$\therefore$ It is non casual system

③ $y(n) = \cancel{x(n)} \cdot \sum_{k=-\infty}^{1} \underbrace{x(n+k)}_{\text{Future value}}$

$\therefore$ It is non casual system.

(4) $y(n) = x(n) + u(n+1)$

↓

present value

∴ It is ~~non~~ casual system.

Stable & un stable :

⇒ For a stable system o/p should be bounded for bounded i/p at each and every instant of time.

BIBO ⟹ $-\alpha$ to α

i/p Amplitude is finite

&

o/p Amplitude is finite.

Bounded signal :- dc, sint, cost

-1 to 1, -1 to $+1$

$u(t)$

BIBO satisfied means ⟹ stable system

BIBO not satisfied ⟹ un stable.

$$\sum_{n=-\alpha}^{\alpha}$$

(1) $y(t) = \sin[x(t)]$

Static or dynamic

A continuous or discrete time system is said to be static (or) memory less system if the o/p at any instant of time depends on the i/p at that instant only

Otherwise it is called Dynamic.

(i) $y(t) = x(t-3)$

$y(0) = x(-3)$

↓ depends on past values of i/P.

$\therefore$ system is called dynamic

(ii) $y(t) = x(2t)$

$y(0) = x(0)$

$y(1) = x(2)$

depends on future value of i/P

$\therefore$ It is called dynamic

(iii) $y(n) = x^2(n)$

$y(0) = x^2(0)$

$y(1) = x^2(1)$

$y(2) = x^2(2)$

$\therefore$ It is static (or) memory less ~~dynamic~~ system.

(iv) $y(n) = x(n+2)$

$y(0) = x(2)$

↓ depends on Future values

$\therefore$ It is dyamic system.

(v) $y(n) = x(n-2) + x(n)$

$y(0) = x(-2) + x(0)$

$\uparrow$ past values of i/p

$\uparrow$ on istant

∴ It is Dynamic

(vi) $y(t) = \frac{d^2 x(t)}{dt} + 2x(t)$

differencial eqn. means system is dynamic systen.

Stable & unstable problems.

Type 1. BIBO

~~$y(t) = \cos[x(t)]$~~

condition for stability for an LTI-CT system

$$\int_{-\infty}^{\infty} h(t)\, dt < \infty.$$

For LTI-DT system

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty.$$

Check whether the following system are stable ~~or~~ or not.

(i) $= t\,x(t)$

(ii) $y(t) = 5x(t) + 3$

(iii) $h(n) = u(n)$

(iv) $h(t) = e^{-4|t|}$

Unit-II Analysis of continuous Time signals.

⟹ Fourier series for periodic signals.
⟹ Fourier transform
⟹ properties
⟹ Laplace transform
⟹ properties.

Fourier series representation of periodic signals

* For eg:-
sinusoidal signal $x(t) = A \sin \Omega_0 t$
with period $T = \frac{2\pi}{\Omega_0}$

* $x(t) = a_0 + a_1 \cos(\Omega_0 t) + a_2 \cos(2\Omega_0 t)$

$+ a_3 \cos 3\Omega_0 t + \ldots a_k \cos k\Omega_0 t$

$+ b_1 \sin(\Omega_0 t) + b_2 \cos(2\Omega_0 t)$

$+ b_3 \cos(3\Omega_0 t) + \ldots b_k \sin(k\Omega_0 t)$

$= a_0 + \sum_{n=1}^{k} a_n \cos(n\Omega_0 t) + b_n \sin(n\Omega_0 t)$

where $a, a_1, a_2 \ldots a_k$ and $b_1, b_2, b_3 \ldots b_k$ are constants.

Ω_0 - fundamental freq.

* If signal $x(t)$ is to be periodic, it has to satisfy the condition.

$$x(t+T) = x(t)$$

$$x(t+T) = a_0 + \sum_{n=1}^{K} \left[a_n \cos n\Omega_0(t+T) + b_n \sin n\Omega_0(t+T) \right]$$

$$= a_0 + \sum_{n=1}^{K} \left[a_n \cos(n\Omega_0 t + 2n\pi) + b_n \sin(n\Omega_0 t + 2n\pi) \right]$$

$$= a_0 + \sum_{n=1}^{K} \left[a_n \cos(n\Omega_0 t) + b_n \sin(n\Omega_0 t) \right]$$

$$= x(t)$$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$x(t) = a_0 + \sum_{n=1}^{K} \left[a_n \cos(n\Omega_0 t) + b_n \sin(n\Omega_0 t) \right]$$

a_0 – dc component

a_n, b_n – constants → fourier coefficients.

Evaluation of fourier coefficients :-

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 \int_{t_0}^{t_0+T} dt + \int_{t_0}^{t_0+T} \sum_{n=1}^{\alpha} \left[a_n \cos(n\Omega_0 t) + b_n \sin(n\Omega_0 t)\, dt \right]$$

$$= a_0 T + \sum_{n=1}^{\alpha} a_n \int_{t_0}^{t_0+T} \cos(n\Omega_0 t)\,dt + b_n \int_{t_0} \sin(n\Omega_0 t)\,dt$$

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 T$$

$$\boxed{a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\,dt}$$

To Evaluate a_n b_n .

$$\int_{t_0}^{t_0+T} \sin(n\Omega_0 t) \cos(m\Omega_0 t)\,dt = 0 \text{ for all } m,n$$

$$\int_{t_0}^{t_0+T} \cos(n\Omega_0 t) \cos(m\Omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \neq 0 \end{cases}$$

$$\int_{0}^{T_0+T} \sin(n\Omega_0 t) \sin(m\Omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \neq 0 \end{cases}$$

To find fourier coefficients a_n find period

by $\cos(m\Omega_0 t)$ and integrate over one period

$$\int_{t_0}^{t_0+T} x(t)\cos(m\Omega_0 t)\,dt = a_0\int_{t_0}^{t_0+T}\cos(m\Omega_0 t)\,dt$$

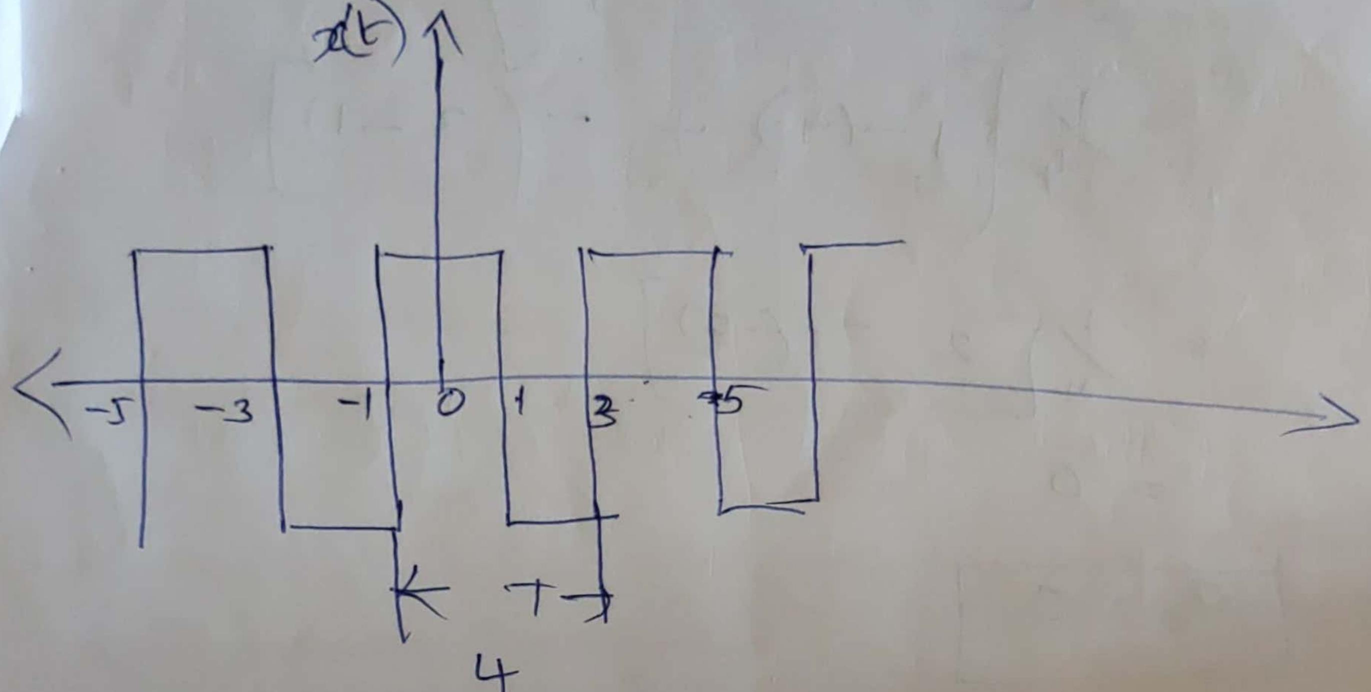
$$+\sum_{n=1}^{\alpha} a_n\left[\int_{t_0}^{t_0+T}\cos(n\Omega_0 t)\cos(m\Omega_0 t)\,dt\right]+$$

$$\sum_{n=1}^{\alpha} b_n\left[\int_{t_0}^{t_0+T}\sin(n\Omega_0 t)\cos(m\Omega_0 t)\,dt\right]$$

$$\boxed{a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\Omega_0 t)\,dt}$$

$$\boxed{b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(n\Omega_0 t)\,dt.}$$

Find the trigonometric fourier series for the periodic signal $x(t)$



$T = 4$

$t = -1$ to $t = 3$ (or) 0 to 4

$\Omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \pi/2$

$$x(t) = \begin{cases} 1 & \text{for } -1 < t < 1 \\ -1 & \text{for } 1 \leq t \leq 3 \end{cases}$$

$$a_0 = \frac{1}{T} = \frac{1}{4} \int_{-1}^{3} x(t)\,dt$$

$$= \frac{1}{4}\left[\int_{-1}^{1} dt + \int_{1}^{3} -dt\right.$$

$$= \frac{1}{4}\left[[t]_{-1}^{1} + [-t]_{1}^{3} \right]$$

$$= \frac{1}{4}\left[1-(-1) + -(3-1) \right]$$

$$= \frac{1}{4}\left[2 + (-2) \right]$$

$$= 0$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{2}{T}\int_{t_0}^{t+T_0} x(t) \cos n(\Omega_0 t)\, dt$$

$$= \frac{2}{T}\int_{-1}^{3} x(t) \cos\left(\frac{n\pi t}{2}\right) dt$$

$$= \frac{2}{T}\left[\int_{-1}^{1} \cos\left(\frac{n\pi}{2} t\right) dt + \int_{1}^{3} (-1) \cos\left(\frac{n\pi}{2}\right) dt \right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin \frac{n\pi}{2} t}{n\pi/2} \right)_{-1}^{1} - \left(\frac{\sin \frac{n\pi}{2}}{n\pi/2} \right)_{1}^{3} \right.$$

$$= \frac{1}{T}\left\{ \frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right) - \sin\frac{3\pi}{2} + \sin\frac{n\pi}{2} \right.\right.$$

⟹ Any periodic fn of time $f(t)$ can be represented by an infinite series is called fourier series.

⟹ Fourier series analysis is also called as hormonic analysis.

⟹ periodic waveform may be expressed in the form of fourier series.

⟹ Non periodic waveforms may be expressed in the form of fourier transform.

Types of fourier series.

1. Trignometric (or) quadrature fourier series.
2. polar fourier series.
3. exponential fourier series.

Quadrature / Trigonometric Fourier Series

want

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_0}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{T_0}\right)$$

$$a_0 = \frac{2}{T_0} \int_{t}^{t+T_0} x(t)\, dt$$

want

$$a_n = \frac{2}{T_0} \int_{t}^{t+T_0} x(t) \cdot \cos\left(\frac{2\pi n t}{T_0}\right) dt$$

$$b_n = \frac{2}{T_0} \int_{t}^{t+T_0} x(t) \cdot \sin\left(\frac{2\pi n t}{T_0}\right) dt.$$

Polar Fourier Series:

$$x(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos\left(\frac{2\pi n t}{T_0} \cancel{+} - \phi_n\right)$$

$$D_0 = a_0 = \frac{1}{T_0} \int_{t}^{t+T_0} x(t)\, dt.$$

$$D_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = -\tan^{-1} b_n / a_n$$

prove the polar Fourier series relations from quradrature fourier series.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_0}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T_0}\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right)\right]$$

~~Use~~ std trigonometric identity

$$a\cos x + b\sin x = \sqrt{a^2+b^2}\cos\left(x - \tan^{-1}\frac{b}{a}\right)$$

$a = a_n \quad b = b_n \quad x = \frac{2\pi nt}{T_0}$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2}\cos\left[\left(\frac{2\pi nt}{T_0}\right) - \tan^{-1}\frac{b_n}{a_n}\right]$$

$D_0 = a_0$

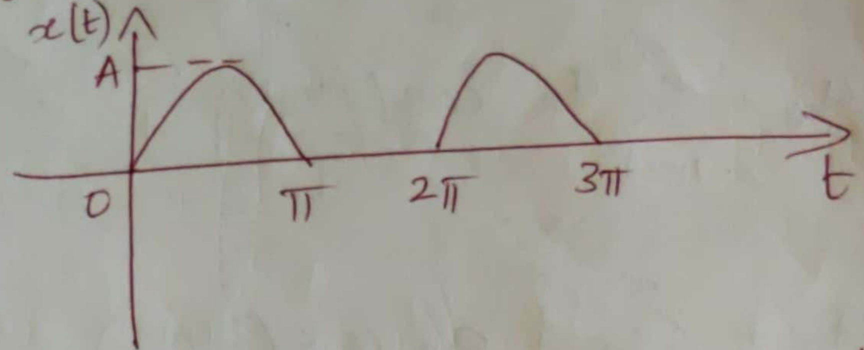
$D_n = \sqrt{a_n^2+b_n^2}$

$\phi_n = -\tan^{-1} b_n/a_n$

$$x(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos\left[\frac{2\pi nt}{T_0} - \phi_n\right]$$

Thus the eqn is proved.

Find cosine Fourier series of half wave rectified sine function.



$$x(t) = A \sin \Omega_0 t \quad \text{for } 0 \leq t \leq \pi$$
$$= 0 \quad \text{fo } \pi \leq t \leq 2\pi$$

$t_0 = 0$

$t_0 + T = 2\pi$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt$$

$$= \frac{1}{2\pi} \int_0^{\pi} A \sin t \, dt$$

$$= \frac{1}{2\pi} \left[\frac{-A \cos t}{1} \right]_0^{\pi}$$

$$= \frac{1}{2\pi} \left[-A \left[\cos \pi - \cos 0 \right] \right.$$

$$= \frac{-A}{2\pi} \left[(-1) - 1 \right]$$

$$= \frac{\cancel{2}A}{\cancel{2}\pi}$$

$$\boxed{a_0 = A/\pi}$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos nt\, dt$$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\boxed{\cos(A+B) = \cos A \cos B - \sin A \sin B}$

$\boxed{\cos(A-B) = \cos A \cos B + \sin A \sin B}$

$$= \frac{2A}{2\pi}\left[\int_0^{\pi} \underset{A}{\sin t}\ \underset{B}{\cos nt}\, dt\right]$$

$$= \frac{A}{\pi}\left[\int_0^{\pi}\frac{1}{2}\left[\sin(1+n)t + \sin(1-n)t\right] dt\right]$$

$$= \frac{A}{2\pi}\left[\left[-\frac{\cos(1+n)t}{1+n}\right]_0^{\pi} - \left[\frac{\cos(1-n)}{1-n}\right]_0^{\pi}\right]$$

$$= \frac{A}{2\pi}\left[\frac{-(\cos\pi\cos n\pi - \sin\pi\sin n\pi)}{1+n} - \left[\frac{\cos\pi\cos n\pi + \sin\pi\sin n\pi}{1-n}\right]\right]$$

$$= \frac{A}{2\pi}\left[\frac{-(-1-1)}{1+n} + \frac{2}{1-n}\right]$$

$$= \frac{A}{2\pi}\left[\frac{2 - \cancel{2n} + 2 + \cancel{2n}}{1-n^2}\right]$$

$$= \frac{A}{2\pi}\left[\frac{4}{1-n^2}\right] = \frac{2A}{\pi[1-n^2]}$$ when n is even

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\,\sin nt\,dt$$

$\sin A \sin B$
$= \frac{1}{2}[\cos(A-B)$

$$= \frac{2}{2\pi}\left[\int_0^{\pi} A \sin t \,\sin nt\,dt\right.$$

$\sin A \sin B$
$= \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$= \frac{A}{2\pi}\left[\int_0^{\pi}\left[\cos(1-n)t - \cos(1+n)t\right]dt\right]$$

$$= \frac{A}{2\pi}\left[\left[\frac{\sin(1-n)t}{1-n}\right]_0^{\pi} - \left[\frac{\sin(1+n)t}{1+n}\right]_0^{\pi}\right]$$

$$= 0$$

for $n=1$

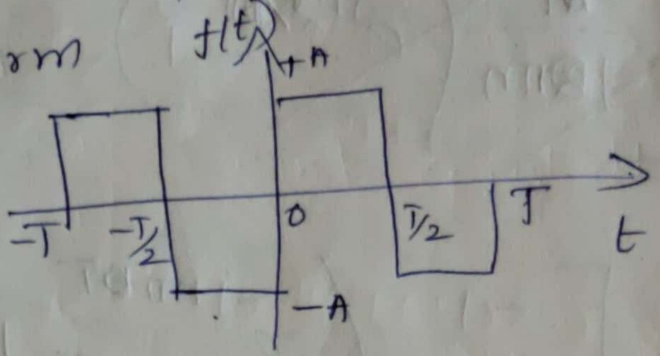
$$a_n = \frac{2}{2\pi}\int_0^{\pi} A \sin t \cos t\,dt$$

$$= \frac{A}{2\pi}\int_0^{\pi} \sin 2t\,dt$$

$$= \frac{-A}{2\pi}\left[\frac{\cos 2t}{2}\right]_0^{\pi}$$

$$= \frac{-A}{2\pi}\left[\frac{\cos 2\pi}{2} - \frac{\cos 0}{2}\right]$$

① Find the exponential fourier series of the waveform



$$x(t) = A \quad \text{for} \quad 0 \leq t \leq T/2$$

$$= -A \quad \text{for} \quad T/2 \leq t \leq T$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{j2\pi nt/T_0}$$

$$C_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \, e^{-j2\pi nt} \, dt$$

$$= \frac{1}{T} \left[\int_0^{T/2} A \, e^{-j2\pi nt} \, dt + \int_{T/2}^{T} -A \, e^{-j2\pi nt} \, dt \right.$$

$$= \frac{A}{T} \left[\left[\frac{e^{-j2n\pi t}}{-j2n\pi} \right]_0^{T/2} - \left[\frac{e^{-j2\pi nt}}{-j2\pi n} \right]_{T/2}^{T} \right.$$

$$= \frac{A}{T} \left[\left(\frac{e^{-j2\pi nT/2}}{-j2n\pi} - 1 \right) + \frac{e^{-j2\pi nT} - e^{-j2\pi nT/2}}{+j2\pi n} \right]$$

$T = 2\pi$

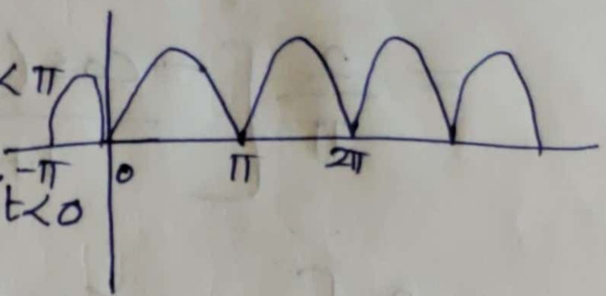
$$= \cancel{\frac{A}{T}\left[\frac{-1}{j2\pi n}\left(e^{jn\pi/2\pi} - 1\right)\right.}$$

$$= \frac{A}{T}\left[\frac{-1}{j2\pi n}\left(e^{-jn\pi\theta T} - 1\right) + \frac{1}{j2\pi n}\right.$$

Full wave rectifier

$x(t) = \sin t \quad 0<t<\pi$

$x(-t) = -\sin t \quad -\pi<t<0$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\,dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x(t)\,dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin t\,dt.$$

$$= \frac{2}{\pi}\left[-\cos t\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\cos\pi + (\cos 0)\right]$$

$$= \frac{2}{\pi}\left[-(-1)+1\right]$$

$$\boxed{a_0 = 4/\pi}$$

$$a_n = 1/\pi \int_{-\pi}^{\pi} x(t)\cos nt\,dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin t \cos nt\,dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi}\left[\sin(1+n)t + \sin(1-n)\right]dt$$

$$= \frac{2}{\pi}\left[-\frac{\cos(1+n)t}{1+n} + \frac{-\cos(1-n)t}{1-n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\cos(1+n)\pi + \cos 0}{1+n} + \frac{-\cos(1-n)\pi + \cos 0}{1-n}\right]$$

$$= \cancel{\frac{2}{\pi}}$$

$$\cos(1+n)\pi = \cos(\pi + n\pi) = -\cos n\pi$$
$$\cos(1-n)\pi = \cos(\pi - n\pi) = -\cos n\pi$$
$$\cos 0 = 1$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi + 1}{1+n} + \frac{\cos n\pi + 1}{1-n}\right]$$

$$= \frac{2}{\pi}\left[\frac{(1-n)(\cos n\pi + 1) + (1+n)(\cos n\pi + 1)}{1^2 - n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi + 1 - \cancel{n\cos n\pi} - \cancel{n} + \cos n\pi + 1 + \cancel{n\cos n\pi} + \cancel{n}}{1^2 - n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\cos n\pi + 2}{1-n^2}\right]$$

$$= \frac{-4}{\pi}\left[\frac{\cos n\pi + 1}{n^2 - 1}\right]$$

n - even
$\cos n\pi = 1$

$a_n = -4$

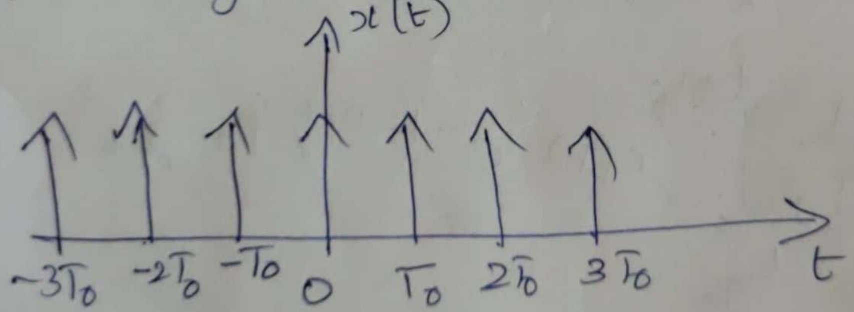
Exponential fourier series (or) complex exponential Fourier series:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0}$$

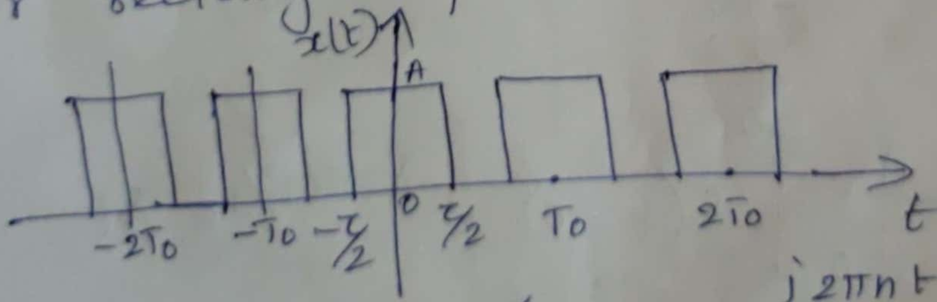
$$C_n = \frac{1}{T_0}\int_{t}^{t_0+T_0} x(t)\, e^{-j2\pi nt/T_0}\, dt$$

C_0 is obtained by putting $n=0$ in above eqn.

① Find out the exponential fourier series for impulse train show in fig. Also plot its magnitude & phase spectrum.



* Let us first find the fourier series for rectangular pulse train.



$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0}$$

$$C_n = \frac{1}{T_0}\int_{t_0}^{t+T_0} x(t)\, e^{-j2\pi nt/T_0}\, dt$$

$$= \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} A \, e^{-j2\pi nt/T_0} \, dt$$

$$=$$

Second method :-

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

$$C_n = \frac{1}{T_0} \int_{t}^{t+T_0} x(t) \, e^{-j2\pi nt/T_0} \, dt$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) \, e^{-j2\pi nt/T_0} . dt$$

$$= \frac{1}{T_0} \int_{-\infty}^{\infty} \delta(t-0) \, e^{-j2\pi nt/T_0} . dt$$

Shifting property is given by

$$\int_{-\infty}^{\infty} x(t) \, \delta(t - t_0) \, dt = x(t_0)$$

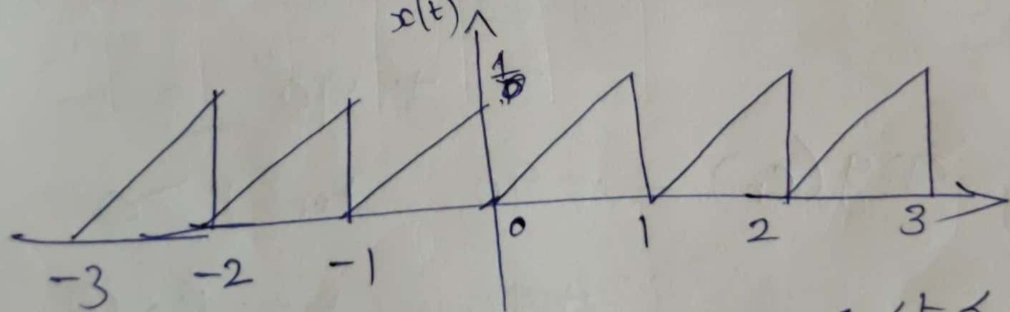
$$C_n = \frac{1}{T_0} e^{-j2\pi nt/T_0}\Big|_{t=0}$$

$$= \frac{1}{T_0}$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0}$$

$$x(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{j2\pi nt/T_0}$$

(2) Find the exponential fourier series and plot the magnitude & phase spectrum for the sawtooth waveform.



$$x(t) = t \quad \text{for } 0 < t < 1$$

$$T_0 = 1$$

$$C_n = \frac{1}{T_0} \int_{t}^{t+T_0} x(t)\, e^{-j2\pi nt/T_0}\, dt$$

$$= \frac{1}{1} \int_{0}^{1} t\, e^{-j2\pi nt/1}\, dt$$

$$\int x e^{ax} dx = e^{ax}\left[\frac{x}{a} - \frac{1}{a^2}\right)$$

$a = -j2\pi n$

$c_n = \frac{-j}{2\pi n}$ for $n \neq 0$

$$C_0 = \frac{1}{T_0}\int_t^{t+T_0} x(t)\,dt$$

$$a_0 = \frac{1}{1}\int_0^1 t\,dt = \frac{1}{2}$$

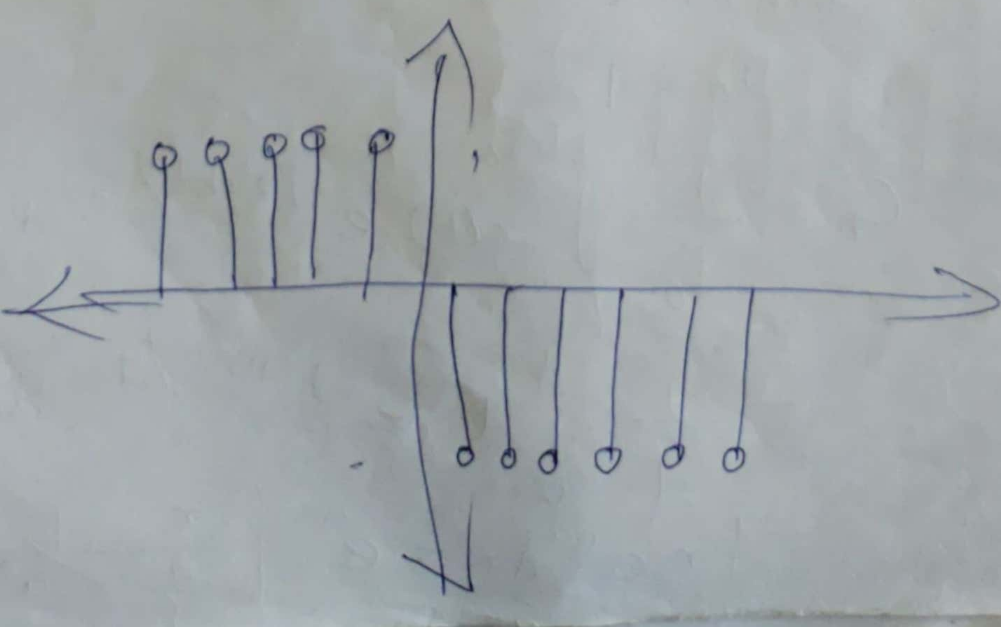
$$|c_n| = \sqrt{0 + \frac{1}{(2\pi n)^2}} = \frac{1}{2\pi n}$$

$$\phi_n = \arg(c_n) = -\tan^{-1}\left(\frac{-1/2\pi n}{0}\right)$$

$$= \pm 90^\circ$$

$\arg(c_n) = -90$ for $n>0$

$+90$ for $n<0$



Fourier Transform :-

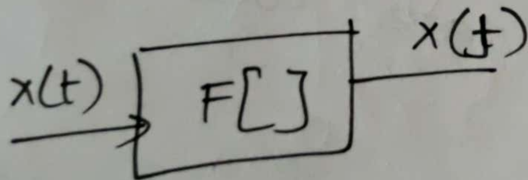
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

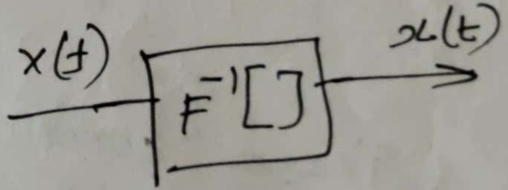
$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, .dt$$

Inverse Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, .df$$





F.T

(i) $|X(-f)| = |X(f)|$

(ii) $\theta(-f) = -\theta(f)$

* For non periodic signals integration is extended to $(-\infty, \infty)$
* For periodic signals integration is over one period

Conditions to obtain its fourier transform

* The fn $x(t)$ should be single-valued in any finite time interval.

* The fn $x(t)$ should have at the most finite no.. of discontinuities in any finite interval.

* The fn $x(t)$ should have finite no.. of maxima & minima

* The fn $x(t)$ should be absolutely Integrable ie

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty.$$

These conditions are also called Dirichlet conditions.

Properties of fourier transform :-

Linearity (superposition) :-

$$c_1 x_1(t) + c_2 x_2(t) \longleftrightarrow c_1 X_1(f) + c_2 X_2(f)$$

Proof :-

$$F\left[c_1 x_1(t) + c_2 x_2(t)\right]$$

$$= \int_{-\infty}^{\infty} \left[c_1 x_1(t) + c_2 x_2(t)\right] e^{-j2\pi f t} \cdot dt$$

$$= c_1 \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi f t} dt + c_2 \int_{-\infty}^{\infty} x_2(t) e^{-j2\pi f t} \cdot dt$$

$$= c_1 X_1(f) + c_2 X_2(f)$$

Time scaling :-

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Duality (or) symmetry property :-

$$x(t) \longleftrightarrow X(-f)$$

$$X(t) \longleftrightarrow x(-f)$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

$$t = -t$$

$$x(-t) = \int_{-\infty}^{\infty} X(f)\, e^{-j2\pi f t} \,.\, df$$

Interchanging t & f we get

$$x(-f) = \int_{-\infty}^{\infty} X(t)\, e^{-j2\pi f t}\, dt = F[X(t)]$$

$$x(-f) \longleftrightarrow X(t)$$

Time shifting :-

$$x(t) \longleftrightarrow X(f)$$

$$x(t-t_0) \longleftrightarrow X(f)\, e^{-j2\pi f t_0}$$

By definition of fourier transform

$$F(x(t-t_0)) = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j2\pi f t} \,.\, dt$$

$$t - t_0 = \tau$$

$$t = \tau + t_0 \longrightarrow dt = d\tau$$

$$F[x(t-t_0)] = \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f (t_0+\tau)} \,.\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f t_0} \,.\, e^{-j2\pi f \tau} \,.\, d\tau$$

$$F[x(t-t_0)] = e^{-j2\pi f t_0} X(f)$$

Freq. shifting :- (modulation theorem)

$$x(t) \longleftrightarrow X(f)$$

$$e^{j2\pi f_c t} \cdot x(t) \longleftrightarrow X(f-f_c)$$

Proof :-

$$F\left[e^{j2\pi f_c t} x(t)\right] = \int_{-\infty}^{\infty} e^{j2\pi f_c t} x(t)\, e^{-j2\pi f t} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi (f-f_c)} dt$$

$$= X(f-f_c)$$

Area under $x(t)$

$$x(t) \longleftrightarrow X(f)$$

$$x(t) = \int_{-\infty}^{\infty} x(t)\, dt = X(0)$$

That is area under $x(t)$ is equal to its fourier transform at zero freq.

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} . dt$$

$f = 0$

$$\boxed{X(0) = \int_{-\infty}^{\infty} x(t)\, dt.}$$

$u = e^{-j2\pi ft}\, dt$

$du =$

$dv = \frac{d}{dt} x(t)$

Area under $X(f)$:-

$$X(f) = \int_{-\infty}^{\infty} X(f)\, df = x(0)$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} . df$$

$t = 0$

$$\boxed{x(0) = \int_{-\infty}^{\infty} X(f) . df}$$

Differentiation in time domain :-

$$\frac{d}{dt} x(t) \longleftrightarrow (j2\pi f)\, X(f)$$

Proof :- $F\left[\frac{d}{dt} x(t)\right] = \int_{-\infty}^{\infty} \overset{dv}{\frac{d}{dt} x(t)}\, \overset{u}{e^{-j2\pi ft}} .dt$

$$= e^{-j2\pi ft} \left[x(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t) (-j2\pi f) e^{-j2\pi ft} \cdot dt$$

$$= j2\pi f \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \cdot dt$$

$$= j2\pi f \; X(f)$$

$$\int u\,dv = uv - \int v\,du$$

$u = \frac{d}{dt} x(t)$

$du = x(t)$

$dv = e^{-j2\pi ft} \cdot dt$

$v = \frac{e^{-j2\pi ft}}{-j2\pi f}$

Integration in time domain :-

$$\int_{-\infty}^{t} x(\tau)\, d\tau \Longleftrightarrow \frac{1}{j2\pi f} X(f)$$

proof :-

$$x(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

$$F\left[x(t)\right] = F\left[\frac{d}{dt}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\}\right]$$

$$= j2\pi f \left[F\left(\int_{-\infty}^{t} x(\tau)\, d\tau\right)\right]$$

$$X(f) = j2\pi f \left[F\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\}\right]$$

$$F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j2\pi f} X(f)$$

Conjugate fn :-

$$x^*(t) \longleftrightarrow X^*(-f)$$

By IFT

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

$$x^*(t) = \int_{-\infty}^{\infty} X^*(f)\, e^{-j2\pi ft} \cdot df$$

now replacing f with $-f$

$$x^*(t) = \int_{-\infty}^{\infty} X^*(-f)\, e^{j2\pi ft}\, df$$

$$= F^{-1}\left[X^*(-f)\right]$$

$$= X^*(-f)$$

Multiplication in Time domain (Multiplication theo

$$* \quad x_1(t)\, x_2(t) \Longleftrightarrow \int_{-\infty}^{\infty} X_1(\lambda)\, X_2(f-\lambda)\, d\lambda$$

ie multiplication of two signals in time domain is transformed into convolution of the fourier transform in freq. domain.

$$x_1(t)\, x_2(t) \longleftrightarrow X_1(f) * X_2(f)$$

proof :-

$$x_1(t)\, x_2(t) \longleftrightarrow X_{12}(f)$$

$$F[x_1(t)\, x_2(t)] = X_{12}(f)$$

$$= \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, e^{-j2\pi f t} \cdot dt \quad \text{—(1)}$$

$$x_2(t) = \int_{-\infty}^{\infty} X_2(f')\, e^{+j2\pi f' t} \cdot df \quad \text{—(2)}$$

(2) in (1)

$$X_{12}(f) = \int_{-\infty}^{\infty} x_1(t) \int_{-\infty}^{\infty} X_2(f')\, e^{+j2\pi f' t}\, df\; e^{-j2\pi f t} \cdot dt$$

$$\lambda = f - f' \qquad f' = f - \lambda$$

$$X_{12}(f) = \int_{-\infty}^{\infty} X_2(-\lambda)\, d\lambda \int_{-\infty}^{\infty} x_1(t)\, e^{-j\pi(f-f)t}\, dt$$

$$= \int_{-\infty}^{\infty} X_2(f-\lambda)\, d\lambda \int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi\lambda t} \cdot dt$$

$$X_{12}(f) = \int_{-\infty}^{\infty} X_2(f-\lambda)\, d\lambda\, X_1(\lambda)$$

$$= \int_{-\infty}^{\infty} X_1(\lambda)\, X_2(f-\lambda)\, d\lambda.$$

This property called as multiplication theorem.

Convolution in Time domain (Convolution theorem)

$$\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \longleftrightarrow X_1(f)\, X_2(f)$$

This property states that convolution of two signals in time domain is transformed into multiplication of their individual fourier transforms in freq. domain.

$$x_1(t) * x_2(t) \Longleftrightarrow X_1(f)\, X_2(f)$$

Proof:-

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right) e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j2\pi f \tau}\, d\tau \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{j2\pi f \tau} \cdot e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j2\pi f \tau}\, d\tau \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j2\pi f (t-\tau)} \cdot dt$$

$$t - \tau = \alpha$$

$$F[x_1(t) * x_2(t)] = X_1(f)\, X_2(f)$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j2\pi f \tau}\, d\tau \int_{-\infty}^{\infty} x_2(\alpha)\, e^{-j2\pi f \alpha}\, d\alpha$$

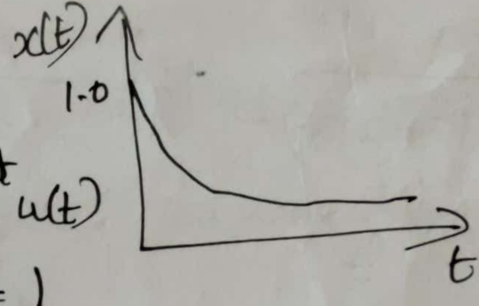
To calculate magnitude and phase spectrum

$$X(f) = \underbrace{A(f)}_{\text{real part}} + j\underbrace{B(f)}_{\text{imaginary part}}$$

$$\text{magnitude} = |X(f)| = \sqrt{A^2(f) + B^2(f)}$$

$$\left.\begin{array}{l}\text{phase}\\ \text{spectrum}\end{array}\right\} = \tan^{-1}\frac{B(f)}{A(f)}$$

① Find the fourier transform of the decaying exponential



$$x(t) = e^{-at} u(t)$$

$$u(t) = 1$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\,.dt$$

$$= \int_{0}^{\infty} e^{-at} u(t)\, e^{-j2\pi f t}\,.dt$$

$$= \int_{0}^{\infty} e^{-(a+j2\pi f)t}\,.dt$$

$$= \left[\frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)}\right]_{0}^{\infty}$$

$$\boxed{x(f) = \frac{1}{a+j2\pi f}}$$

$$x(f) = \frac{1}{a+j2\pi f}$$

$$= \frac{1}{a+j2\pi f} \times \frac{a-j2\pi f}{a-j2\pi f}$$

$$= \frac{a-j2\pi f}{a^2+(2\pi f)^2}$$

$$= \underbrace{\frac{a}{a^2+(2\pi f)^2}}_{A_f} + j\,\underbrace{\frac{-2\pi f}{a^2+(2\pi f)^2}}_{B_f}$$

magnitude $x(f)$ will be

$$|x(f)| = \sqrt{\frac{a^2}{[a^2+(2\pi f)^2]^2} + \frac{(2\pi f)^2}{a^2+(2\pi f)^2}}$$

$$= \sqrt{\frac{\cancel{a^2+(2\pi f)^2}}{[a^2+(2\pi f)^2]^{\cancel{2}}}}$$

$$\Rightarrow \sqrt{\frac{1}{a^2+(2\pi f)^2}}$$

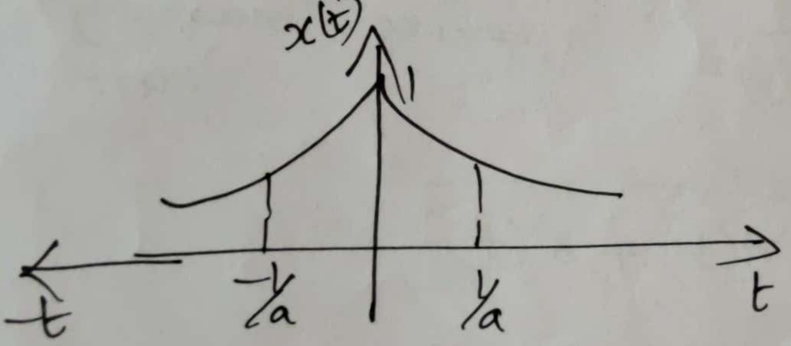
$$\theta(f) = \tan^{-1} \frac{B_f}{A_f}$$

$$= \tan^{-1} \left[\frac{-2\pi f}{\cancel{a^2+(2\pi f)^2}} \times \frac{\cancel{a^2+(2\pi f)^2}}{a} \right]$$

$$= \tan^{-1}\left(\frac{-2\pi f}{a}\right)$$

$$\boxed{\theta(f) = -\tan^{-1}\left(\frac{2\pi f}{a}\right)}$$

(2) With the help of Linearity property obtain FT of double exponential pulse shown below.



$$x(t) = e^{-at} \quad t>0$$
$$= 1 \quad t=0$$
$$= e^{at} \quad t<0$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} \cdot dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j2\pi f t} \cdot dt + \int_{0}^{0} 1 \cdot e^{-j2\pi f t} \cdot dt + \int_{0}^{\infty} e^{-at} e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j2\pi f)t} \cdot dt + \int_{0}^{0} e^{-j2\pi f t} \cdot dt + \int_{0}^{\infty} e^{-(a+j2\pi f)t} \, dt$$

$$= \left[\frac{e^{(a-j2\pi f) \cdot t}}{a-j2\pi f} \right]_{-\infty}^{0} + \left[\frac{e^{-j2\pi f t}}{-j2\pi f} \right]_{0}^{0} + \left[\frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)} \right]_{0}^{\infty}$$

$$= \frac{1}{a-j2\pi f} + 0 + \frac{1}{a+j2\pi f}$$

$$= \frac{a+\cancel{j2\pi f}+a-\cancel{j2\pi f}}{a^2+(2\pi f)^2}$$

$$X(f) = \frac{2a}{a^2+(2\pi f)^2} \Longrightarrow \text{no imaginary part}$$

$$|X(f)| = \sqrt{A^2(f) + B^2(f)}$$

$$= \sqrt{\left(\frac{2a}{a^2+(2\pi f)^2}\right)^2}$$

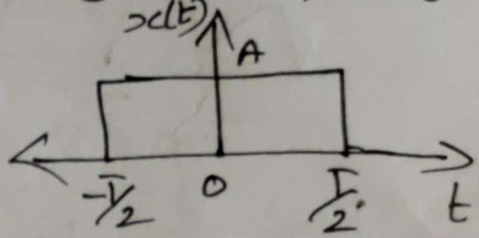
$$= \frac{2a}{a^2+(2\pi f)^2}$$

$$\theta(f) = \tan^{-1} \frac{B_f}{A_f}$$

$$= \tan^{-1} 0/A_f = 0$$

3) obtain the fourier transform of rectangular pulse ~~&~~ Sketch the signal and its fourier transform.

$$\operatorname{rect}\left(\frac{t}{T}\right) = \begin{cases} A & \text{for } -T/2 < t < T/2 \\ 0 & \text{elsewher} \end{cases}$$



$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} \, dt$$

$$= \int_{-T/2}^{T/2} \cancel{x(t)}\, A\, e^{-j2\pi f t}\, dt$$

$$= \frac{A}{-j2\pi f} \left[e^{-j2\pi f t} \right]_{-T/2}^{T/2}$$

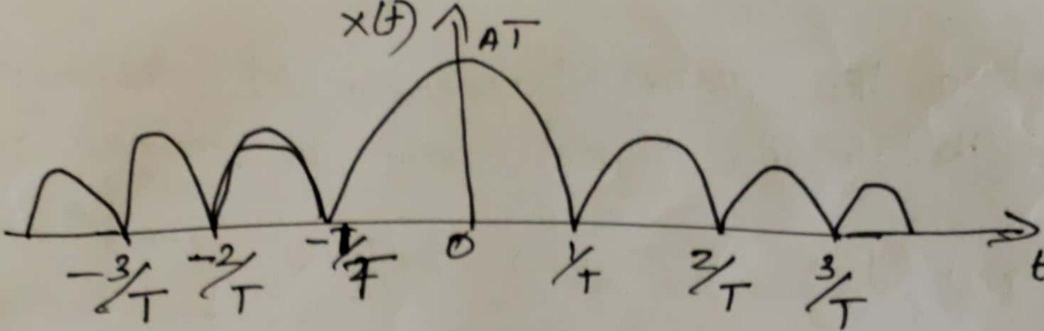
$$= \frac{A}{-j2\pi f} \left[e^{-j\pi f T} - e^{j\pi f T} \right]$$

$$= \frac{A}{+\pi f} \left[\frac{e^{j\pi f T} - e^{-j\pi f T}}{2j} \right]$$

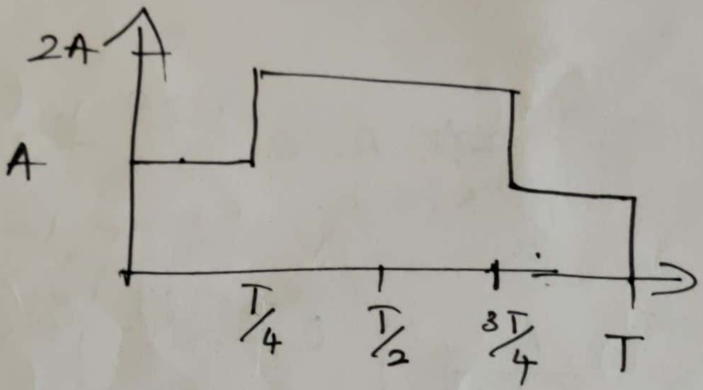
$$= \frac{A}{\pi f} \sin(\pi f T)$$

$$= \frac{AT}{\pi f T} \sin \pi f t \qquad \operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$\boxed{X(f) = AT \operatorname{sinc}(ft)}$$



(4) Find the fourier transform of the signal shown in figure.



* Here $x_1(t)$ is a rectangular pulse of Amplitude A

* It is fourier transform

$$X_1(f) = AT \sin(fT)$$

* $x_2(t)$ is rectangular pulse of Amplitude A and duration $T/2$

* It is fourier transform

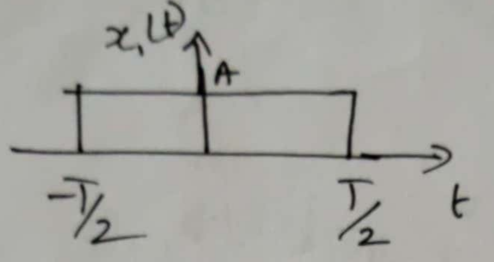
$$X_2(f) = \frac{AT}{2} \operatorname{sinc}\left(\frac{fT}{2}\right)$$

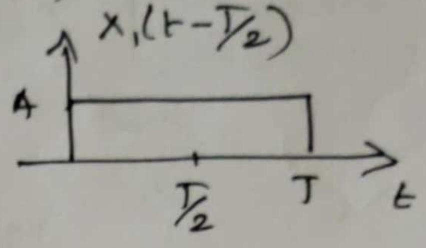
* Shifting property states that

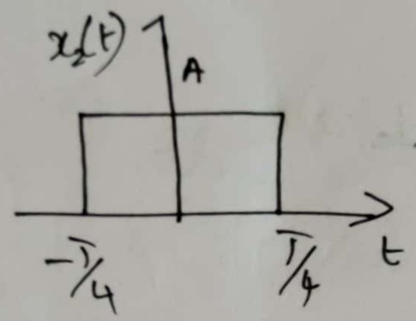
$$x(t-t_0) \xleftrightarrow{FT} X(f)\, e^{-j2\pi f t_0}$$

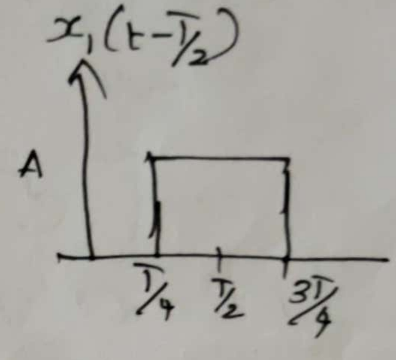
$$x_1(t-T/2) \xleftrightarrow{FT} X_1(f)\, e^{-j2\pi f T/2}$$

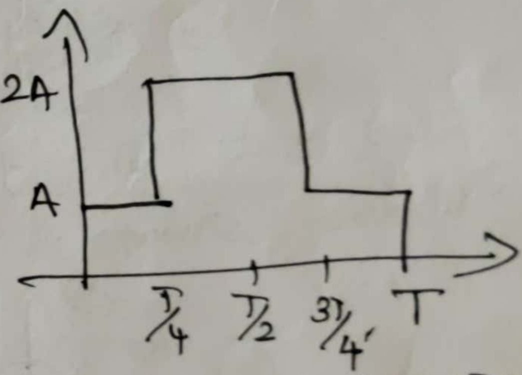
$$x_2(t-T/2) \xleftrightarrow{FF} X_2(f)\, e^{-j2\pi f T/2}$$











$$X(f) = X_1(f)\, e^{-j2\pi f T/2} + X_2(f)\, e^{-j2\pi f T/2}$$

$$= AT \operatorname{sinc}(fT)\, e^{-j\cancel{2}\pi fT} + \frac{AT}{2} \operatorname{sinc}\left(\frac{fT}{2}\right) e^{-j\cancel{2}\pi fT}$$

$$\boxed{X(f) = AT\, e^{-j\pi fT}\left[\operatorname{sinc}(fT) + \frac{1}{2}\operatorname{sinc}\left(\frac{fT}{2}\right)\right]}$$

(5) Obtain the fourier transform of $x(t) = e^{j2\pi f_c t}$.

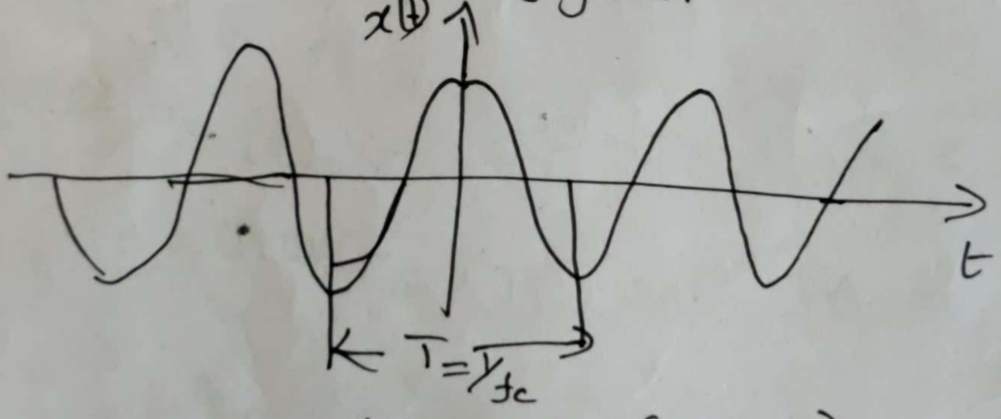
$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt.$$

$$= \int_{-\infty}^{\infty} e^{j2\pi f_c t} \cdot e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{j2\pi (f_c - f)t}\, dt$$

$$= \delta(f - f_c)$$

(6) Find out the fourier transform of cosine wave shown in figure.



$$x(t) = \cos(2\pi f_c t) = \frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}$$

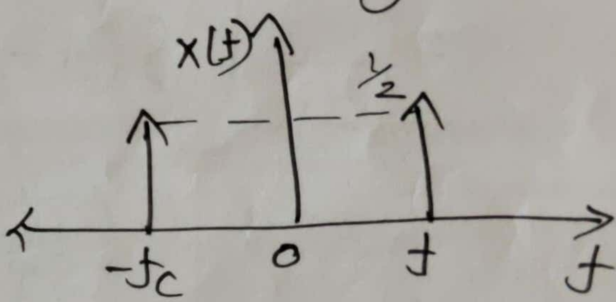
$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right) e^{-j2\pi f t} \, . dt$$

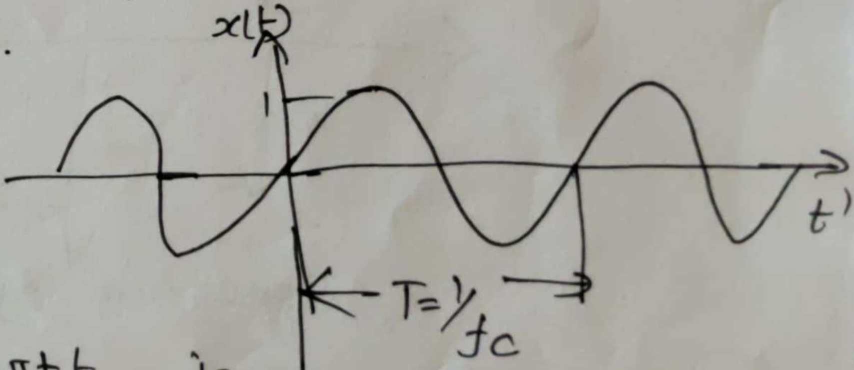
$$= \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{-j2\pi(f - f_c)t} + e^{-j2\pi(f+f_c)t}\right) dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi(f-f_c)t} \, . dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi(f+f_c)dt} \, . dt$$

$$= \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$



(4) Find out the fourier transform of sine fn shown in figure.



$x(t) = \sin 2\pi f_c t$

$$\sin 2\pi f_c t = \frac{e^{j2\pi f_c t} - e^{j2\pi f_c t}}{2j}$$

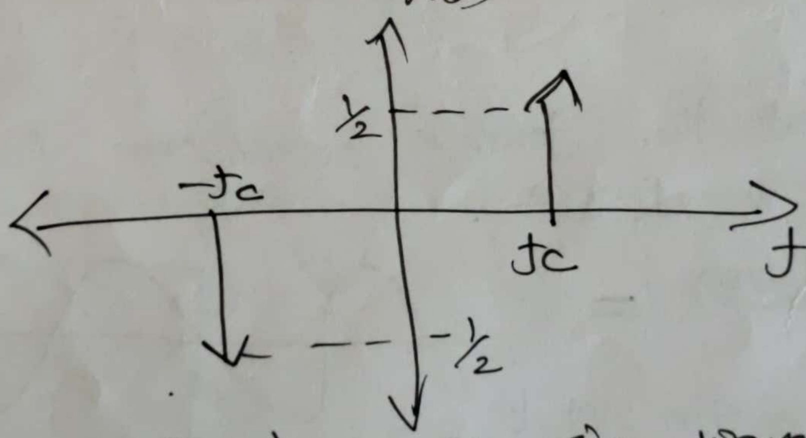
$$X(f) = \int_{-\infty}^{\infty} x(t) \, e^{-j2\pi f t} \, dt.$$

$$= \int_{-\infty}^{\infty} \sin(2\pi f_c t)\, e^{-j2\pi f t} \cdot dt$$

$$= \int_{-\infty}^{\infty} \left(\frac{e^{j2\pi f_c t} - e^{-j2\pi f_c t}}{2j} \right) e^{-j2\pi f t} \cdot dt$$

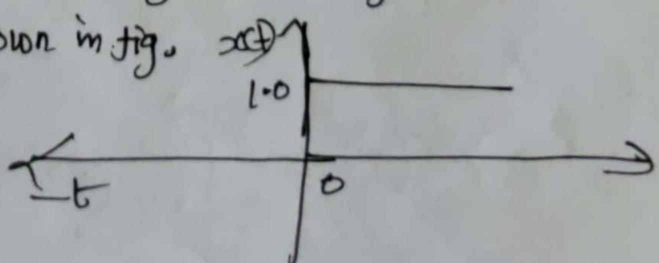
$$= \frac{1}{2j} \int_{-\infty}^{\infty} \left(e^{-j2\pi (f - f_c) t} - e^{-j2\pi (f + f_c) t} \right) dt$$

$$= \frac{1}{2j} \left[\delta(f - f_c) - \delta(f + f_c) \right]$$



Spectrum of sine wave.

(8) Find the fourier transform of the unit step function shown in fig.



$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} \cdot dt.$$

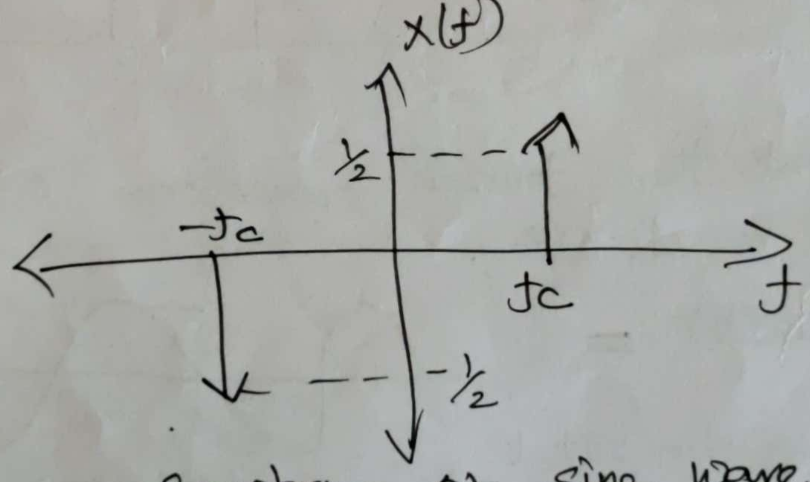
(5) Obtain the fourier tr...

$$= \int_{-\infty}^{\infty} \sin(2\pi f_c t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} \left(\frac{e^{j2\pi f_c t} - e^{-j2\pi f_c t}}{2j} \right) e^{-j2\pi f t}\, dt$$

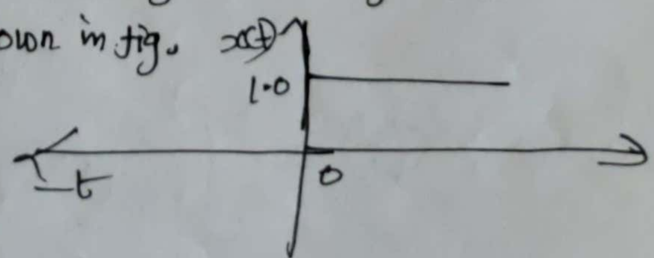
$$= \frac{1}{2j} \int_{-\infty}^{\infty} \left(e^{-j2\pi(f - f_c)t} - e^{-j2\pi(f + f_c)t} \right) dt$$

$$= \frac{1}{2j} \left[\delta(f - f_c) - \delta(f + f_c) \right]$$



Spectrum of sine wave.

(8) Find the fourier transform of the unit step function shown in fig.



$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt.$$

(12)

$$= \int_0^{\infty} 1 \cdot e^{-j2\pi f t} \cdot dt$$

$$= \left[\frac{e^{-j2\pi f t}}{-j2\pi f} \right]_0^{\infty}$$

$$= \frac{1}{-j2\pi f} \left[e^{-\infty} - e^{0} \right]$$

$$= \frac{1}{j2\pi f}$$

$$|X(f)| = \sqrt{A_f^2 + B_f^2}$$

$$= \sqrt{0 + \left(\frac{1}{2\pi f}\right)^2}$$

$$= \frac{1}{2\pi f}$$

$$\theta(f) = -\tan^{-1} \frac{1}{2\pi f}$$

Laplace transform

12

* Laplace transform is another mathematical tool used for analysis of signals and systems.

* LT can be used for analysis of unstable system.

* There are two types

(i) Bilateral (or) two sided LT

(ii) Unilateral (or) single sided "

$$F(s) = \int_{-\alpha}^{\alpha} f(t)\, e^{-st}.dt$$

$$F(s) = \int_{0}^{\alpha} f(t)\, e^{-st}.dt.$$

$$f(t) \xleftrightarrow{LT} F(s)$$

$s = \sigma + j\omega$

σ → attenuation constant

ω → angular freq.

$$F(s) = \int_{-\alpha}^{\alpha} f(t)\, e^{-(\sigma+j\omega)t}.dt$$

$$= \int_{-\alpha}^{\alpha} f(t)\, e^{-\sigma t}.e^{-j\omega t}.dt$$

$$= \int_{-\alpha}^{\alpha} \left(f(t)e^{-\sigma t}\right) e^{-j\omega t}.dt$$

$F(s)$ is the FT of $f(t)e^{-\sigma t}$

This is the relation between laplace Transform and FT

* condition for $f(t)$ to be LT

$$\int_0^\infty |f(t)|\, e^{-\sigma t}\, dt < \infty$$

for real & +ve values of σ

<u>Inverse LT:-</u>

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)\, e^{st}\, dt$$

<u>properties LT:-</u>

(1) Linearity

$$L[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

$a_1\ a_2$ - constants.

<u>Proof:-</u>

$$L[a_1 f_1(t) + a_2 f_2(t)] = \int_0^\infty [a_1 f_1(t) + a_2 f_2(t)]\, e^{-st}\, dt$$

$$= a_1 \int_0^\infty f_1(t)\, e^{-st}\, dt + a_2 \int_0^\infty f_2(t)\, e^{-st}\, dt$$

$$= a_1 F_1(s) + a_2 F_2(s).$$

2. Shifting theorem:- (Translation in Time domain) 13

$$f(t) \xleftrightarrow{LT} F(s)$$

$$\mathcal{L}[f(t-t_0)] = e^{-st_0} F(s)$$

t_0 - constant.

$$\mathcal{L}[f(t-t_0)] = \int_{t_0}^{\infty} f(t-t_0)\, e^{-st} . dt$$

$\tau = t - t_0$

$d\tau = dt$

$t = t_0 \quad \tau = 0$

$t = \infty \quad \tau = \infty$

$$= \int_0^{\infty} f(\tau)\, e^{-s(\tau + t_0)} . d\tau$$

$$= e^{-st_0} \int_0^{\infty} f(\tau)\, e^{-s\tau} . d\tau$$

$$= e^{-st_0} F(s)$$

3. complex translation (or) Translation in freq. domain.

$$F[s-a] = \mathcal{L}[e^{at} f(t)], \quad F(s+a) = \mathcal{L}[e^{-at} f(t)]$$

Proof:-

$$\mathcal{L}[e^{at} f(t)] = \int_0^{\infty} e^{at} f(t)\, e^{-st} . dt$$

$$= \int_0^{\infty} f(t)\, e^{-(s-a)} . dt = F(s-a)$$

Differentiation Theorem :-

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$$

Proof :- $\quad \int u\,dv = uv - \int v\,du$

$$\frac{d}{dt} f(t) = f'(t)$$

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$u = e^{-st}$

$\frac{du}{dt} = e^{-st}(-s)$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\,dt$$

$dv = f'(t)$

$v = f(t)$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty f(t)\, e^{-st}.dt$$

$$= \left[e^{-\infty} f(\infty) - e^0 f(0)\right] + sF(s)$$

$$= sF(s) - f(0)$$

Integration Theorem :-

$u = \int_0^t f(t)\,dt$

$dv = e^{-st}dt$

$V = \frac{e^{-st}}{-s}$

$du = f(t)\,dt$

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \int_0^\infty e^{-st}\underbrace{\left[\int_0^t f(t)\,dt\right]}_{u} dt$$

$$= \left[\frac{e^{-st}}{-s}\int_0^t f(t)\,dt\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} f(t)\,dt$$

$$= 0 + \frac{1}{s}\int_0^{t} e^{-st} f(t)\, dt$$

$$= \frac{F(s)}{s}$$

$$L\left[\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_n} f(t)\, dt_1\, dt_2 \ldots dt_n\right] = \frac{F(s)}{s^n}$$

Differenation by s :-

$$L[t\, f(t)] = -\frac{d}{ds} F(s)$$

$$F(s) = \int_0^{\alpha} f(t)\, e^{-st}\, dt$$

$$\frac{d}{ds} F(s) = \int_0^{\alpha} f(t) \cdot \frac{d}{ds} e^{-st} \cdot dt$$

$$= -\int_0^{\alpha} t\, f(t)\, e^{-st} \cdot dt$$

$$= -L[t\, f(t)]$$

Initial value theorem :-

$$f(0^+) = \lim_{T \to 0} f(t) = \lim_{s \to \alpha} [s\, F(s)]$$

Proof :-

$$L\left[\frac{d}{dt} f(t)\right] = s\, F(s) - f(0^-)$$

$$\lim_{s \to \alpha} L\left[\frac{d}{dt} f(t)\right] = \lim_{s \to \alpha} \{s F(s) - f(0^-)\}$$

consider LHS

$$\lim_{s \to \alpha} L\left[\frac{d}{dt}(f(t))\right] = \lim_{s \to \alpha} \int_0^{\alpha} \frac{d}{dt} f(t)\, e^{-st}\, dt$$

$$= 0$$

$$0 = \lim_{s \to \alpha} \left\{ sF(s) - f(0^-) \right\}$$

$$f(0^-) = \lim_{s \to \alpha} \left[sF(s) \right]$$

$f(0^-)$ — the value of $f(t)$ just before $t=0$

$f(0^+)$ — " " just after $t=0$

$f(0^+) = f(0^-)$ for $f(t)$ is continuous at time $t=0$

$$\boxed{f(0^+) = \lim_{s \to \alpha} \left[sF(s) \right]}$$

This is used to determine the initial value of $f(t)$ and its derivative.

Final value Theorem :-

$$\lim_{t \to \alpha} f(t) = \lim_{s \to 0} \left[sF(s) \right]$$

Proof :-

$$\lim_{s \to \alpha} f(t) = \lim_{s \to 0} sF(s)$$

$$\mathcal{L}\left[\frac{d}{dt}(f(t))\right] = s F(s) - f(0^-)$$

$$= \lim_{s \to 0}\left[s F(s) - f(0^-)\right]$$

consider LHS

$$\lim_{t \to \alpha} \mathcal{L}\left[\frac{d}{dt} f(t)\right] = \lim_{s \to 0} \int_0^{\alpha} \frac{d}{dt} f(t)\, e^{-st}\, dt$$

$$= \int_0^{\alpha} \frac{d}{dt} f(t) \cdot dt \text{ since } \lim_{s \to 0} e^{-st} = 1$$

$$= \left[f(t)\right]_0^{\alpha}$$

$$= \lim_{t \to \alpha} f(t) - f(0^-)$$

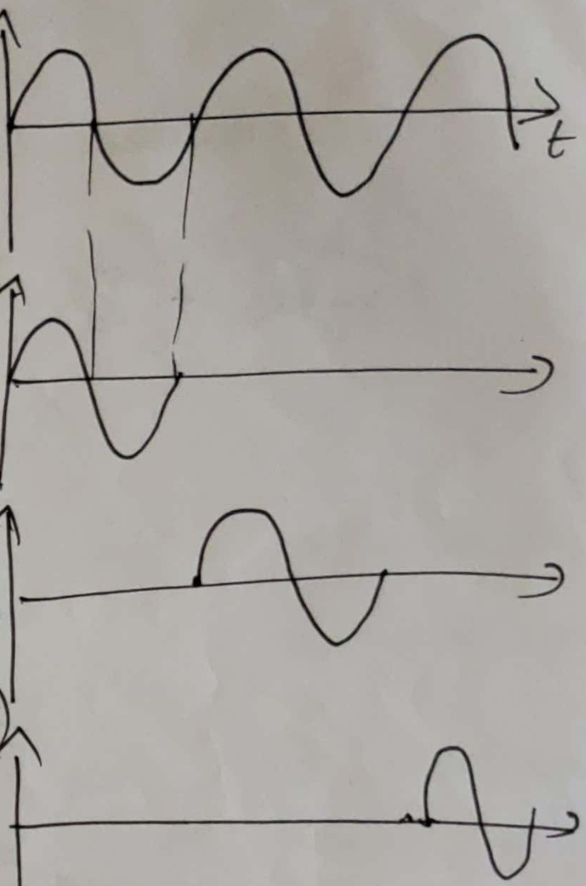
$$\lim_{t \to \alpha} f(t) - \cancel{f(0^-)} = \lim_{s \to 0} s F(s) - \cancel{f(0^-)}$$

$$\boxed{\lim_{t \to \alpha} f(t) = \lim_{s \to 0} s F(s)}$$

The final value Theorem is usefull for in analysis and design of feedback control systems.

Laplace transform of periodic function :-

$$F(s) = \frac{1}{1-e^{-stT}} F_1(s)$$



proof :-

$f_2(t) = f_1(t-T) \; u(t-T)$

$u(t-T) = 1$ for $t \geq 0$

$f_2(t) = 0$ for $t \leq 0$.

$f_3(t) = f_1(t-2T) \, u(t-2T)$

$$f(t) = f_1(t) + f_1(t-T)\, u(t-T) + f_2(t-2T)\, u(t-2T) + f_3(t-3T)\, u(t-3T) + \cdots$$

$$= F_1(s) + e^{-Ts} F_1(s) + e^{-2Ts} F_1(s) + e^{-3Ts} F_1(s) + \cdots$$

$$= F_1(s) \left[1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + e^{-TS} + \cdots \right\}$$

$$= F_1(s) \frac{1}{1-e^{-Ts}}$$

$$= \frac{F_1(s)}{1-e^{-sT}}$$

This is required expression.

Convolution Theorem :-

$$\mathcal{L}\left[f_1(t) * f_2(t)\right] = F_1(s) \cdot F_2(s)$$

ie the laplace transform of convolution of two functions is equivalent to multiplication of their laplace transforms.

Proof :-

$$u(t-\tau) = 1 \quad \text{for } t \geq \tau$$
$$= 0 \quad \text{for } t < \tau$$

$$f_1(t) * f_2(t) = \int_0^{\infty} f_1(t-\tau)\, u(t-\tau) f_2(\tau)\, d\tau$$

$$\mathcal{L}\left[f_1(t) * f_2(t)\right] = \int_0^{\infty} \left[\int_0^{\infty} f_1(t-\tau)\, u(t-\tau) f_2(\tau)\, d\tau\right] e^{-st}\, dt$$

$$= \int_0^{\infty} e^{-st} \int_0^{\infty} f_1(t-\tau)\, u(t-\tau)\, f_2(\tau)\, d\tau\, dt$$

put $x = t-\tau$ $\quad e^{-st} = e^{-s(x+\tau)}$

$t = x+\tau$ $\quad = e^{-sx} \cdot e^{-s\tau}$

$dx = dt$

$$\mathcal{L}\left[f_1(t) * f_2(t)\right] = \int_0^{\infty}\int_0^{\infty} f_1(x) \cdot u(x)\, f_2(\tau)\, e^{-st} \cdot e^{-sx} \cdot d\tau\, dx$$

$$= \int_0^{\infty} f_1(x) \underbrace{u(x)}_{1} \cancel{f_2(\tau)}\, e^{-sx} \cdot dx \int_0^{\infty} f_2(\tau) e^{-s\tau} d\tau$$

$$= \int_0^{\infty} f_1(x) e^{-sx}\, dx \cdot \int_0^{\infty} f_2(\tau) e^{-s\tau} \cdot d\tau$$

$$= F_1(s) \cdot F_2(s)$$

Time scaling :-

$x(t) \xrightarrow{L} X(s)$

$$x(at) \xrightarrow{L} \frac{1}{|a|} X\left(\frac{s}{a}\right) \qquad ROC : R/a$$

$$L[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-st}\, dt$$

$$at = \tau \qquad t = \frac{\tau}{a}$$

$$dt = \frac{d\tau}{a}$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau/a} \cdot \frac{1}{a}\, d\tau$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-(s/a)\tau}\, d\tau$$

$$= 1/a\; X\left(\frac{s}{a}\right)$$

Let us consider −ve value of a i.e

$$L[x(-at)] = \int_{-\infty}^{\infty} x(-at)\, e^{-st}\, dt$$

$$-at = \tau \qquad t = -\frac{\tau}{a}$$

$$dt = -\frac{1}{a}\, d\tau$$

$$L[x(-at)] = \int_{-\infty}^{\infty} x(\tau)\, e^{-s(-\tau/a)} \left(-1/a\right) d\tau$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-\left(\frac{s}{-a}\right)}\, d\tau$$

$$= \frac{1}{a} X\left(\frac{s}{-a}\right), \quad ROC : \frac{R}{-a}$$

1. Find out the laplace transform of an exponential fn.

$f(t) = e^{at}$

$e^{at} u(t), \quad u(t) = 1 \text{ for } t \geq 0$

$$\mathcal{L}[e^{at}] = \int_0^{\infty} e^{at} e^{-st} . dt$$

$$= \int_0^{\infty} e^{-(s-a)t} . dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty}$$

$$= -\frac{1}{s-a} \left[e^{-\infty} - e^{0} \right]$$

$$\boxed{\mathcal{L}[e^{at}] = \frac{1}{s-a}}$$

2. Find out the LT of unit Step fn

$u(t) = 1 \quad \text{for } t \geq 0$

$\quad\;\; = 0 \quad \text{for otherwise}$

$$\mathcal{L}[u(t)] = \int_0^{\infty} 1 . e^{-st} . dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^{\infty}$$

$$= \frac{1}{-s} \left[e^{-\infty} - e^{0} \right]$$

$$= \frac{1}{-s} (-1)$$

$$= 1/s$$

③ Find out the LT of ramp fn

$$r(t) = t \quad \text{for } t \geq 0$$
$$= 0 \quad \text{Otherwise}$$

$$r(t) = t\, u(t)$$

$$\mathcal{L}[r(t)] = \int_0^\infty t\, e^{-st}\, dt$$

$$= \left[t \cdot \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-st}}{-s} \cdot dt$$

$$= \left[\underbrace{\frac{t\, e^{-st}}{-s}}_{\to 0} \right]_0^\infty - \left[\frac{e^{-st}}{-s^2} \right]_0^\infty$$

$$= \frac{1}{s^2}$$

④ Find the LT of impulse fn.

$$\delta(t) = \frac{d}{dt}\, u(t)$$

$$\mathcal{L}[\delta(t)] = \mathcal{L}\left[\frac{d}{dt}\, u(t)\right]$$

$$= s\, F(s) - f(0_-)$$

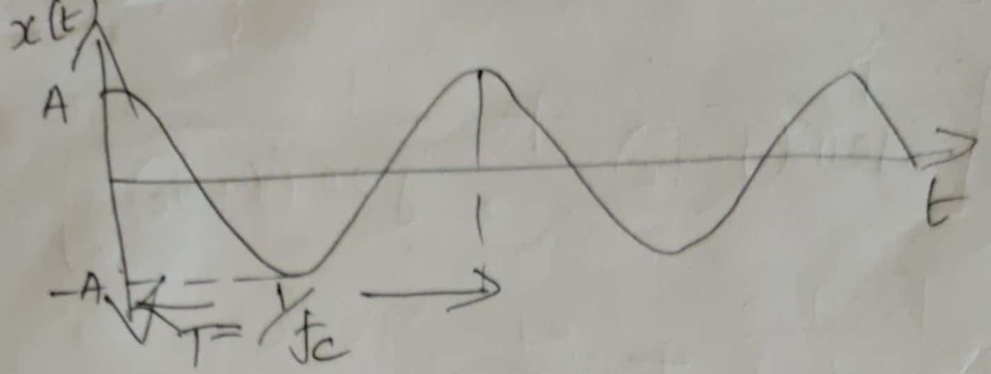
$$F(s) = \mathcal{L}[u(t)] = 1/s$$

$$f(0_-) = u(t)\Big|_{t=0} = 0$$

$$\mathcal{L}[\delta(t)] = s \cdot \frac{1}{s} - 0 = 1$$

$$\mathcal{L}[\delta(t)] = 1$$

Find out the fourier transform of cosine wave form represented in figure.



$$x(t) = A \cos 2\pi f_c t \; u(t)$$

$$X(f) = \int_{-\alpha}^{\alpha} x(t)\, e^{-j2\pi f t} . dt$$

$$= \int_{0}^{\alpha} A \cos 2\pi f_c t \; e^{-j2\pi f t} . dt$$

$$= A \int_{0}^{\alpha} \left(\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2} \right) e^{-j2\pi f t} . dt$$

$$= \frac{A}{2} \left[\int_{0}^{\alpha} e^{j2\pi f_c t} . e^{-j2\pi f t} dt + \int_{0}^{\alpha} e^{-j2\pi f_c t} e^{-2\pi f t} . dt \right]$$

$$= \frac{A}{2} \left[\int_{0}^{\alpha} e^{-j2\pi (f - f_c) t} . dt + \int_{0}^{\alpha} e^{-j2\pi (f + f_c) t} . dt \right]$$

$$= \frac{A}{2} \left[\left[\frac{e^{-j2\pi (f - f_c) t}}{-j2\pi (f - f_c)} \right]_{0}^{\alpha} + \left[\frac{e^{-j2\pi (f + f_c) t}}{-j2\pi (f + f_c)} \right]_{0}^{\alpha} \right]$$

$$= \frac{A}{2}\left[\left(0 - \left(\frac{1}{-j2\pi(f-f_c)}\right)\right) + \left(0 - \frac{1}{-j2\pi(f+f_c)}\right)\right]$$

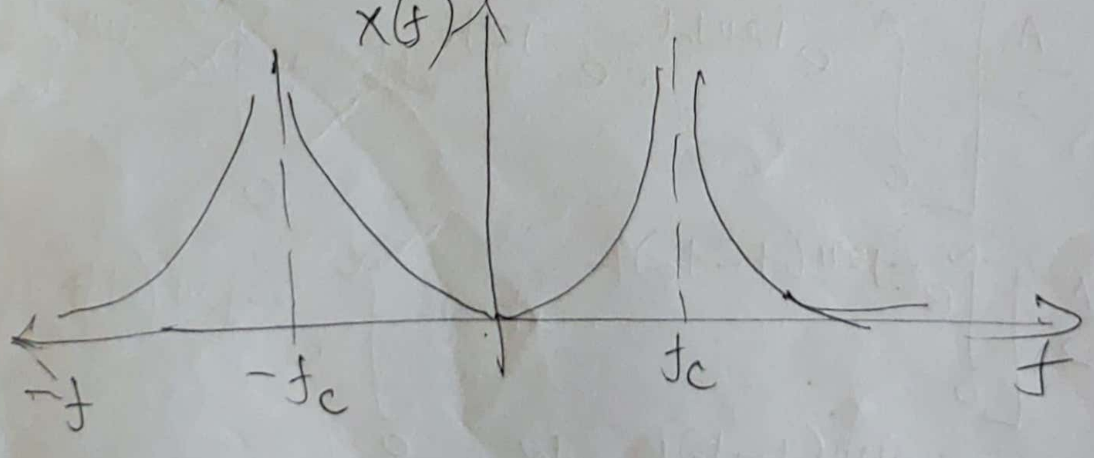
$$= \frac{A}{2}\left[\frac{1}{j2\pi(f-f_c)} + \frac{1}{j2\pi(f+f_c)}\right]$$

$$= \frac{A}{j4\pi}\left[\frac{1}{f-f_c} + \frac{1}{f+f_c}\right]$$

$$= \frac{A}{j4\pi}\left[\frac{f+\cancel{f_c}+f-\cancel{f_c}}{f^2-f_c^2}\right]$$

$$= \frac{A}{j\cancel{4}_{2}\pi}\left[\frac{\cancel{2}f}{f^2-f_c^2}\right]$$

$$\boxed{X(f) = \frac{A}{j2\pi}\frac{f}{f^2-f_c^2}}$$



(5) Find. The LT of sinewave

$$f(t) = A \sin w_o t$$

$$\sin w_o t = \frac{1}{2j}\left[e^{jw_o t} - e^{-jw_o t}\right]$$

$$f(t) = \frac{A}{2j}\left[e^{jw_o t} - e^{-jw_o t}\right]$$

$$L\left[f(t)\right] = \frac{A}{2j}\left[L\left[e^{jw_o t}\right] - L\left[e^{-jw_o t}\right]\right]$$

$$L\left[e^{at}\right] = \frac{1}{s-a}$$

$$L\left[e^{jw_o t}\right] = \frac{1}{s-jw_o}$$

$$L\left[e^{-jw_o t}\right] = \frac{1}{s+jw_o}$$

$$= \frac{A}{2j}\left[\frac{1}{s-jw_o} - \frac{1}{s+jw_o}\right]$$

$$= \frac{A}{2j} \cdot \frac{\cancel{s}+jw_o - \cancel{s}+jw_o}{s^2+w_o^2}$$

$$= \frac{A}{2\cancel{j}} \frac{2\cancel{j}w_o}{s^2+w_o^2}$$

$$L\left[A \sin w_o t\right] = \frac{Aw_o}{s^2+w_o^2}$$

(6) Find the LT of cosine wave

$$f(t) = A \cos w_0 t$$

$$\cos w_0 t = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2}$$

$$f(t) = \frac{A}{2}\left[e^{jw_0 t} + e^{-jw_0 t}\right]$$

$$F(s) = \int_0^{\alpha} f(t) e^{-st} \cdot dt$$

$$= \int_0^{\alpha} \frac{A}{2}\left[e^{jw_0 t} + e^{-jw_0 t}\right] e^{-st} \cdot dt$$

$$= \frac{A}{2}\Big[$$

(1) Determine the initial value $x(0^+)$ for the following Laplace transform

(i) $X(s) = \dfrac{3}{s^2+5s-1}$

(ii) $X(s) = \dfrac{2s+3}{s(s^2+5s+6)}$

(i) $x(0^+) = \lim\limits_{s \to \infty} sX(s)$

$$= \lim_{s \to \infty} s\left[\frac{3}{s^2+5s-1}\right]$$

$$s = 1/x$$

$$= \lim_{x \to 0} \frac{1}{x}\left[\frac{3}{1/x^2 + 5/x - 1}\right]$$

$$= \lim_{x \to 0} \frac{1}{x}\left[\frac{3}{\frac{1+5x-x^2}{x^2}}\right]$$

$$= \lim_{x \to 0} \frac{1}{\cancel{x}}\left[\frac{3x^{\cancel{2}}}{1+5x-x^2}\right]$$

$$= 0$$

(ii) $x(0^+) = \lim\limits_{s \to \infty} sX(s)$

$$= \lim_{s \to \infty} \cancel{s}\left[\frac{2s+3}{\cancel{s}(s^2+5s+6)}\right]$$

$$= \lim_{x \to 0}\left[\frac{2\,1/x + 3}{1/x^2 + 5/x + 6}\right]$$

$$= \lim_{x\to 0} \frac{\frac{2}{x}+3}{\frac{1+5x+6x^2}{x^2}}$$

$$= \lim_{x\to 0} \frac{2+3x}{\cancel{x}} \frac{x^{\cancel{2}}}{(1+5x+6x^2)}$$

$$= \lim_{x\to 0} \frac{2x+3x^2}{6x^2+5x+1}$$

$$= 0$$

(2) Find $x(\infty)$ if $X(s)$ is given by

(i) $\frac{1}{s-2}$ (ii) $\frac{s-1}{s(s+1)}$ (iii) $\frac{1}{s^2+4}$

(i) The poles lies in right half of the S-plane $\therefore x(\infty) = \infty$

(ii) $x(\infty) = \lim_{s\to 0} \cancel{s} \frac{(s-1)}{\cancel{s}(s+1)} = \frac{0-1}{0+1} = -1$

(iii) $X(s) = \frac{1}{s^2+4} = \frac{1}{(s+2j)(s-2j)}$

The poles are lying on the imaginary axis $\therefore x(\infty)$ is not defined.

Unit-III Linear Time invariant Continuous Time systems

Impulse Response :-

⟹ impulse response
⟹ convolution integrals
⟹ Differential equation
⟹ Fourier and Laplace Transforms in analysis of CT.
⟹ systems connected in series/parallel.

* Impulse response is the o/p of the system for a unit impulse input.
* If i/p $x(t) = \delta(t)$
  o/p $y(t) = h(t)$

$$L[\delta(t)] = 1$$

$$F[\delta(t)] = 1$$

* $H(\omega)$ of an ~~LTI system known in~~ LTI system is known in freq. domain.
  * Impulse response of the system can be found by finding the inverse fourier transform of $H(\omega)$

$$h(t) = F^{-1}[H(\omega)]$$

$$h(t) = L^{-1}[H(s)]$$

Step response :-

* The step response can be obtained by using convolution integral.

* If $u(t)$ is i/p
  $h(t)$ is impulse.

The step response

$$s(t) = h(t) * u(t)$$

* If the system is non causal

$$s(t) = \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau$$

$$\therefore s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

* If the system is causal

$$s(t) = \int_{0}^{t} h(\tau)\, d\tau.$$

ie The unit step response of a Continuous time LTI system is the running integral of its impulse respons.

Stability :-

* A system is stable if every bounded i/p produces a bounded o/p.

* The BIBO stability of an LTI system can be easily determind from its impulse response.

* For CT LTI system to be BIBO stable, its implulse response $h(t)$ must be absolutely integrable.

$$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty.$$

① Consider a stable LTI system characterized by differential equations

$$\frac{dy(t)}{dt} + 5y(t) = x(t)$$ Find its impulse response.

$$\frac{dy(t)}{dt} + 5y(t) = x(t)$$

$$s\,Y(s) + 5Y(s) = X(s)$$

$$Y(s)[s+5] = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+5}$$

The impulse response $h(t) = L^{-1}[H(s)]$

$$= L^{-1}\left[\frac{1}{s+5}\right]$$

$$h(t) = e^{-5t}u(t)$$

2. Find whether the following systems with impulse response h(t) are stable or not.

(i) $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$

(ii) $h(t) = e^{-3|t|}$

(iii) $h(t) = t e^{-t} u(t)$

(iv) $h(t) = e^{2t} u(t-2)$

(i) $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$

For the system to be BIBO Stable, its impulse response must be absolutely integrable.

ie $\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$.

$$= \int_{-\infty}^{\infty} \frac{1}{RC} e^{-t/RC} u(t) \, dt$$

$$= \frac{1}{RC} \left[\frac{e^{-t/RC}}{-1/RC} \right]_0^{\infty}$$

$$= \frac{1}{\cancel{RC}} \left[\frac{e^{-\infty} - e^{0}}{-\cancel{1/RC}} \right] = \frac{-1}{-1}$$

$= 1 < \infty$

frequency response of the system.

$Y(\omega)$ — F.T of o/p signal $y(t)$

$X(\omega)$ — " i/p " $x(t)$

* Transfer fn is defined as the ratio of the fourier transform of the o/p to the fourier " of the i/p.

* If i/p $x(t)$ is an impulse $\delta(t)$

$$X(\omega) = 1$$

$$Y(\omega) = H(\omega)$$

$F^{-1}[H(\omega)] = h(t)$ is called impulse response of the system.

$$H(\omega) = |H(\omega)| \angle H(\omega)$$

$|H(\omega)|$ — magnitude response

$\angle H(\omega)$ — phase response.

$$H(\omega) = H_r(\omega) + j H_I(\omega)$$

$H_r(\omega)$ — real part of $H(\omega)$

$H_I(\omega)$ — Imaginary part of $H(\omega)$

$$= \int_0^{\infty} t e^{-t} dt$$

$\cancel{\int uv = u dv - \int v du}$

$$= \left[t \frac{e^{-t}}{-t} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-t}}{-t} \cdot 1 \, dt$$

$\cancel{u = t}$

$\int u \, dv = uv - \int v \, du$

$$= 0 - \left[\frac{e^{-t}}{t} \right]_0^{\infty}$$

$u = t \quad du = 1$

$dv = e^{-t} dt$

$$= -1 e^{-\infty} - e^{0}$$

$V = \frac{e^{-t}}{-t}$

(iv) $h(t) = e^{2t} u(t-2)$

$$\int_{-\infty}^{\infty} |h(t)| \, dt = \int_{-\infty}^{\infty} e^{2t} u(t-2) dt$$

$$= \int_2^{\infty} e^{2t} dt$$

$$= \left[\frac{e^{2t}}{2} \right]_2^{\infty}$$

$$= e^{\infty} - \frac{e^4}{2}$$

$$= \infty - \frac{e^4}{2}$$

$$= \infty$$

The o/p is unbounded

$\therefore$ The system is unstable

2. Find the step response for given impulse response

(i) $h(t) = t^3 u(t)$

(ii) $h(t) = u(t+3) - u(t-5)$

(iii) $h(t) = e^{-4t} u(t)$

(i) $h(t) = t^3 u(t)$, $s(t) = \int_{-\infty}^{t} h(\tau) d\tau$

$$s(t) = \int_{-\infty}^{t} \tau^3 u(\tau) d\tau$$

$$= \int_{0}^{t} \tau^3 d\tau$$

$$= \frac{\tau^4}{4} \quad \text{for } t \geq 0$$

$$\boxed{\therefore s(t) = \frac{t^4}{4} u(t)}$$

(ii) $h(t) = u(t+3) - u(t-5)$

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

$$= \int_{-\infty}^{t} [u(\tau+3) - u(\tau-5)] d\tau$$

$$= \int_{-3}^{t} d\tau - \int_{5}^{t} d\tau$$

$$= [\tau]_{-3}^{t} - [\tau]_{5}^{t}$$

$$= \cancel{t} + 3 \cancel{-t} + 5$$

$$= 8$$

$$s(t) = 8.$$

(iii) $h(t) = e^{-4t} u(t)$

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$= \int_{-\infty}^{t} e^{-4\tau} u(\tau)\, d\tau$$

$$= \int_{0}^{t} e^{-4\tau}\, d\tau$$

$$= \left[\frac{e^{-4\tau}}{-4}\right]_{0}^{t}$$

$$= \frac{e^{-4t} - e^{0}}{-4}$$

$$= \frac{e^{-4t}}{-4} + \frac{1}{4} \quad \text{for } t \geq 0$$

$$s(t) = -\frac{1}{4} e^{-4t} u(t) + \frac{1}{4} u(t).$$

Convolution Integral :-

* Convolution is a mathematical operation which is used to express the i/p o/p relationship of an LTI system.

* correlation is a mathematical operation is similar to convolution.

* Types

1. cross correlation
2. auto correlation

* when one signal is correlated with another signal to form third signal it is called cross correlation.

* when a signal is correlated with itself to form another signal it is called auto correlation.

Any arbitary signal $x(t)$ can be represented as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau.$$

The system o/p is

$$y(t) = H[x(t)]$$

$$\therefore y(t) = H\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right]$$

For Linear system

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, H[\delta(t-\tau)]\, d\tau$$

* If the response of the system due to impulse $\delta(t)$ is $h(t)$, then the response of the system due to delayed impulse is

$$h(t,\tau) = H[\delta(t-\tau)]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t,\tau)\, d\tau$$

* For time Invariant system, the o/p due to input delayed by τ second is equal to the o/p delayed by τ sec

ie $h(t,\tau) = h(t-\tau)$

$$\therefore y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$y(t) = x(t) * h(t)$$

Properties of convolution integral :-

Commutative property :-

$h_1(t)$ & $h_2(t)$ connected in series.

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow y(t)$$

$$=$$

$$x(t) \longrightarrow \boxed{h_2(t)} \longrightarrow \boxed{h_1(t)} \longrightarrow y(t)$$

$$h_1(t) * h_2(t) = h_2(t) * h_1(t)$$

$$h_1(t) * h_2(t) = \int_{-\infty}^{\infty} h_1(\tau)\, h_2(t-\tau)\, d\tau$$

$$t - \tau = t' \qquad d\tau = dt'$$

$$\tau = t - t'$$

$$= \int_{-\infty}^{\infty} h_1(t-t')\, h_2(t')\, dt'$$

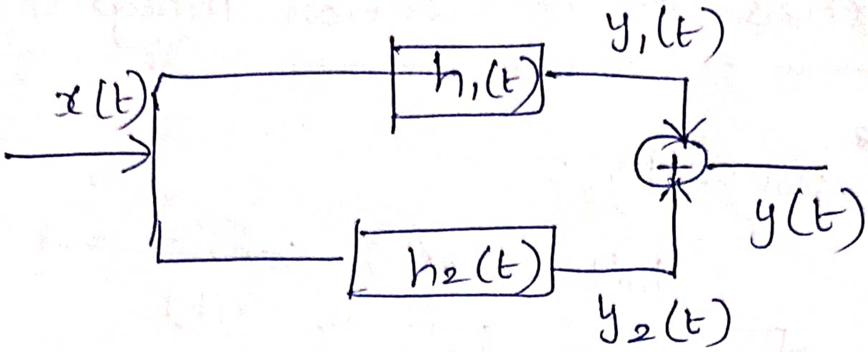
$$= \int_{-\infty}^{\infty} h_2(t')\, h_1(t-t')\, dt'$$

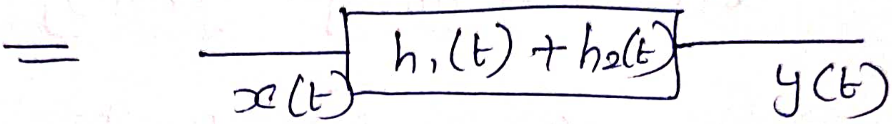
$$= h_2(t) * h_1(t)$$

Distributive property :-

$$x(t) * \left[h_1(t) + h_2(t) \right] = x(t) * h_1(t) + x(t)\, h_2(t)$$

$h_1(t)$ & $h_2(t)$ connected in parallel.





The o/p of the First system

$$y_1(t) = x(t) * h_1(t)$$

The o/p of the second system

$$y_2(t) = x(t) * h_2(t)$$

$$y(t) = y_1(t) + y_2(t)$$

$$= x(t) * h_1(t) + x(t) * h_2(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h_1(t-\tau)\, d\tau + \int_{-\infty}^{\infty} x(\tau)\, h_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[h_1(t-\tau) + h_2(t-\tau) \right] d\tau$$

$$\boxed{h(t-\tau) = h_1(t-\tau) + h_2(t-\tau)}$$

$$h(t) = h_1(t) + h_2(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

$$= x(t) * h(t)$$

If two systems are connected in parallel then the impulse of the system to the i/p signal $x(t)$ is equal to the sum of the two impulse response.

Associative property :-

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

Let us consider two continuous time LTI systems with impulses $h_1(t)$ and $h_2(t)$ connected in series. 18

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow \; = \; x(t) \longrightarrow \boxed{h_1(t) * h_2(t)} \longrightarrow y(t)$$

Proof :-

The o/p of the 1st system

$$y_1(t) = x(t) * h_1(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h_1(t-\tau)\, d\tau.$$

The o/p of the 2nd system

$$y_2(t) = y_1(t) * h_2(t)$$

$$= \int_{-\infty}^{\infty} y_1(k)\, h_2(t-k)\, dk$$

$$y(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, h_1(k-\tau)\, d\tau \right] h_2(t-k)\, dk$$

$$y(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, h_1(k-\tau)\, h_2(\cancel{t-k})\, dk\, d\tau$$

$$(k-\tau) = t'$$

$$dk = dt'$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h_1(t')\, h_2(t-(t'+\tau))\, dt' \right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h_1(t')\, h_2[(t-\tau)-t']\, dt' \right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

wher

$$h(t-\tau) = \int_{-\infty}^{\infty} h_1(t')\, h_2[(t-\tau)-t']\, dt' = h_1(t) * h_2(t)$$

$$y(t) = x(t) * h(t)$$

$$y(t) = x(t) * [h_1(t) * h_2(t)]$$

Convolution with impulse response

$x(t) \longrightarrow \delta(t) \longrightarrow y(t)$

$\therefore y(t) = x(t) * \delta(t) = x(t)$

Proof:-

$$y(t) = x(t) * \delta(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$= x.(t)$$

Convolution with step response :-

Convolution of a unit step signal with an impulse response is given by

$$y(t) = x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau.$$

Proof:-

$$y(t) = x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$u(t-\tau) = 1$ for $t > 0$

$$= \int_{-\infty}^{\infty} x(\tau)\, d\tau$$

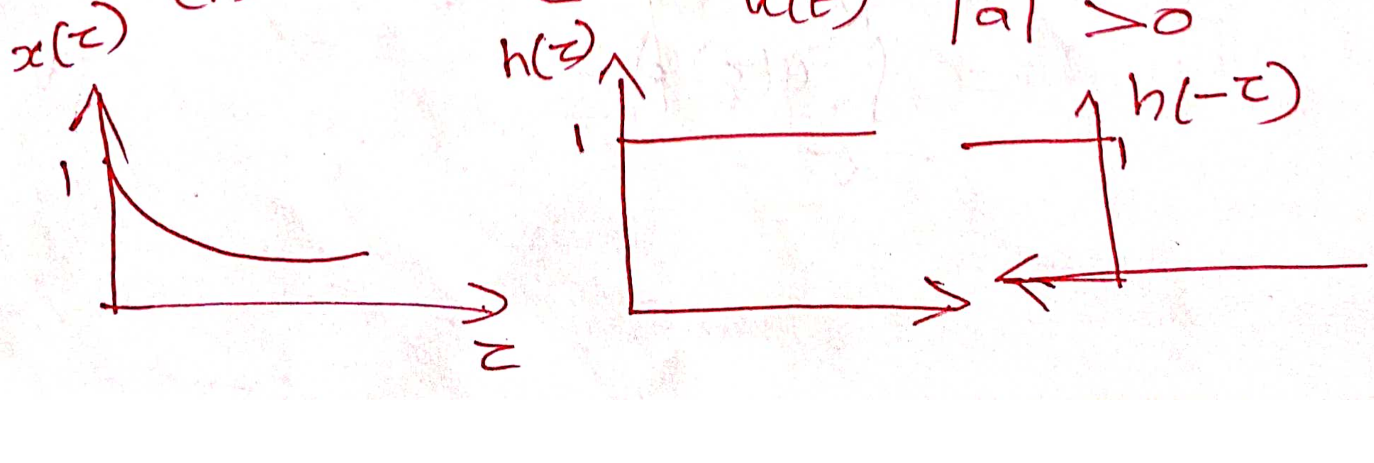
Graphical method procedure to perform convolution :-

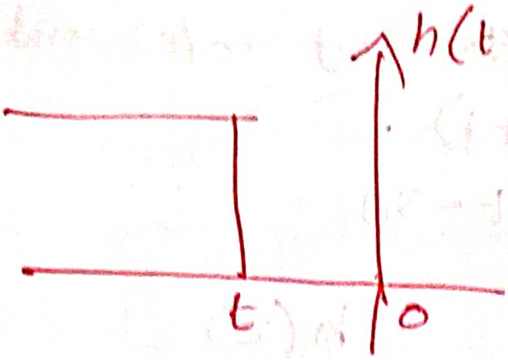
1. plot the given signal $x(\tau)$ and impulse $h(\tau)$ by replacing t by dummy variable τ
2. obtain $h(t-\tau)$ by folding $h(\tau)$ about $\tau=0$ and by shifting by time t
3. multiply signal $x(\tau)$ and impulse response $h(t-\tau)$ and integrate over the overlapped area to obtain $y(t)$
4. Use the value of t such that the fn of $x(\tau)$ and $h(t-\tau)$ changes calculate $y(t)$
5. Repeat step 4 ~~to~~ 5 for all intervals

1. Find the convolution of $x(t)$ and $h(t)$

(i) $x(t) = u(t)$

(ii) $h(t) = e^{-at}\, u(t)$ $|a| > 0$





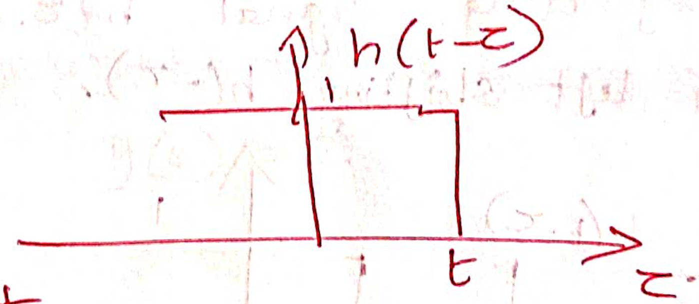
$x(\tau)$ and $h(t-\tau)$ are not overlapped.

$$\therefore y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)$$

$$= 0$$

For $t > 0$

$$x(\tau)\, h(t-\tau) = \begin{cases} e^{-at}, & 0<\tau<t \\ 0 & \text{otherwise} \end{cases}$$



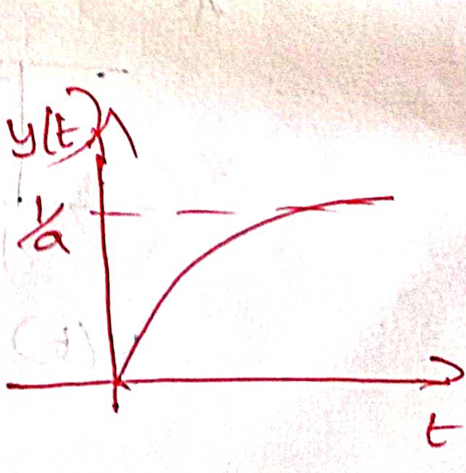
$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau.$$

$$= \int_0^t e^{-a\tau}\, d\tau$$

$$= \left[\frac{e^{-a\tau}}{-a}\right]_0^t$$

$$= -\frac{1}{a}\left[e^{-at} - e^{0}\right]$$

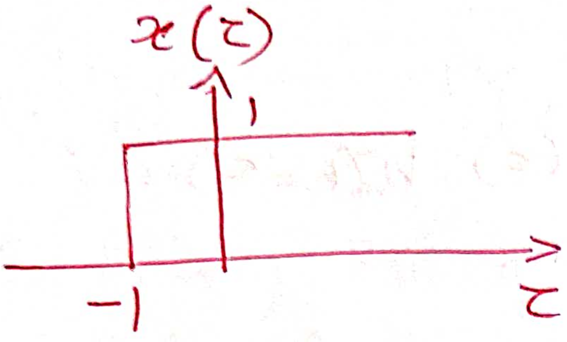
$$= \frac{1}{a}\left[1 - e^{-at}\right]$$

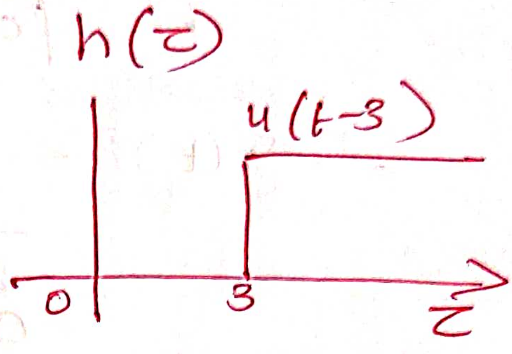


2. Find the convolution of $x(t)$ and $h(t)$

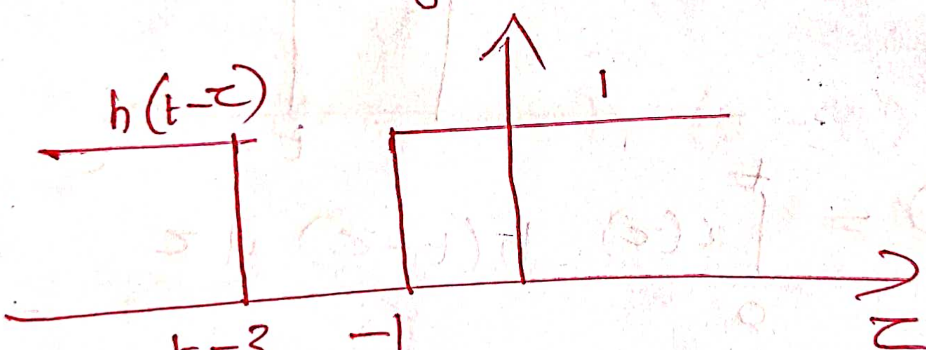
$x(t) = u(t+1)$.

$h(t) = u(t-3)$

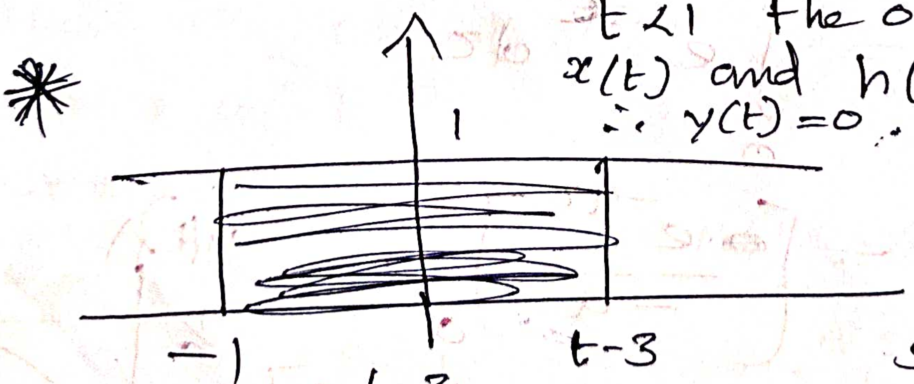




* The signals $x(t)$ and $h(t-\tau)$ on the the same axis. The signal $h(t-\tau)$ is obtained by folding signal $h(\tau)$ about $\tau = 0$ then left shifting $h(-\tau)$.



*



$t<1$ the overlap area of $x(t)$ and $h(t-\tau)$ is zero $\therefore y(t) = 0$. Now shift the signal $h(t-\tau)$. This overlap is shown by shaded area.

$$y(t) = \int_{-1}^{t-3} x(\tau)\, h(t-\tau)\, d\tau.$$

$$= \int_{-1}^{t-3} d\tau$$

$$= \left[\tau\right]_{-1}^{t-3}$$

$$= t - 3 + 1$$

$$= t - 2$$

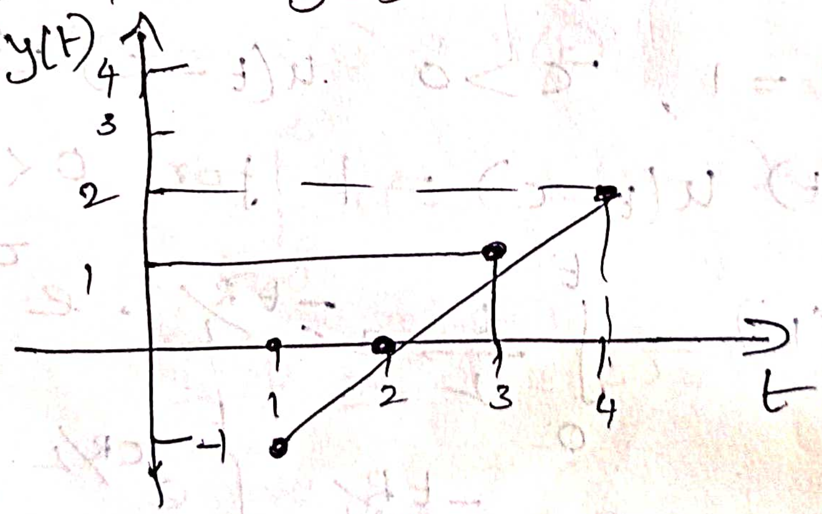
$$y(t) = \begin{cases} 0 & \text{for } t < 1 \\ t-2 & \text{for } t \geq 1 \end{cases}$$

$t = 1 \quad y(1) = -1$

$t = 2 \quad y(2) = 0$

$t = 3 \quad y(3) = 1$

$t = 4 \quad y(4) = 2$



① By using continuous time convolution integral obtain response of the system to unit step signal given the impulse response
$h(t) = \frac{R}{L} e^{-tR/L} u(t)$

Sol :-

$$x(t) = u(t)$$
$$h(t) = \frac{R}{L} e^{-tR/L} u(t)$$
$$y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} u(\tau) \frac{R}{L} e^{-(t-\tau)R/L} u(t-\tau)\, d\tau$$

$u(t) = 1$, $\tau > 0$ $\quad u(t-\tau) = 1$ for $t > \tau$

ie $u(t)\, u(t-\tau) = 1$ for $0 < \tau < t$

$$y(t) = \int_0^t \frac{R}{L} e^{-tR/L} \cdot e^{\tau R/L}\, d\tau.$$
$$= \frac{R}{L} e^{-tR/L} \int_0^t e^{\tau R/L}\, d\tau.$$
$$= \cancel{\frac{R}{L}} e^{-tR/L} \left[\frac{e^{\tau R/L}}{\cancel{R/L}} \right]_0^t$$
$$= e^{-tR/L} \left[e^{tR/L} - e^{0} \right]$$

$$= e^{-tR/L} e^{tR/L} - 1 \cdot e^{-tR/L}$$

$$= e^{0} - e^{-tR/L}$$

$$\boxed{y(t) = 1 - e^{-tR/L}}$$

(2) The i/p signal $x(t)$ and impulse response $h(t)$ of the system are described by

$x(t) = e^{-3t} u(t)$ and $h(t) = u(t-1)$

Evaluate the o/p using convolution.

$$x(t) = e^{-3t} u(t)$$

$$h(t) = u(t-1)$$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-3\tau} u(\tau)\, u(t-\tau-1)\, d\tau$$

$u(\tau) = 1$ for $\tau > 0$

$u(t-\tau-1) = 1$ for $t-1 > \tau$

ie $u(\tau)\, u(t-\tau-1) = 1$ for $0 < \tau < t-1$

$$y(t) = \int_0^{t-1} e^{-3\tau} d\tau$$

$$= \left[\frac{e^{-3\tau}}{-3}\right]_0^{t-1}$$

$$= \frac{e^{-3(t-1)} - e^0}{-3}$$

$$= \frac{e^{-3t+3} - 1}{-3}$$

$$\boxed{y(t) = \frac{1}{3}\left[1 - e^{-3(t-1)}\right]}$$

③ The signal $x(t) = u(t-3) - u(t-6)$ is fed thro' an LTI system with an impulse response $h(t) = e^{-3t} u(t)$. Determine the o/p response.

$$x(t) = u(t-3) - u(t-6)$$

$$h(t) = e^{-3t} u(t)$$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\cancel{= \int_{-\infty}^{\infty} [u(\tau-3) - u(\tau-6)] \cdot e^{-3(t-\tau)}\, u(t-\tau)\, d\tau}$$

① Find the convolution of the the following signal.

① $x_1(t) = e^{-2t} u(t)$

$x_2(t) = e^{-5t} u(t)$

② $x_1(t) = t\, u(t)$ $x_2(t) = t\, u(t)$

②

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_0^t \tau\, u(\tau)\, (t-\tau)\, u(t-\tau)\, d\tau$$

$u(\tau) = 1$

$u(t-\tau) = 1$

$$= \int_0^t \tau\,(t-\tau)\, d\tau$$

$$= \int_0^t (\tau t - \tau^2)\, d\tau$$

$$= \left[\frac{t\tau^2}{2} - \frac{\tau^3}{3}\right]_0^t$$

$$= \frac{t^3}{2} - \frac{t^3}{3}$$

$$= \frac{3t^3 - 2t^3}{6} = t^3/6$$

$$\boxed{x_1(t) * x_2(t) = \frac{t^3}{6}\, u(t)}$$

① $x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau.$

$$= \int_0^t e^{-3\tau} u(\tau)\; e^{-5(t-\tau)} u(t-\tau)\, d\tau$$

$$= \int_0^t e^{-3\tau}\; e^{-5(t-\tau)}\; d\tau.$$

$$= \int_0^t e^{-3\tau}\; e^{-5t}\; e^{5\tau}\; d\tau$$

$$= e^{-5t}\left[\int_0^t e^{\tau(5-3)}\, d\tau\right]$$

$$= e^{-5t}\left[\int_0^t e^{2\tau}\, d\tau\right]$$

$$= e^{-5t}\left[\frac{e^{2\tau}}{2}\right]_0^t$$

$$= e^{-5t}\left[\frac{e^{2t}}{2} - \frac{1}{2}\right]$$

$$= \frac{e^{-3t}}{2} - \frac{e^{-5t}}{2}$$

$$= \left(\frac{e^{-3t} - e^{-5t}}{2}\right) u(t)$$

Differential equations :-

* A CT System is modelled by Linear differential equation, it can be represented as

$$\sum_{K=0}^{N} a_K \frac{d^K y(t)}{dt^K} = \sum_{K=0}^{M} \frac{d^K x(t)}{dt^K}$$

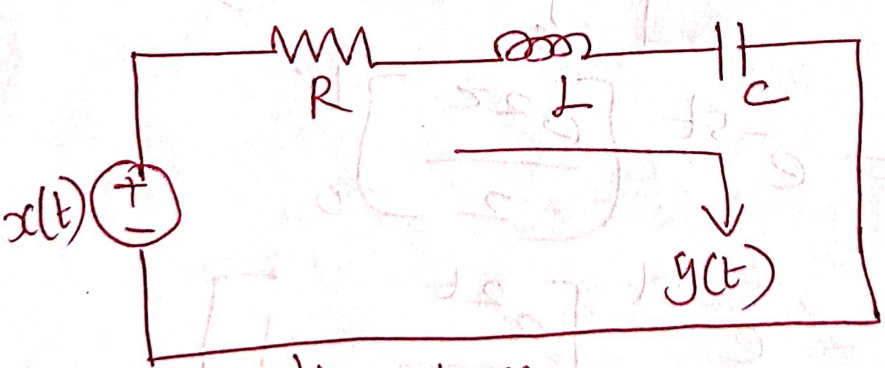
$y(t)$ - o/p response of the system

$x(t)$ - i/p " " " "

N - order of the differential equation

a_K, b_K - Constants.

* Consider physical model relating the input voltage $x(t)$ to the o/p ct $y(t)$ thro' the circuit element R, L, C



apply KVL

$$x(t) = R\,y(t) + L\frac{d\,y(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} y(t)\,dt$$

* Rearranging and differentiating eqn w.r.t t

$$R \frac{dy(t)}{dt} + L \frac{d^2 y(t)}{dt^2} + \frac{1}{C} \int_{-\infty} y(t) = x(t)$$

$$L \frac{d^2 y(t)}{dt^2} + R \frac{d\, y(t)}{dt} + \frac{1}{C} y(t) = x(t)$$

$$N = 2$$

* The general form of N^{th} order differential eqn is given by

$$a_n \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{y(t)}{dt^{N-1}} + \cdots + a_1 \frac{dy(t)}{dt}$$

$$+ a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

It consists of two components

(i) Natural response (zero i/p response)
(ii) forced response (zero state response)

Total response = Natural response + forced response.

Natural response (free response) :-

* The natural response of the system o/p can be obtained by the initial conditions while calculating the natural response of the system o/p, the i/p is made zero.

* ∴ Differential eqns is reduced to homogeneous equation is

$$\frac{d^N y(t)}{dt^N} a_{N-1} + \frac{d^{N-1} y(t)}{dt^{N-1}} + \ldots a_1 \frac{dy(t)}{dt} + a_0 = 0$$

$$a_N = 1$$

The solution to the homogeneous eqn is the form

$$y_h(t) = c\, e^{\lambda t}$$

$$\frac{d\, y_h(t)}{dt} = c\, \lambda\, e^{\lambda t}$$

$$\frac{d^2 y_h(t)}{dt^2} = c\, \lambda^2 e^{\lambda t}$$

$$\vdots$$

$$\frac{d^N y_n(t)}{dt^N} = c\, \lambda^N . e^{\lambda t}$$

Substituting these values in eqn

$$c\lambda^N e^{\lambda t} + a_{N-1} c\lambda^{N-1} e^{\lambda t} + \dots$$

$$a_1 c\lambda e^{\lambda t} + a_0 c\lambda e^{\lambda t} = 0$$

$$ce^{\lambda t}\left[\lambda^N + a_{N-1}\lambda^{N-1} + \dots a_1\lambda + a_0\right] = 0$$

$$ce^{\lambda t} \neq 0$$

$$\left[\lambda^N + a_{N-1}\lambda^{N-1} + \dots a_1\lambda + a_0\right] = 0$$

This polynomial is called the characteristics eqn of the system.

① A LTICT system is given by

$$\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 3y(t) = \frac{dx(t)}{dt} + x(t)$$

input $x(t) = e^{-4t}u(t)$

Find (i) natural response of the system for initial $y(0^+) = 2$

$$\frac{dy(0^+)}{dt} = 0$$

(ii) Force response of the system

(iii) Total response

(i) ~~The homogeneous equation is of the~~

The homogenons equation can be obtained by equating i/p to zero

$$\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 3y(t) = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

× +

−3 −2

+1×−3 −2

$$(\lambda+1)(\lambda-3) = 0$$

$$\lambda = 3, \lambda = -1$$

$$y_h(t) = c_1 e^{3t} + c_2 e^{-1t} \quad \text{— ①}$$

For $t=0$ $\quad y_h(0) = c_1 + c_2 \quad$ — ②

Differentiating eqn (1)

$$\frac{d\,y_h(t)}{dt} = c_1\, 3e^{3t} + c_2(-1)e^{-t}$$

For $t=0$

$$\frac{d\,y_h(t)}{dt} = 3C_1 - C_2$$

$$y(0^+) = 2 \qquad \frac{dy(0^+)}{dt} = 0$$

$$C_1 + \cancel{C_2} = 2$$

$$3C_1 - \cancel{C_2} = 0$$

$$4C_1 = 2$$

$$\boxed{C_1 = \tfrac{1}{2}} \qquad \boxed{C_2 = \tfrac{3}{2}}$$

$$y_h(t) = \tfrac{1}{2}e^{3t} + \tfrac{3}{2}e^{-t}$$

$$\boxed{y_h(t) = \tfrac{1}{2}e^{3t}u(t) + \tfrac{3}{2}e^{-t}u(t)}$$

(ii) **Forced response:-**

Force response = homogeneous solution + particular integral

$$\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 3y(t) = 0$$

$$y_h(t) = c_1 e^{3t} + c_2 e^{-t}$$

$$y_p(t) = c \cdot e^{-4t}$$

$$\frac{d y_p(t)}{dt} = c(-4) e^{-4t}$$

$$\frac{d^2 y_p(t)}{dt^2} = 16 c e^{-4t}$$

$$c\, 16 e^{-4t} - 2c(-4) e^{-4t} - 3ce^{-4t} = \frac{d e^{-4t}}{dt} + e^{-4t}$$

$$16 c e^{-4t} + 8 c e^{-4t} - 3ce^{-4t} = -4 e^{-4t} + e^{-4t}$$

$$\cancel{21}_{7}\, c \cancel{e^{-4t}} = -\cancel{3} \cancel{e^{-4t}}$$

$$c = \frac{-3}{21}$$

$$\therefore y_p(t) = \frac{-3}{21} e^{-4t}$$

$$y_f(t) = y_h(t) + y_p(t)$$

Fourier transforms in analysis of CT system

Consider an LTI system described by the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Taking FT on both sides

$$F.T\left[\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right] = F.T\left[\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right]$$

$$\sum_{k=0}^{N} a_k \, F.T\left[\frac{d^k y(t)}{dt^k}\right] = \sum_{k=0}^{M} b_k \, F.T\left[\frac{d^k x(t)}{dt^k}\right]$$

By using differentiation property

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

$H(\omega)$ is a transfer function or

The magnitude response is defined as

$$|H(\omega)| = \sqrt{H_r^2(\omega) + H_i^2(\omega)}$$

The phase response is defined as

$$\angle H(\omega) = \tan^{-1}\left[\frac{H_i(\omega)}{H_r(\omega)}\right]$$

properties of frequency response :-

* $H(\omega)$ takes on values for all ω i.e. on a continum of ω
* $H(\omega)$ is periodic with ω with period of 2π
* The magnitude response $|H(\omega)|$ is an even function of ω and symmetrical about π
* The phase response $\angle H(\omega)$ is an odd function of ω and antisymmetrical about π

(1) Find the differential eqns. for the system having impulse response $h(t) = [3e^{-3t} - 2e^{-2t}]u(t)$

$$h(t) = \left[3e^{-3t} - 2e^{-2t}\right]u(t)$$

$$= 3\frac{1}{j\omega+3} - 2\frac{1}{j\omega+2}$$

$$= \frac{3(j\omega+2) - 2(j\omega+3)}{(j\omega+3)(j\omega+2)}$$

$$H(\omega) = \frac{3j\omega \cancel{+6} - 2j\omega \cancel{-6}}{(j\omega)^2 + 5(j\omega) + 6}$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{j\omega}{(j\omega)^2 + 5j\omega + 6}$$

$$(j\omega)^2 Y(\omega) + 5(j\omega) Y(\omega) + 6 Y(\omega) = j\omega X(\omega)$$

Taking inverse FT

$$\frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} + 6 y(\omega) = \frac{dx(t)}{dt}$$

(2) Find the differential eqn for the system having impulse response $h(t) = [e^{-t} - e^{-2t}]u(t)$

$$h(t) = [e^{-t} - e^{-2t}]u(t)$$

Taking FT on Both sides

$$H(\omega) = \frac{1}{j\omega + 1} - \frac{1}{j\omega + 2}$$

$$= \frac{\cancel{j\omega} + 2 - \cancel{j\omega} - 1}{(j\omega+1)(j\omega+2)}$$

$$= \frac{1}{(j\omega)^2 + 2j\omega + j\omega + 2}$$

$$= \frac{1}{(j\omega)^2 + 3j\omega + 2}$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 3j\omega + 2}$$

$$(j\omega)^2 Y(\omega) + 3(j\omega) Y(\omega) + 2 Y(\omega) = X(\omega)$$

Taking inverse FT

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

3. The i/p and o/p of a casual LTI system are related by the differential eqn

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = x(t)$$

(i) Find the impulse response of the system

(ii) what is the response of the system if $x(t) = e^{-3t}u(t)$

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = x(t)$$

Take FT on both sides.

$$(j\omega)^2 Y(\omega) + 6(j\omega) Y(\omega) + 8Y(\omega) = X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + 6(j\omega) + 8\right] = X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 6(j\omega) + 8}$$

(i) The impulse response

$$h(t) = FT^{-1}[H(\omega)]$$

$$H(\omega) = \frac{1}{(j\omega)^2 + 6(j\omega) + 8}$$

$$= \frac{1}{(j\omega + 2)(j\omega + 4)}$$

Using partial fraction expansion method

$$H(\omega) = \frac{A}{j\omega + 2} + \frac{B}{j\omega + 4}$$

$$A = \left.\frac{1}{j\omega + 4}\right|_{j\omega = -2}$$

$$= \frac{1}{-2 + 4} = \frac{1}{2}$$

$$B = \left.\frac{1}{j\omega + 2}\right|_{j\omega = -4} = \frac{-1}{2}$$

$$H(\omega) = \frac{1/2}{j\omega + 2} - \frac{1/2}{j\omega + 4}$$

$$\therefore h(t) = FT^{-1}[H(\omega)]$$

$$\boxed{h(t) = \frac{1}{2} e^{-2t} u(t) - \frac{1}{2} e^{-4t} u(t)}$$

(ii) $x(t) = e^{-3t}u(t)$

$$X(w) = \frac{1}{jw+3}$$

$$H(w) = \frac{Y(w)}{X(w)} = \frac{1}{(jw+2)(jw+4)}$$

$$Y(w) = H(w)\,X(w)$$

$$= \frac{1}{(jw+2)(jw+4)(jw+3)}$$

using partial fraction expansion

$$= \frac{A}{jw+2} + \frac{B}{jw+3} + \frac{C}{jw+4}$$

$$A = \left.\frac{1}{(jw+3)(jw+4)}\right|_{jw=-2}$$

$$= \frac{1}{(-2+3)(-2+4)}$$

$$\boxed{A = \frac{1}{2}}$$

$$B = \left.\frac{1}{(jw+2)(jw+4)}\right|_{jw=-3}$$

$$= \frac{1}{(-3+2)(-3+4)}$$

$$\boxed{B = -1}$$

$$C = \frac{1}{(j\omega+2)(j\omega+3)}\Bigg|_{j\omega=-4}$$

$$= \frac{1}{(-4+2)(\cancel{j\omega}\,-4+3)}$$

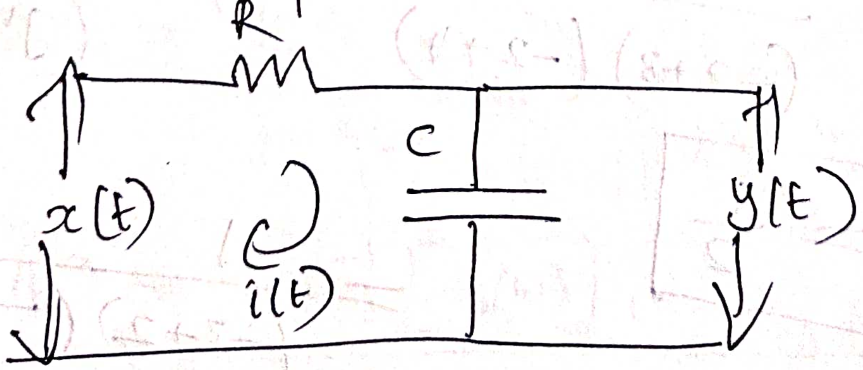
$$= \frac{1}{(-2)(-1)}$$

$$\boxed{C = \frac{1}{2}}$$

$$Y(\omega) = \frac{1/2}{j\omega+2} - \frac{1}{j\omega+3} + \frac{1/2}{j\omega+4}$$

$$\boxed{\therefore y(t) = \frac{1}{2}e^{-2t}u(t) - e^{-3t}u(t) + \frac{1}{2}e^{-4t}u(t)}$$

(4) Find the freq. response of the RC circuit shown in figure. plot the magnitude and phase response for RC = 1 Find the impulse response of the circuit



* Applying KVL to the loop

$$x(t) = R.\, i(t) + \frac{1}{C}\int i(t)\, dt$$

$$y(t) = \frac{1}{C}\int i(t)\, dt$$

Taking FT on both sides of the above equations

$$X(\omega) = R\, I(\omega) + \frac{1}{C}\frac{I(\omega)}{j\omega}$$

$$\boxed{X(\omega) = I(\omega)\left[R + \frac{1}{j\omega C}\right]} \quad \text{—— ①}$$

$$Y(\omega) = \frac{1}{C}\,\frac{I(\omega)}{j\omega}$$

$$\boxed{Y(\omega) = \frac{I(\omega)}{j\omega C}} \quad \text{—— ②}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\dfrac{\cancel{I(\omega)}}{j\omega C}}{\cancel{I(\omega)}\left[R + \dfrac{1}{j\omega C}\right]}$$

$$= \frac{\dfrac{1}{\cancel{j\omega C}}}{\dfrac{jR\omega C + 1}{\cancel{j\omega C}}}$$

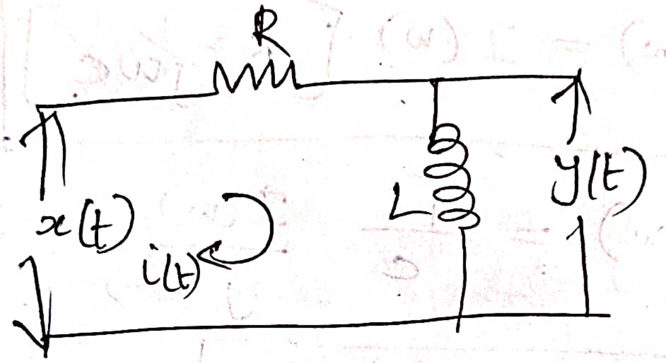
$$H(\omega) = \frac{1}{1 + j\omega RC}$$

$$h(t) = FT^{-1}\left[H(\omega)\right]$$

$$= FT^{-1}\left[\frac{1}{RC\left[j\omega + \frac{1}{RC}\right]}\right]$$

$$\boxed{h(t) = \frac{1}{R} e^{-\left(\frac{1}{RC}\right)t} u(t)}$$

5. Find the frequency response of RL n/w shown in figure.



Applying KVL to the Loop

$$x(t) = R\,i(t) + L\frac{di(t)}{dt}$$

$$y(t) = L\frac{di(t)}{dt}$$

Taking FT transform on both sides

$$X(\omega) = R\,I(\omega) + L(j\omega)I(\omega)$$

$$= I(\omega)\left[R + L(j\omega)\right]$$

$$Y(\omega) = L(j\omega)\,I(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{L(j\omega)\cancel{I(\omega)}}{\cancel{I(\omega)}\left[R + L(j\omega)\right]}$$

$$H(\omega) = \frac{j\omega L}{R + j\omega L}$$

$$h(t) = FT^{-1}\left[H(\omega)\right]$$

$$= FT^{-1}\left[\frac{\cancel{j\omega} L}{\cancel{j\omega}\left[\frac{R}{j\omega} + L\right]}\right]$$

$$= FT^{-1}\left[\frac{L}{R + j\omega L}\right]$$

$$= FT\left\{ \frac{L}{\cdot} \right.$$

① $\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} - 3y(t) = \frac{dx(t)}{dt} + 2x(t)$

Find the system transfer fn, frequency response and impulse response.

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} - 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Take FT on Both sides.

$$(j\omega)^2 Y(\omega) + (j\omega)Y(\omega) - 3Y(\omega) = j\omega X(\omega) + 2X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + j\omega - 3\right] = X(\omega)\left[j\omega + 2\right]$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega + 2}{(j\omega)^2 + j\omega - 3}$$

① The i/p and o/p are related by the differential eqn.

$$\frac{d^2y(t)}{dt^2} + \frac{6\,dy(t)}{dt} + 8y(t) = 2x(t)$$

Find the impulse response of the system

Take FT on Both sides

$$(j\omega)^2 Y(\omega) + 6(j\omega)Y(\omega) + 8Y(\omega) = 2X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + 6(j\omega) + 8\right] = 2X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{2}{(j\omega)^2 + 6(j\omega) + 8}$$

$\times$ 8, $+$ 6

2×4, $2+4$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2}{(j\omega+2)(j\omega+4)}$$

$$= 2\left(\frac{A}{j\omega+2} + \frac{B}{j\omega+4}\right)$$

$$A = \left.\frac{1}{j\omega+4}\right|_{j\omega=-2} \qquad B = \left.\frac{1}{j\omega+2}\right|_{j\omega=-4}$$

$$= \frac{1}{-2+4} = \frac{1}{2} \qquad = \frac{1}{-4+2}$$

$$\boxed{A = 1/2} \qquad \boxed{B = \frac{-1}{2}}$$

$$= 2\left(\frac{1/2}{j\omega+2} - \frac{1/2}{j\omega+4}\right)$$

$$= \frac{2/2}{j\omega+2} - \frac{2/2}{j\omega+4}$$

$$H(\omega) = \frac{1}{j\omega+2} - \frac{1}{j\omega+4}$$

$$h(t) = F^{-1}\left[H(\omega)\right]$$

$$\boxed{h(t) = e^{-2t}u(t) - e^{-4t}u(t)}$$

(2) The i/p $x(t)$ and o/p $y(t)$ for a system satisfy the differential equation.

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$$(j\omega)^2 Y(\omega) + 3(j\omega)Y(\omega) + 2Y(\omega) = X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + 3(j\omega) + 2\right] = X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 3(j\omega) + 2}$$

1x2

$$\boxed{H(\omega) = \frac{1}{(j\omega+1)(j\omega+2)}}$$

$$= \frac{A}{j\omega+1} + \frac{B}{j\omega+2}$$

$$A = \frac{1}{j\omega+2}\Big|_{j\omega=-1} \qquad B = \frac{1}{j\omega+1}\Big|_{j\omega=-2}$$

$$= \frac{1}{-1+\cancel{1}2} = \cancel{2} -1 \qquad = \frac{1}{-2+1} = -1$$

$$\boxed{\cancel{A = \tfrac{1}{3}}} \quad \boxed{A = -1} \qquad \boxed{B = -1}$$

$$H(\omega) = \frac{-1}{(j\omega+1)} + \frac{-1}{j\omega+2}$$

3/2

Take inverse FT on

$$\boxed{h(t) = -e^{-t}u(t) - e^{-2t}u(t)}$$

1/2

1/2

3/2

① A stable LTI system is characterised by differential equation. Find the impulse response using FT

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$$(j\omega)^2 Y(\omega) + 4(j\omega)Y(\omega) + 3Y(\omega) = j\omega X(\omega) + 2X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + 4(j\omega) + 3\right] = X(\omega)\left[j\omega + 2\right]$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3}$$

3 4

1×3 1+3

$$= \frac{j\omega + 2}{(j\omega+1)(j\omega+3)}$$

$$= \frac{2}{(j\omega+1)(j\omega+3)} + \frac{j\omega}{(j\omega+1)(j\omega+3)}$$

$$= 2\left(\frac{1}{(j\omega+1)(j\omega+3)}\right) + j\omega\left(\frac{1}{(j\omega+1)(j\omega+3)}\right)$$

$$= 2\left(\frac{A}{j\omega+1} + \frac{B}{j\omega+3}\right) + j\omega\left(\frac{A}{(j\omega+1)} + \frac{B}{j\omega+3}\right)$$

$$A = \frac{1}{j\omega+3}\bigg|_{j\omega=-1} \qquad B = \frac{1}{j\omega+1}\bigg|_{j=-3}$$

$$A = \frac{1}{-1+3} = \frac{1}{2} \qquad B = \frac{1}{-3+1} = \frac{1}{-2}$$

$$\boxed{A = \frac{1}{2}} \qquad \boxed{B = -\frac{1}{2}}$$

$$H(\omega) = 2\left(\frac{\frac{1}{2}}{j\omega+1} - \frac{\frac{1}{2}}{j\omega+3}\right) + j\omega\left(\frac{\frac{1}{2}}{j\omega+1} - \frac{\frac{1}{2}}{j\omega+3}\right)$$

$$= \frac{1}{j\omega+1} - \frac{1}{j\omega+3} + j\omega\left(\frac{\frac{1}{2}}{j\omega+1} - \frac{\frac{1}{2}}{j\omega+3}\right)$$

take $FT^{-1}[H(\omega)]$.

using differentiation property

$$h(t) = e^{-t}u(t) - e^{-3t}u(t) + \frac{1}{2}F^{-1}\left[\frac{j\omega}{j\omega+1}\right] - \frac{1}{2}F^{-1}\left[\frac{j\omega}{j\omega+3}\right]$$

$$= e^{-t}u(t) - e^{-3t}u(t) + \frac{1}{2}\frac{d}{dt}\left[e^{-t}\right]u(t) - \frac{1}{2}\frac{d}{dt}\left[e^{-3t}u(t)\right]$$

$$= e^{-t}u(t) - e^{-3t}u(t) - \frac{1}{2}e^{-t}u(t) + \frac{3}{2}e^{-3t}$$

$$= \frac{1}{2} e^{-t} u(t) + \frac{1}{2} e^{-3t} u(t).$$

Laplace transform in analysis of CT system

Let us consider a LTI system with i/p $x(t)$ and o/p $y(t)$ by a differential equation is

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots a_0 y(t)$$

$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M}} + \dots b_0 x(t)$$

$$y(0^-) = \frac{dy(0^-)}{dt} = \dots \frac{d^{N-1} y(0^-)}{dt^{N-1}} = 0$$

$$x(0^-) = \frac{dx(0^-)}{dt} = \dots \frac{d^{M-1} y(0^-)}{dt^{M-1}} = 0$$

Applying Laplace transform on both sides

$$a_N s^N Y(s) + a_{N-1} s^{N-1} Y(s) + \dots a_0 Y(s)$$

$$= b_M s^M X(s) + b_{M-1} s^{M-1} X(s) + \dots b_0 X(s)$$

$$Y(s)\left[a_N s^N + a_{N-1} s^{N-1} + \dots a_0\right] =$$

$$X(s)\left[b_M s^M + b_{M-1} s^{M-1} + \dots b_0\right]$$

$$\therefore H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^M + b_{m-1} s^{M-1} + \dots b_0}{a_N s^N + a_{N-1} s^{N-1} + \dots a_0}$$

$H(s)$ — transfer fn (or) system fn.

$Y(s)$ — laplace transform of the o/p

$X(s)$ = " " " " i/p

poles and zeros of the s-plane:

$$H(s) = \frac{N(s)}{D(s)} = \frac{K(s-z_1)(s-z_2)\dots(s-z_M)}{(s-p_1)(s-p_2)\dots(s-p_N)}$$

K — real number

$z_1, z_2 \dots z_M$ are zeros of the system

p_1, p_2, p_3 " poles of the "

① Plot the pole zero diagram of the following transfer fn

(i) $H(s) = \dfrac{s+2}{s^2+4s+13}$

(ii) $H(s) = \dfrac{s+4}{(s^2+2s+2)(s+3)}$

(i) $H(s) = \dfrac{s+2}{s^2+4s+13}$

$\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

Zeros

$Nr = 0$

$s+2 = 0$

$\boxed{s = -2}$

poles

$Dr = 0$

$\underset{a}{s^2} + \underset{b}{4s} + \underset{c}{13} = 0$

$s = \dfrac{-4 \pm \sqrt{16 - 4 \times 13}}{2(1)}$

$= \dfrac{-4 \pm \sqrt{16-52}}{2}$

$= \dfrac{-4}{2} \pm \dfrac{\sqrt{-36}}{2}$

$= \dfrac{-4}{2} \pm j\dfrac{\sqrt{36}}{2}$

$= -2 \pm j6/2$

$= -2 \pm 3j$

The poles are at

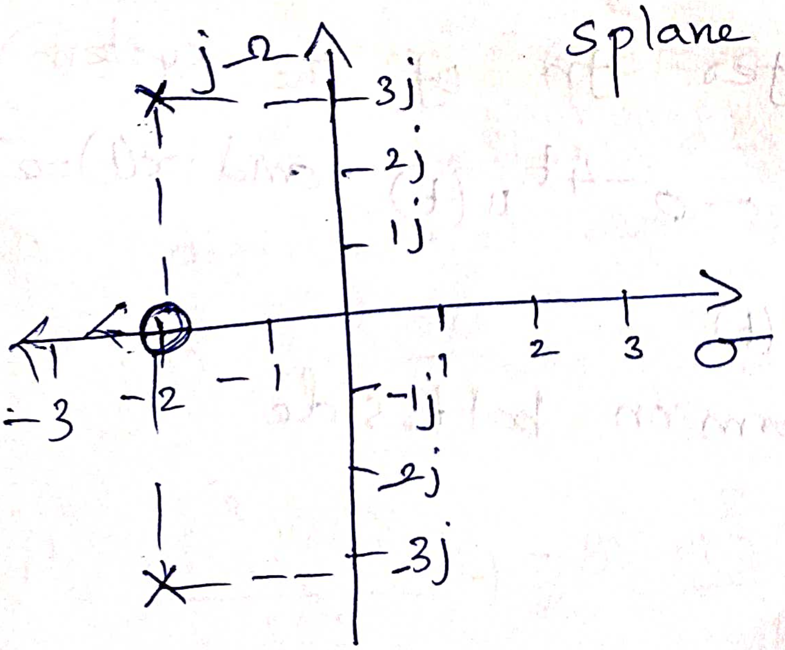
$-2+3j$

&

$-2-3j$

The zeros is at -2

splane



(ii) $H(s) = \dfrac{s+4}{(s^2+2s+2)(s+3)}$

For zero's

$\Rightarrow$ $Nr = 0$

$s+4 = 0$

$\boxed{s = -4}$

For poles

$Dr = 0$

$(s+3) = 0$

$s = -3$

$s^2+2s+2 = 0$

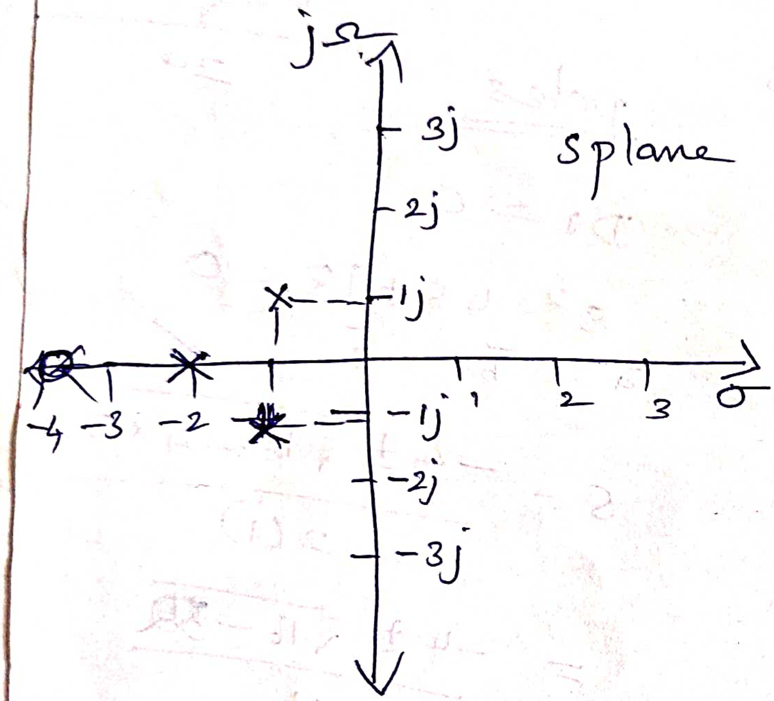
$= \dfrac{-2 \pm \sqrt{4-4\times 2}}{2}$

$= \dfrac{-2 \pm \sqrt{4-8}}{2}$

$= \dfrac{-2}{2} \pm \dfrac{\sqrt{-4}}{2}$

$= -1 \pm \dfrac{j2}{2}$

$= -1 \pm j$



(2) obtain the transfer fn of the system $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$ and $x(t) = e^{-2t}u(t)$

$x(t) = e^{-2t}u(t)$

Take Laplace transform on both side

$X(s) = \dfrac{1}{s+1}$

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t)$$

. Take Laplace transform on both side

$$Y(s) = \frac{1}{s+3} - \frac{1}{s+4}$$

$$= \frac{\cancel{s}+4 \cancel{-s}-3}{(s+3)(s+4)}$$

$$Y(s) = \frac{1}{(s+3)(s+4)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{(s+3)(s+4)}}{1/s+2}$$

$$\boxed{H(s) = \frac{s+2}{(s+3)(s+4)}}$$

(3) Find the system transfer fn for each of the system described by the following differential eqn :-

(i) $$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + 5x(t)$$

(ii) $$\frac{d^3y(t)}{dt^3} + \cancel{3}5\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 7y(t) = 4\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + 7x(t)$$

(i) $$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + 5x(t)$$

Take LT on both side.

$$s^2 Y(s) + 5s Y(s) + 4Y(s) = sX(s) + 5X(s)$$

$$Y(s)\left[s^2 + 5s + 4\right] = X(s)\left[s+5\right]$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{s+5}{s^2+5s+4}}$$

(ii) $$\frac{d^3y(t)}{dt^3} + 3\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 7y(t) = 4\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + 7x(t)$$

Take LT

$$s^3 Y(s) + 3s^2 Y(s) + 5s Y(s) + 7Y(s)$$

$$= 4s^2 X(s) + s X(s) + 7X(s)$$

$$Y(s)\left[s^3 + 3s^2 + 5s + 7\right] = X(s)\left[4s^2 + s + 7\right]$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{4s^2+s+7}{s^3+3s^2+5s+7}}$$

4) Solve the integral differential eqn

$$\frac{dy(t)}{dt} + 6\,y(t) + 5\int_0^t y(t)\,dt = e^{-3t},$$

$y(0) = 0$

$$\frac{dy(t)}{dt} + 6\,y(t) + 5\int_0^t y(t\;dt = e^{-3t}$$

Taking LT on both side

$$s\,Y(s) + 6\,Y(s) + 5\,\frac{Y(s)}{s} = \frac{1}{s+3}$$

$$Y(s)\left[s + 6 + 5/s\right] = 1/s+3$$

$$Y(s)\left[\frac{s^2+6s+5}{s}\right] = \frac{1}{s+3}$$

$$Y(s) = \frac{s}{(s+3)(s^2+6s+5)}$$

$\times$ $+$
5 6
1×5 $1+5$

$$= \frac{s}{(s+3)(s+1)(s+5)}$$

Taking partial fraction

$$Y(s) = \frac{A}{s+3} + \frac{B}{(s+1)} + \frac{C}{s+5}$$

$$A = \frac{s}{(s+5)(s+1)}\bigg|_{s=-3}$$

$$= \frac{-3}{(-3+5)(-3+1)} = \frac{-3}{2(-2)} = 3/4$$

$$\boxed{A = 3/4}$$

$$B = \frac{s}{(s+3)(s+1)}\bigg|_{s=-5}$$

$$C = \frac{s}{(s+3)(s+5)}\bigg|_{s=-1}$$

$$= \frac{-5}{(-5+3)(-5+1)}$$

$$= \frac{-1}{(-1+3)(-1+5)}$$

$$= \frac{-5}{(-2)(-4)}$$

$$= \frac{-1}{2(4)}$$

$$\boxed{B = -5/8}$$

$$\boxed{C = -1/8}$$

$$Y(s) = \frac{3/4}{s+3} - \frac{5/8}{s+5} - \frac{1/8}{s+1}$$

Taking inverse Laplace transform

$$y(t) = 3/4\, e^{-3t} u(t) - 5/8\, e^{-5t} u(t) - 1/8\, e^{-t} u(t)$$

Unit IV

Analysis of discrete Time Signals.

13/24

- ⟹ Base band signal sampling
- ⟹ Fourier transform of Discrete Time signals
- ⟹ properties of DTFT
- ⟹ Z-transform
- ⟹ Its properties.

Sampling of continuous Time signals

* The process of converting CT signal into DT signal is called sampling.
* After sampling the signal is defined at discrete instants of time.
* Time interval between two successive sampling instants is called sampling period (or) sampling interval.
* Sampling rate $\propto \frac{1}{\text{sampling period}}$

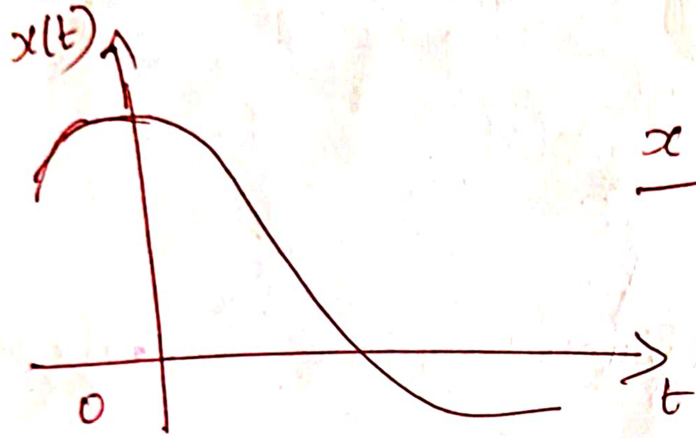
Sampling theorem :- A continuous time signal may be completely represented in its samples and recovered back if the sampling freq.

is $f_s \geq 2f_m$.

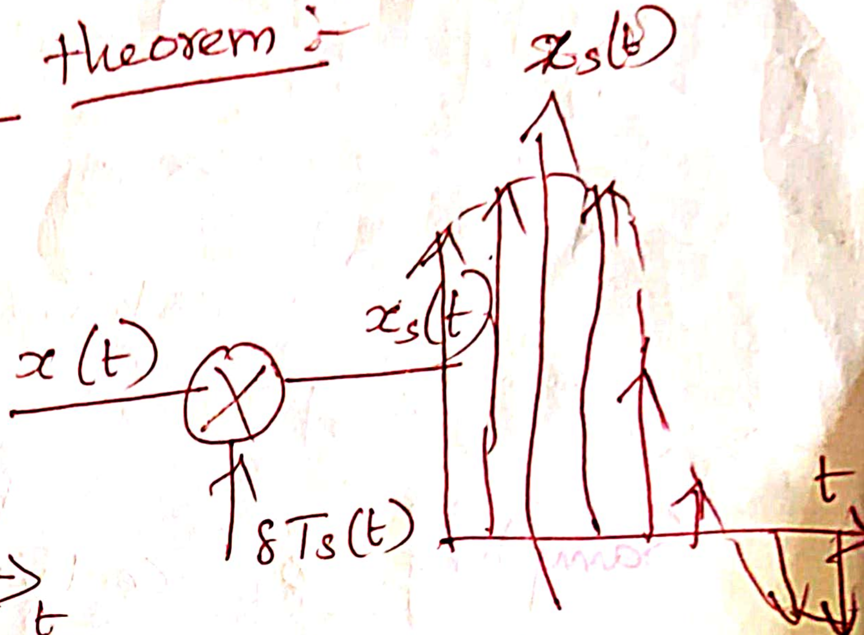
f_s — sampling freqnency

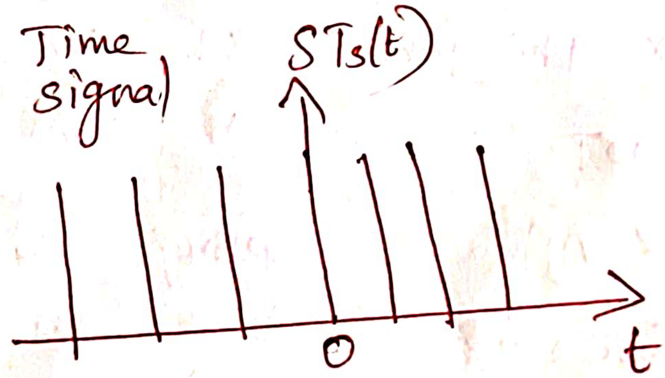
f_m — maximum freq.

Proof of sampling theorem :-



Continuous Time signal





* Let $x(t)$ be the continuous band limited signal which has no freq. components higher than f_m Hz.

* $$\delta T_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

* $$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s) \quad \text{— (1)}$$

* The exponential form of FT of $\delta_{T_s}(t)$

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_s t}$$

$$C_n = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-jn\omega_s t} . dt$$

$$= \frac{1}{T_s}$$

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} 1/T_s \; e^{jn\omega_s t}$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

~~Taking FT on~~

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{1}{T_s} e^{jn\omega_s t}$$

Taking FT on both side

$$X_s(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} FT\left[x(nT_s) e^{jn\omega_s t} \right]$$

using freq. shifting property

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

$$X_s(f) = \cancel{\frac{1}{T_s}}\, f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

f_s – sampling rate

$$f_s = \frac{1}{T_s}$$

$$X(f - nf_s) \Rightarrow x(t)$$

$$nf_s = 0, \pm f_s, \pm 2f_s \ldots\ldots$$

$$X_s(f) = f_s X(f) + f_s X(f + f_s) + f_s X(f + 2f_s) + \ldots\ldots$$

$$X_s(f) = f_s X(f) + \sum_{n=-\infty}^{\infty} f_s X(f - nf_s)$$

$\therefore$

Applying FT on eqn (1)

$$F.T\left[x_s(t)\right] = F.T\left[\sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)\right]$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\; F.T\left[\delta(t - nT_s)\right]$$

$$F.T\left[\delta(t - nT_s)\right] = \int_{-\infty}^{\infty} \delta(t - nT_s)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega nT_s} = e^{-j2\pi f nT_s}$$

$$X_s[f] = \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f nT_s}$$

$$X_0(f) = \frac{1}{f_s} X_s(f)$$

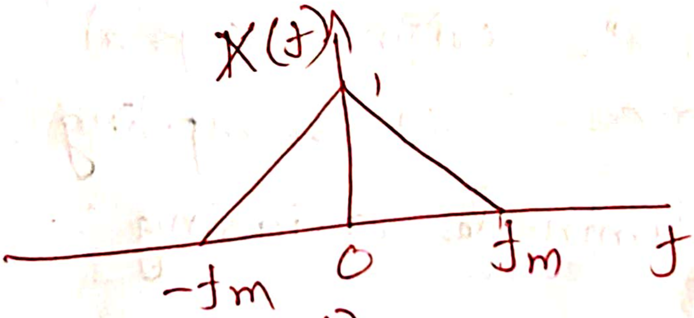
$$X(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s}$$

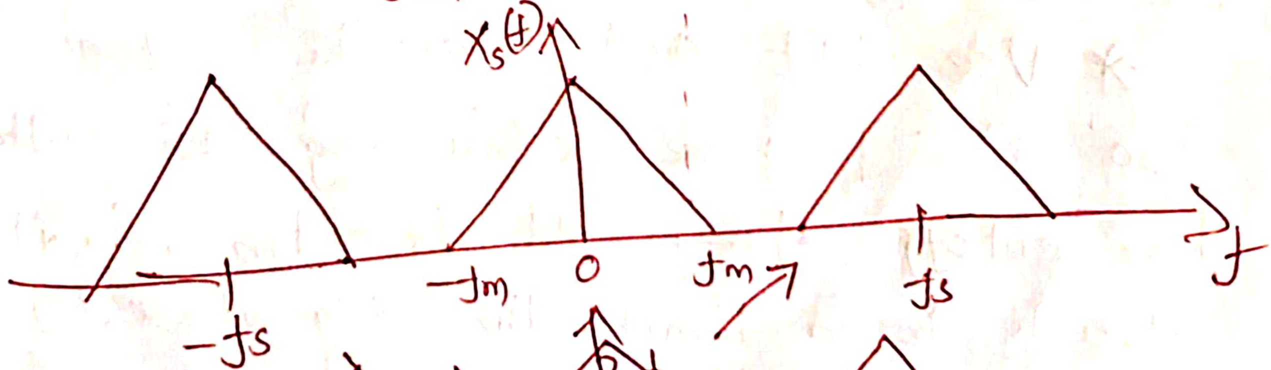
$$f_s = 2f_m \quad T_s = \frac{1}{2f_m}$$

$$X(f) = \frac{1}{2f_m} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) e^{-j\pi f n/f_m}$$

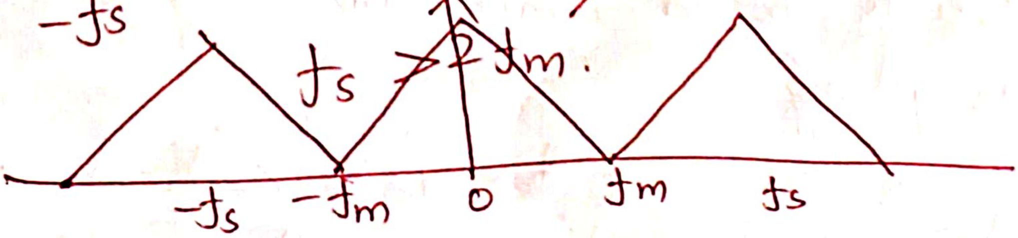
$x(t)$ may be recovered from $X(f)$ by taking inverse FT

$$x(t) = \text{IFT}\left[\frac{1}{2f_m} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) e^{-j\pi f n/f_m}\right]$$

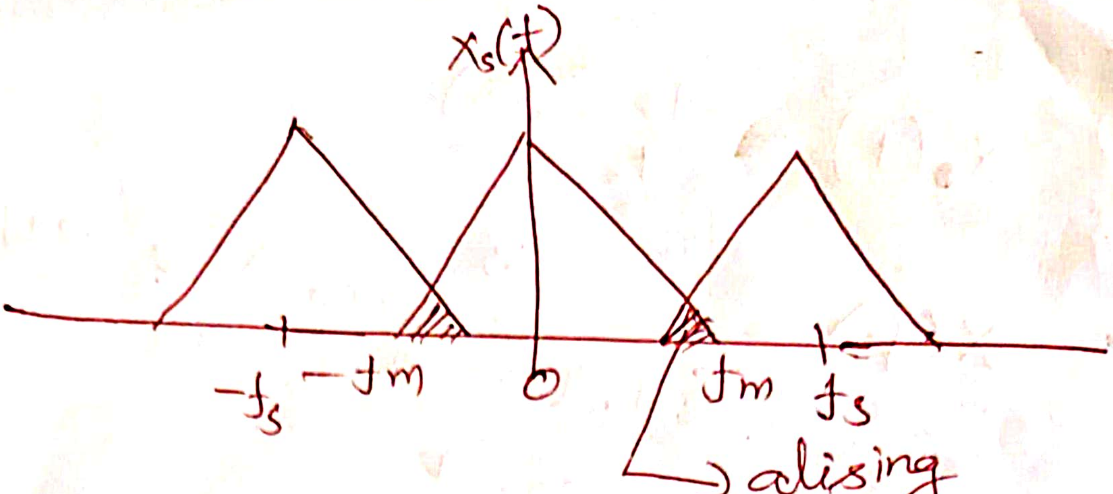




$f_s > 2f_m$.



$f_s = 2f_m$



Alising can be aviodad by if $f_s > 2f_m$.

<u>Nyquist rate</u> and <u>Nyquist interval</u> :-

$$\boxed{f_s = 2f_m}$$

$$T_s = \frac{1}{2f_m}$$

<u>Effect of aliasing</u> :-

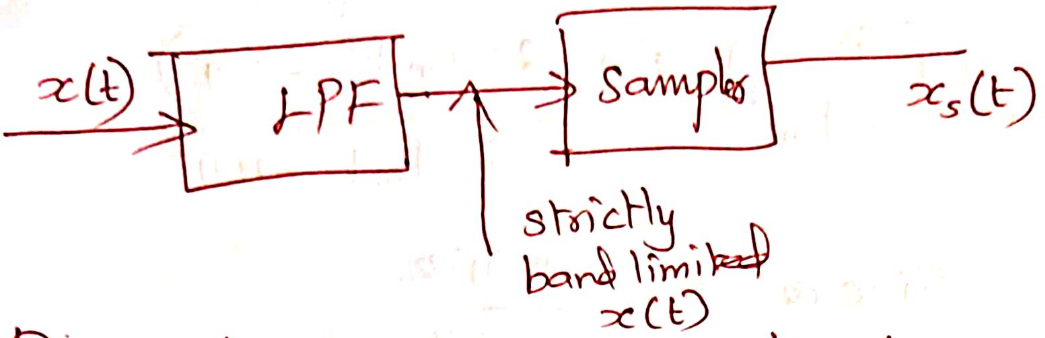
Due to aliasing some of the information contained in the original signal $x(t)$ is lost int process of sampling.

<u>How to eliminate aliasing</u> :-

* Use LPF and pass the signal $x(t)$ thro' it before sampling. This filter has cutoff freq. at $f_s = f_m$. ∴ it will strictly band limit the signal $x(t)$ before sampling.

* This filter is called as antialising filter (or) prealiasing filter.

(ii) use sampling freq f_s is greater than $2f_m$.



Reconstruction of signals from samples:-

By Taking inverse FT of $X(f)$

$$x(t) = \frac{1}{2f_m}\int_{-f_m}^{f_m} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{2f_m}\right) e^{-j\pi f n/f_m}\, e^{j2\pi f t}\, df$$

Interchanging the order of summation & Integration.

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \frac{1}{2f_m}\int_{-f_m}^{f_m} e^{j2\pi f\left(t-\frac{n}{2f_m}\right)}\, df$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \frac{1}{2f_m}\left[\frac{e^{j2\pi f\left(t-\frac{n}{2f_m}\right)}}{j2\pi t - \frac{j2\pi n}{2f_m}}\right]_{-f_m}^{f_m}$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \frac{1}{\cancel{2f_m}\left[\frac{j2\pi t\cdot 2f_m - j2\pi n}{\cancel{2f_m}}\right]}\left[e^{j2\pi f_m\left(t-n/2f_m\right)} - e^{-j2\pi f_m\left(t-n/2f_m\right)}\right]$$

$$= \sum_{n=-\alpha}^{\alpha} x\left(\frac{n}{2f_m}\right) \frac{e^{j2\pi f_m\left(t-\frac{n}{2f_m}\right)} - e^{-j2\pi f_m\left(t-\frac{n}{2f_m}\right)}}{2j\left[2\pi f_m t - n\pi\right]}$$

$$= \sum_{n=-\alpha}^{\alpha} x\left(\frac{n}{2f_m}\right) \frac{\sin\left(2\pi f_m t - n\pi\right)}{2\pi f_m t - n\pi}$$

$$\text{Sinc } a = \frac{\sin \pi a}{\pi a}.$$

$$= \sum_{n=-\alpha}^{\alpha} x\left(\frac{n}{2f_m}\right) \frac{\sin \pi\left(2f_m t - n\right)}{\pi\left(2f_m t - n\right)}$$

$$x(t) = \sum_{n=-\alpha}^{\alpha} x\left(\frac{n}{2f_m}\right) \text{sinc}\left(2f_m t - n\right)$$

Several methods are used to reconstruct the original signal.

* Zero order hold $\Rightarrow$ most commonly used method
* First " "
* Linear interpolator.

DTFT

DTFT of $x(n)$ is defined as

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Inverse DTFT of $X(\omega)$ is defined as

$$F^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

Freq. spectrum of discrete Time signal

$$X(\omega) = \underbrace{X_r(\omega)}_{\text{real part}} + j\underbrace{X_i(\omega)}_{\text{imaginary part}}$$

magnitude spectrum is defined as

$$|X(\omega)| = \sqrt{X_r^2(\omega) + X_i^2(\omega)}$$

phase spectrum is defined as

$$\angle X(\omega) = \tan^{-1}\left[\frac{X_i(\omega)}{X_r(\omega)}\right]$$

1) Find the DTFT of $x(n) = \delta(n)$

$$x(n) = \delta(n)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\omega n}\Big|_{n=0}$$

$$= 1$$

② Find the DTFT of $x(n) = u(n)$

$$x(n) = u(n)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} u(n) e^{-j\omega n} \qquad u(n) = \begin{cases} 1 & n>0 \\ 0 & n<0 \end{cases}$$

$$= \sum_{n=-\infty}^{\infty} 1\, e^{-j\omega n}$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$= \frac{1}{1-e^{-j\omega}}$$

③ Find the DTFT of the following sequences

(i) $x(n) = \{1, 2, -1, 3\}$

(ii) $x(n) = 3^n u(n)$

(iii) $x(n) = (0.5)^n u(n) + 2^n u(-n-1)$

(iv) $x(n) = \left(\frac{1}{3}\right)^n u(n+1)$

(v) $x(n) = \left(\frac{1}{3}\right)^{n-3} u(n-3)$

(i) $x(n) = \{1, 2, -1, 3\}$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Frequency Shifting property

$$X(\omega) = x(0) + x(1)e^{-j\omega} + x(2)e^{-2j\omega} + x(3)e^{-3j\omega}$$

$x(0) = 1$

$x(1) = 2$

$x(2) = -1$

$x(3) = 3$

$$X(\omega) = 1 + 2e^{-j\omega} - e^{-2j\omega} + 3e^{-3j\omega}$$

(i) Find the DTFT of (i) $x(n) = (0.5)^n u(n) + 2^n u(-n-1)$

(ii) $x(n) = (1/3)^n u(n+1)$

(iii) $x(n) = (1/3)^{n-3} u(n-3)$

(i) $x(n) = (0.5)^n u(n) + 2^n u(-n-1)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[(0.5)^n u(n) + 2^n u(-n-1)\right] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5)^n e^{-j\omega n} + \sum_{n=1}^{\infty} 2^{-n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5e^{-j\omega})^n + \sum_{n=1}^{\infty} (2^{-1} e^{-j\omega})^n$$

$$= \frac{1}{1-0.5e^{-j\omega}} + \frac{2^{-1}e^{-j\omega}}{1-2^{-1}e^{-j\omega}}$$

$$= \frac{1}{1-0.5e^{-j\omega}} + \frac{1}{(1-2^{-1}e^{-j\omega})2e^{j\omega}}$$

$$= \frac{1}{1-0.5\,e^{-j\omega}} + \frac{1}{2e^{j\omega} - 1}$$

$$= \frac{1}{1-0.5\,e^{-j\omega}} - \frac{1}{1 \cancel{+} 2e^{+j\omega}}$$

Q4 $\quad x(n) = \left(\frac{1}{3}\right)^n u(n+1)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n u(n+1)\, e^{-j\omega n}$$

$$= \sum_{n=-1}^{\infty} \left(\frac{1}{3}\right)^n e^{-j\omega n}$$

$$= \sum_{n=-1}^{\infty} \left(\frac{1}{3} e^{-j\omega}\right)^n$$

$$= \sum_{n=-1}^{\infty} \left(\frac{1}{3} e^{-j\omega}\right)^n$$

$$= \left(\frac{1}{3} e^{-j\omega}\right)^{-1} \sum_{n=0}^{\infty} \left(\frac{1}{3} e^{-j\omega}\right)^n$$

$$= \left(\frac{1}{3} e^{-j\omega}\right)^{-1} \frac{1}{1 - \frac{1}{3} e^{-j\omega}}$$

$$= \frac{1}{\left(\frac{1}{3} e^{-j\omega}\right)} \frac{1}{\left(1 - \frac{1}{3} e^{-j\omega}\right)}$$

$$= \frac{1}{\frac{1}{3} e^{-j\omega}}$$

(1) Find the DTFT of $x(n) = \left(\frac{1}{2}\right)^n u(n)$ and plot its spectrum.

$$X(\omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-j\omega}\right)^n$$

$$= \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

$$= \frac{1}{1 - \frac{1}{2}(\cos\omega + j\sin\omega)}$$

$$= \frac{1}{1 - \frac{1}{2}\cos\omega - \frac{j}{2}\sin\omega}$$

magnitude of spectrum is

$$X(\omega) = \frac{1}{\sqrt{\left(1 - \frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}}$$

$$= \frac{\quad}{\sqrt{1^2 + \frac{1}{4}\cos^2\omega - \cos\omega + \frac{1}{4}\sin^2\omega}}$$

$$= \frac{1}{\sqrt{1 + \frac{1}{4}\left[\sin^2 \omega + \cos^2 \omega\right] - \cos \omega}}$$

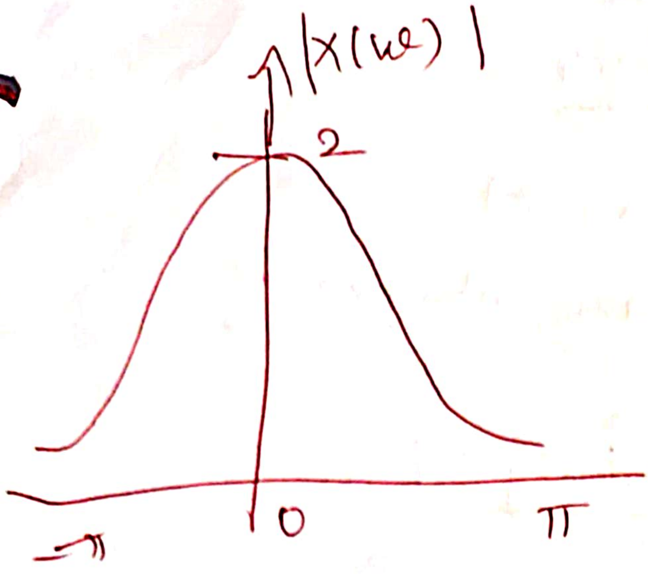
$$= \frac{1}{\sqrt{1 + \frac{1}{4} - \cos \omega}}$$

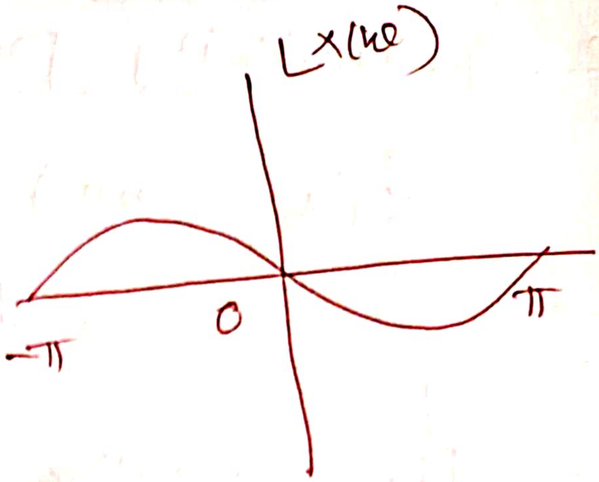
$$= \frac{1}{\sqrt{\frac{5}{4} - \cos \omega}}$$

$$\boxed{|X(\omega)| = \frac{2}{\sqrt{5 - 4\cos \omega}}}$$

$$\angle X(\omega) = \tan^{-1} \frac{\frac{1}{2}\sin \omega}{1 - \frac{1}{2}\cos \omega}$$

ω	$X(\omega)$	$\angle X(\omega)$
$-\pi$	0.67	0
$-\pi/2$	0.89	26.6
$-\pi/4$	1.36	28.7
0	2	~~-26.6~~ 0°
$\pi/4$	1.36	-28.7
$\pi/2$	0.89	-26.6
π	0.67	-0°





Properties of DTFT :-

1. Linearity property :-

$$F[x_1(n)] = X_1(\omega), \quad F[x_2(n)] = X_2(\omega)$$

$$F[a x_1(n) + b x_2(n)] = a X_1(\omega) + b X_2(\omega)$$

Proof :-

$$LHS = F[a x_1(n) + b x_2(n)]$$

$$= \sum_{n=-\infty}^{\infty} (a x_1(n) + b x_2(n)) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a x_1(n) e^{-j\omega n} + \sum_{n=-\infty}^{\infty} b x_2(n) e^{-j\omega n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) e^{-j\omega n} + b \sum_{n=-\infty}^{\infty} x_2(n) e^{-j\omega n}$$

$$= a X_1(\omega) + b X_2(\omega)$$

2. periodicity property

$$X(\omega + 2n\pi) = X(\omega)$$

for any integer n

3. Time shifting property :-

$$F[x(n-n_0)] = e^{-j\omega n_0} X(\omega)$$

proof :-

$$LHS = F[x(n-n_0)]$$

$$= \sum_{n=-\infty}^{\infty} x(n-n_0) e^{-j\omega n}$$

$$n - n_0 = k$$

$$n = k + n_0$$

$$= \sum_{k=-\infty}^{\infty} x(k) e^{-j\omega(k+n_0)}$$

$$= \sum_{k=-\infty}^{\infty} x(k) e^{-j\omega k} e^{-j\omega n_0}$$

$$= e^{-j\omega n_0} \left[\sum_{k=-\infty}^{\infty} x(k) e^{-j\omega k} \right]$$

$$= e^{-j\omega n_0} [X(\omega)]$$

4. Frequency shifting property

$$F[x(n)\, e^{j w_0 n}] = X(w - w_0)$$

Proof

$$LHS = F\left[x(n)\, e^{j w_0 n}\right]$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)\, e^{j w_0 n}\right] e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(w - w_0)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(w - w_0)n}$$

$$= X(w - w_0)$$

5. Time reversal property

$$F[x(n)] = X(w)$$

$$F[x(-n)] = X(-w)$$

Proof :-

$$LHS = F[x(-n)]$$

$$= \sum_{n=-\infty}^{\infty} x(-n)\, e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{jwn} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(-w)}$$

$$= X(-w)$$

$K = -\infty \qquad p = -\infty$

6. Differentiation in the freq. domain

$$F[x(n)] = X(\omega)$$

$$F[n\,x(n)] = j \frac{d}{d\omega}[X(\omega)]$$

Proof:-

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Differentiating both sides w.r.t ω

$$\frac{d}{d\omega}[X(\omega)] = \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{d}{d\omega}(e^{-j\omega n})$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\,(-jn)$$

$$= -j \sum_{n=-\infty}^{\infty} n\,x(n)\, e^{-j\omega n} \qquad j^2 = -1$$

$$j\frac{d}{d\omega}[X(\omega)] = \sum_{n=-\infty}^{\infty} n\,x(n)\, e^{-j\omega n}$$

$$j\frac{d}{d\omega}[X(\omega)] = F[n\,x(n)]$$

7. Time convolution property of DTFT :

$$F[x_1(n)] = X_1(\omega)$$

$$F[x_2(n)] = X_2(\omega)$$

$$F[x_1(n) * x_2(n)] = X_1(\omega) X_2(\omega)$$

Proof :-

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

$$F[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right\} e^{-j\omega n}$$

put $n-k = p$

$n = p+k$

Interchanging the order of summation

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k) e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(p) e^{-j\omega(p+k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{p=-\infty}^{\infty} x_2(p) e^{-j\omega p} \cdot e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) e^{-j\omega k} \cdot \sum_{p=-\infty}^{\infty} x_2(p) e^{-j\omega p}$$

$F[x_1(n) * x_2(n)] = X_1(\omega)\ X_2(\omega)$

8. <u>Frequency convolution / multiplication</u>

$F[x_1(n) * x_2(n)] = X_1(\omega) * X_2(\omega)$

~~IDFT~~ $\cancel{x(n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\alpha)\, X_2(\omega-\alpha)\, d\alpha$

<u>Proof :-</u>

$$F[x_1(n)\, x_2(n)] = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\alpha)\, e^{j\alpha n}\, d\alpha\right] e^{-j\omega n}\, x_2(n)$$

Interchanging the order of summation and integration.

$$F[x_1(n)\, x_2(n)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\alpha) \left[\sum_{n=-\infty}^{\infty} e^{-j(\omega \ldots}\right.$$

$$F[x_1(n)\, x_2(n)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\alpha)\, X_2(\omega-\alpha)\, d\alpha.$$

9. <u>Modulation Theorem :-</u>

$$F[x(n)\cos\omega_0 n] = \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{e^{j\omega_0 n} e^{-j\omega n}}{2} + \frac{e^{-j\omega_0 n} e^{-j\omega n}}{2} \right]$$

$$= \frac{1}{2} \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega-\omega_0)n} + \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+\omega_0)n} \right]$$

$$= \frac{1}{2} \left[X(\omega-\omega_0) + X(\omega+\omega_0) \right]$$

10. <u>parseval's theorem.</u>

$$F[x(n)] = X(\omega)$$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2$$

<u>Proof :-</u>

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n)\, x^*(n)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \right\}^*$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) e^{-j\omega n} \right\}$$

Interchanging the order of summation and Integration.

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega) \left\{ \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} d\omega \right\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\, X(\omega)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega.$$

1. Using the properties of DTFT, find the DTFT of the following.

(i) $\left(\frac{1}{2}\right)^{|n-2|} u(n-2)$

(ii) $\delta(n-3) - \delta(n+3)$

(iii) $u(-n)$

(iv) $n\, 3^n\, u(n)$

(v) $x(n) = (n-1)^2 x(n)$

(i) $x(n) = \left(\frac{1}{2}\right)^{|n-2|} u(n-2)$

Using Time shifting property

$$= e^{-j\omega n_0} F[x(n)]$$

$$= e^{-j2\omega} F\left[\left(\tfrac{1}{2}\right)^n u(n)\right]$$

$$F\left[\left(\tfrac{1}{2}\right)^n u(n)\right] = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$F\left[\left(\tfrac{1}{2}\right)^{(n-2)} u(n-2)\right\} = e^{-2j\omega}\left\{\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right\}$$

(ii) $x(n) = \delta(n-3) - \delta(n+3)$

Using time shifting property

$$F[\delta(n-3) - \delta(n+3)] = F[\delta(n-3)] - F[\delta(n+3)]$$

$$= e^{-3j\omega} F[\delta(n)] - e^{j3\omega} F[\delta(n)]$$

$$= e^{-3j\omega} \cdot 1 - e^{j3\omega} \cdot 1$$

$$= e^{-3j\omega} - e^{j3\omega}$$

$$= -2j \sin 3\omega$$

(iii) $x(n) = u(-n)$

$$F[u(-n)] = F[u(n)]_{\omega = -\omega}$$

$$= \left\{\frac{1}{1-e^{-j\omega}}\right\}_{\omega=-\omega}$$

$$= \frac{1}{1-e^{j\omega}}$$

(iv) $x(n) = n\,3^n u(n)$

Using the differentiation in the freq. domain property.

$$F\left[n\,3^n u(n)\right] = j\frac{d}{d\omega}\left[F(3^n u(n))\right]$$

$$= j\frac{d}{d\omega}\left[\frac{1}{1-3e^{-j\omega}}\right]$$

$$= j\frac{d}{d\omega}\left[\frac{1}{1-3e^{-j\omega}}\right]$$

$$= j\left\{\frac{-(-3e^{-j\omega}(-j))}{(1-3e^{-j\omega})^2}\right\}$$

$$= \frac{3e^{-j\omega}}{(1-3e^{-j\omega})^2}$$

(v) $x(n) = (n-1)^2 x(n)$

Using differentiation in the freq. domain property.

$$F\left[n x(n)\right] = j\frac{d}{d\omega}X(\omega)$$

$$F\left[n^2 x(n)\right] = -\frac{d^2}{d\omega^2}X(\omega)$$

i.e. $x(n) = (n^2 - 2n + 1)\, x(n)$

$$F\left[(n^2 - 2n + 1)x(n)\right] = F\left[n^2 x(n)\right] - F\left[2n\,x(n)\right] + F\left[x(n)\right]$$

$$= -\frac{d^2}{d\omega^2}X(\omega) - 2j\frac{d}{d\omega}X(\omega) + X(\omega)$$

(2) Find $\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$ for sequence

$x(n) = \{2, -1, \underset{\uparrow}{-2}, 3, 1\}$

From Parseval's theorem,

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-2}^{2} |x(n)|^2$$

$$= |x(-2)|^2 + |x(-1)|^2 + |x(0)|^2 + |x(1)|^2 + |x(2)|^2$$

$$= 2^2 + (-1)^2 + (-2)^2 + (3)^2 + 1^2$$

$$= 4 + 1 + 4 + 9 + 1$$

$$= 19$$

$$\sum_{n=-\infty} |x(n)|$$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(\omega)|^2 d\omega = 19$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega = 19$$

$$\int_{-\pi}^{\pi}|X(\omega)|^2 d\omega = 2\pi \times 19$$
$$= 38\pi$$

3. Find the convolution of two signals using DTFT.

$$x_1(n) = (\tfrac{1}{2})^n u(n) \text{ and } x_2(n) = (\tfrac{1}{4})^n u(n)$$

$$x_1(n) = (\tfrac{1}{2})^n u(n) \Longrightarrow X_1(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$x_2(n) = (\tfrac{1}{4})^n u(n) \Longrightarrow X_2(\omega) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

Using convolution property.

$$F[x_1(n) * x_2(n)] = X_1(\omega) \cdot X_2(\omega)$$

$$Y(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{1}{1 - \frac{1}{2}e^{-j\omega} - \frac{1}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

Z - transform :-

* Z - transform is a powerful mathematical tool used to convert the difference eqn into algebric eqn.

* Types

⇒ bilateral (or) two sided z-transform

⇒ Unilateral (or) single sided z-transform.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

* Relation between DTFT & Z-transform

DTFT is

$$X(w) = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$$

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Z is a complex variable

$$z = re^{jw}$$

r - radious of a circle

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) (re^{jw})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\left[[x(n)]\, r^{-n} \right] \right] e^{-j\omega n}$$

For the existance of z-transform the above summation should converge $x(n)\, r^{-n}$ must be absolutely summable.

$$\sum_{n=-\infty}^{\infty} x(n)\, r^{-n} < \infty.$$

For the DTFT exist

ie $$\sum_{n=-\infty}^{\infty} x(n) < \infty.$$

ROC - Region of convergence :-

For a given sequence, the z-transform may not converge. The set of values of z or the set of points in z-plane for which X(z) converges is called of Region of Convergence (ROC) of X(z)

Properties of ROC :-

* The ROC of X(z) is a ring or disk in z-plane, with centre at the origin.
* The ROC cannot contain any poles.
* If x(n) is finite duration right sided sequence, then the ROC is entire z-plane except z=0

* If $x(n)$ is finite duration left sided sequence then ROC is entire z-plane except $z = \infty$

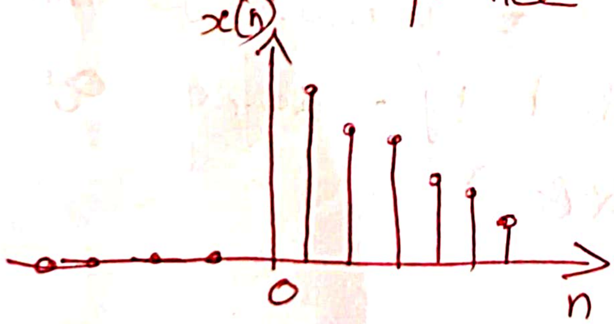
* If $x(n)$ is finite duration two sided sequence then the ROC is entire z-plane except $z = 0$ & $z = \infty$

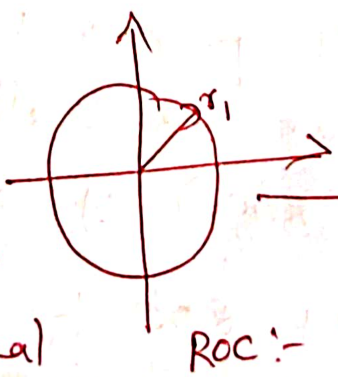
* If $x(n)$ is infinite duration right sided sequence then the ROC is exterior of a circle of radious r_1

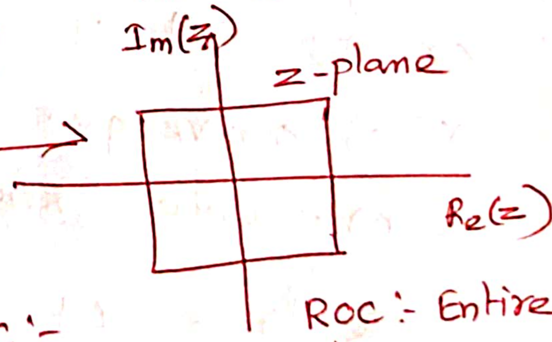
* If $x(n)$ is infinite duration left sided sequence then ROC is interior of a circle of radious r_2.

casual sequence

ROC's



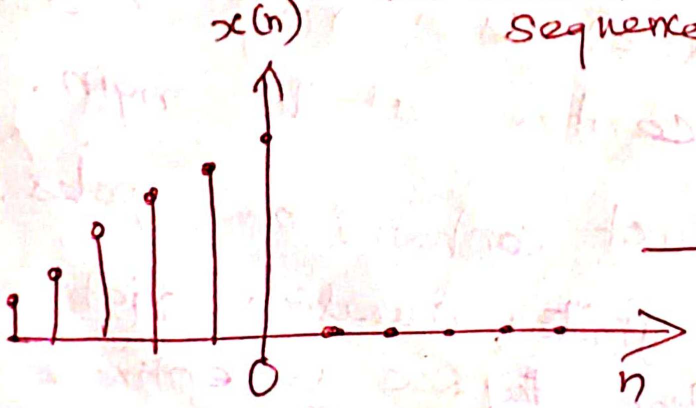


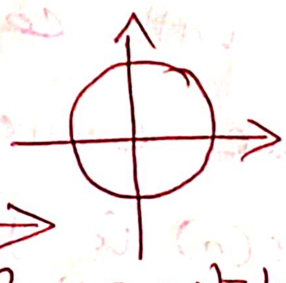


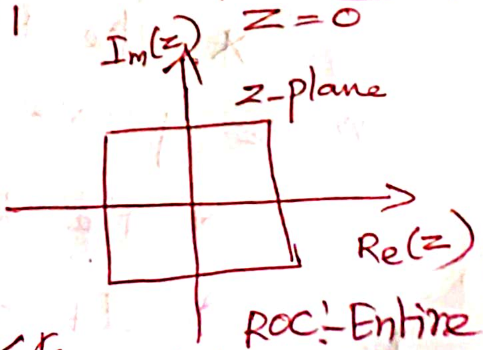
ROC :- $|z| < r_1$

ROC :- Entire z plane except $z = 0$

non casual sequence



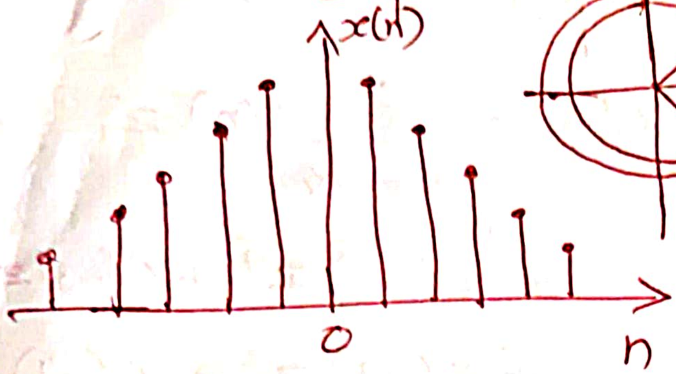


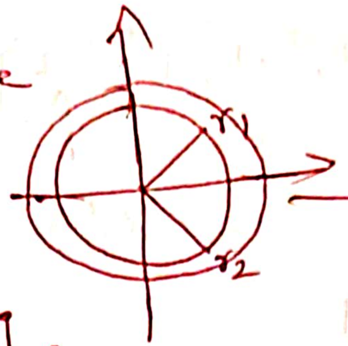


ROC :- $|z| < r_2$

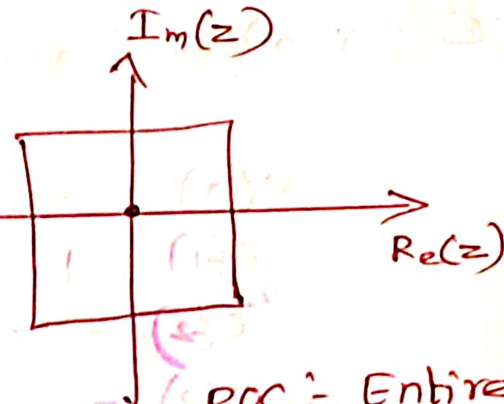
ROC :- Entire z plane except $z = \infty$

Two sided sequence





ROC :- $r_1 < |z| < r_2$



ROC :- Entire z-plane except $z=0$ and $z=\infty$

① Find the ROC and z-transform of following sequences

(a) $x(n) = \{\underset{\uparrow}{1}, 0, 3, 5, 4\}$

(b) $x(n) = \{5, 3, -2, 4\}$

(c) $x(n) = \{3, 2, -1, 1, \underset{\uparrow}{2}\}$

(d) $x(n) = \{2, 1, -4, \underset{\uparrow}{2}, 1, 5\}$

(a) $x(n) = \{\underset{\uparrow}{1}, 0, 3, 5, 4\}$

$x(0) = 1$
$x(1) = 0$
$x(2) = 3$
$x(3) = 5$
$x(4) = 4$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x(0) z^{-0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4}$$

$$= 1 + 0 \cdot z^{-1} + 3z^{-2} + 5z^{-3} + 4z^{-4}$$

$$= 1 + 3z^{-2} + 5z^{-3} + 4z^{-4}$$

ROC :- Entire z-plane except at $z=0$

(6) $x(n) = \{3, 2, -1, 1, \underset{\uparrow}{-2}\}$

$x(0) = -2$
$x(-1) = 1$
$x(-2) = -1$
$x(-3) = 2$
$x(-4) = 3$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + x(-1)z^1 + x(0)z^0$$

$$X(z) = 3z^4 + 2z^3 - 1z^2 + z - 2$$

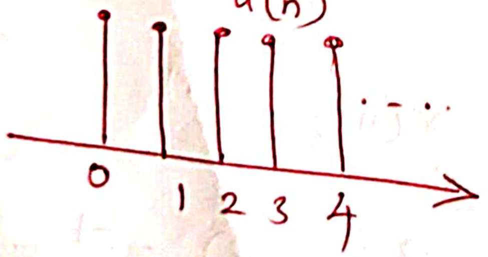
(2) Find the ROC and z-transform of following Sequence.

(a) $x(n) = u(n) - u(n-2)$

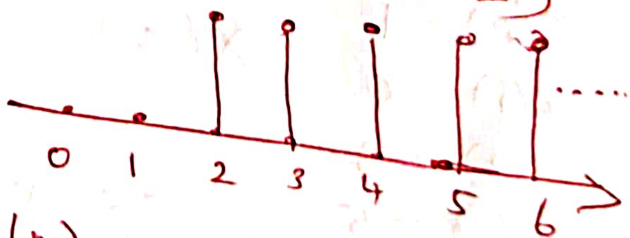
(b) $x(n) = u(-n) - u(-n-3)$

(a) $x(n) = u(n) - u(n-2)$

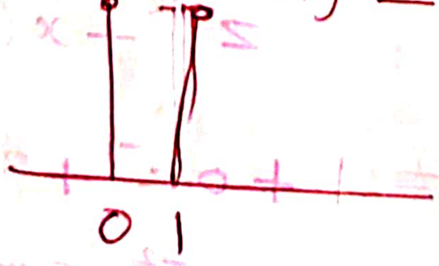
$u(n)$



$u(n-2)$



$u(n) - u(n-2)$



$$x(n) = \begin{cases} 1 & \text{for } 0 \le n \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{1} 1 \cdot z^{-n}$$

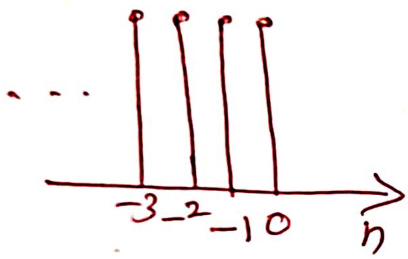
$$\boxed{X(z) = 1 + z^{-1}}$$

ROC :- Entire z-plane except z = 0

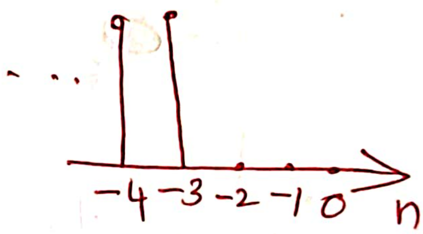
(b) x(-n) − u(-n-3)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

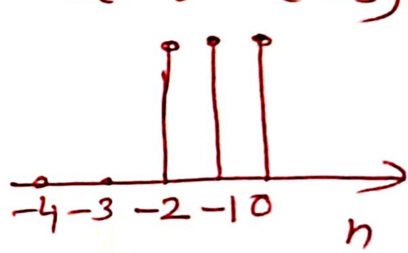
u(−n)



u(−n−3)



u(−n) − u(−n−3)



$x(n) = 1$ for $-2 \leq n \leq 0$

$\quad\;\; 0$ otherwise.

$$X(z) = \sum_{n=-2}^{0} 1 \cdot z^{-n}$$

$$= z^{2} + z^{1} + 1$$

$$\boxed{X(z) = 1 + z + z^{2}}$$

(3) Find the z-transform and RO of following signals.

(a) $x(n) = a^{n} u(n)$

(b) $x(n) = -a^{n} u(-n-1)$.

(a) $x(n) = a^n u(n)$

$$x(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

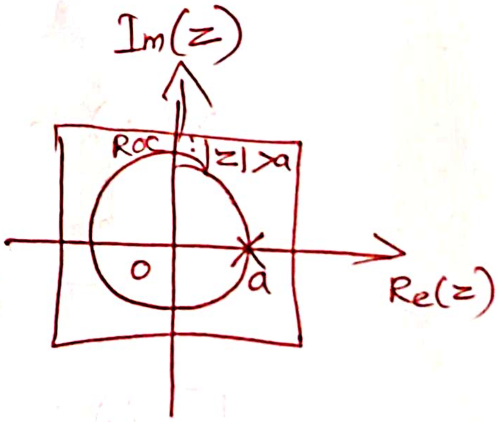
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \qquad \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1-az^{-1}}$$

$$= \frac{1}{1-a/z}$$



$$= \frac{1}{\frac{z-a}{z}}$$

$$= \frac{z}{z-a} \quad : \text{ROC} : |z| > |a|$$

which implies that the ROC is exterior to the circle of radious a

(b) $x(n) = -a^n u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$u(-n-1) = \begin{cases} 1 & \text{for } n \leq -1 \\ 0 & n \geq 0 \end{cases}$

$$= \sum_{n=-\infty}^{-1} -a^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{\infty} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$= -\left[\sum_{n=0}^{\infty} (a^{-1}z)^n - 1 \right]$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$= 1 - \frac{1}{1-a^{-1}z}$$

$$= \frac{\cancel{1} - a^{-1}z - \cancel{1}}{1-a^{-1}z}$$

$$= \frac{-a^{-1}z}{1-a^{-1}z}$$

$$= \frac{z}{z-a} \quad \text{ROC}: |z| < |a|$$

$$\frac{-z/a}{1-z/a}$$

$$= \frac{\frac{-z}{\cancel{a}}}{\frac{a-z}{\cancel{a}}}$$

$$= \frac{z}{z-a}$$

④ Determine the z-transform ROC and pole zero locations of $X(z)$ for

$$x(n) = \left(\frac{2}{3}\right)^n u(n) + \left(-\frac{1}{2}\right)^n u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty}\left[\left(\frac{2}{3}\right)^n u(n) + \left(-\frac{1}{2}\right)^n u(n)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n z^{-n} + \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{2}{3}z^{-1}\right)^n + \sum_{n=0}^{\infty}\left(-\frac{1}{2}z^{-1}\right)^n$$

$$= \frac{1}{1-\frac{2}{3}z^{-1}} + \frac{1}{1+\frac{1}{2}z^{-1}}$$

$$= \frac{1}{1-\frac{2/3}{z}} + \frac{1}{1+\frac{1/2}{z}}$$

$$= \frac{1}{\frac{z-2/3}{z}} + \frac{1}{\frac{z+1/2}{z}}$$

$$= \frac{z}{z-2/3} + \frac{z}{z+1/2}$$

$$= \frac{z(z+1/2) + z(z-2/3)}{(z-2/3)(z+1/2)}$$

$$= \frac{z^2 + \frac{1}{2}z + z^2 - \frac{2}{3}z}{\left(z-\frac{2}{3}\right)\left(z+\frac{1}{2}\right)}$$

$\frac{1}{2} - \frac{2}{3}$

$$= \frac{2z^2 - \frac{1}{6}z}{\left(z-\frac{2}{3}\right)\left(z+\frac{1}{2}\right)}$$

$\frac{3-4}{6}$

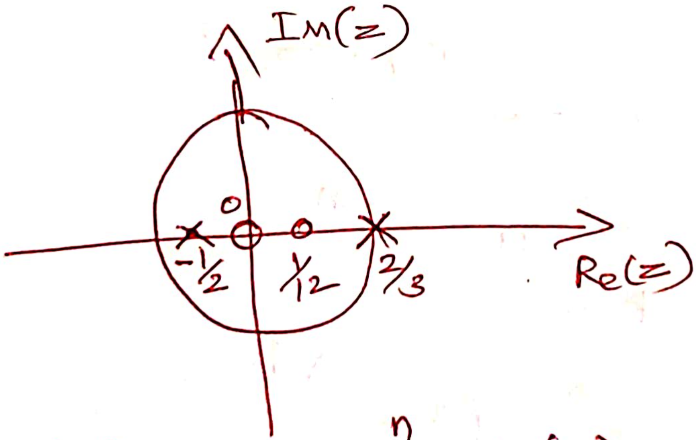
$$X(z) = \frac{z\left(2z - \frac{1}{6}\right)}{\left(z-\frac{2}{3}\right)\left(z+\frac{1}{2}\right)}$$

The poles of $X(z)$ at $z = \frac{2}{3}$

$z = -\frac{1}{2}$

The zeros of $X(z)$ at $z = 0$

$z = \frac{1}{12}$



(5) $x(n) = 0.5^n u(n) - \left(\frac{1}{3}\right)^n u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty} \left[0.5^n u(n) - \left(\frac{1}{3}\right)^n u(-n-1)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} - \sum_{n=-\infty}^{-1} \left(\frac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n - \sum \left(\left(\frac{1}{3}\right) z^{-1}\right)^n$$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n - \sum_{n=0}^{\infty} b$$

---

(6) Find the z-transform of $x(n) = a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1-az^{-1}}$$

$$= \frac{1}{1-a/z} = \frac{z}{z-a}$$

(6) Find the convolution of two sequences

$x(n) = \{1\ 1\ 1\ 1\}$ $h(n) = \{2, 2\}$

		$x(n)$			
		1	1	1	1
$h(n)$	2	2	2	2	2
	2	2	2	2	2

$y(n) = \{2, 4, 4, 4, 2\}$

(7) Determine the convolution of signals

$x(n) = \{2, -1, 3, 2\}$ $h(n) = \{1, -1, 1, 1\}$

		$x(n)$			
		2	−1	3	2
$h(n)$	1	2	−1	3	2
	−1	−2	1	−3	−2
	1	2	−1	3	2
	1	2	−1	3	2

$y(n) = \{2, -3, 6, 0, 0, 5, 2\}$

8. $y(n) = \{3, 8, 14, 8, 3\}$

$h(n) = \{1, 2, 3\}$

	$h(n)$: 1	2	3
a	a	2a	3a
b	b	2b	3b
c	c	2c	3c

$y(n) = \{a, b+2a, c+2b+3a, 2c+3b, 3c\}$

$= \cancel{\{3, 8,}$

$\boxed{a=3}$

$b+2a=8$

$b=8-2a$

$=8-6$

$\boxed{b=2}$

$c+4+9=14$

$c=14-13$

$\boxed{c=1}$

$x(n) = \{3, 2, 1\}$

9. Find the Z transform of the sequence $x(n) = \cos wn\, u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} \cos wn\, u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \cos wn\; z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right) z^{-n}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} + \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^{n} + \sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^{n}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j\omega}z^{-1}} + \frac{1}{1-e^{-j\omega}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{j\omega}} + \frac{z}{z-e^{-j\omega}}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - ze^{-j\omega} + z^2 - ze^{j\omega}}{(z-e^{j\omega})(z-e^{-j\omega})}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z(e^{j\omega} + e^{-j\omega})}{z^2 - ze^{-j\omega} - ze^{j\omega} + 1}\right]$$

$$= \frac{\cancel{1}}{\cancel{2}}\left[\frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}\right]$$

$$\boxed{X(z) = \frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}}$$

(10) Find the z-transform $x(n) = \sin \omega n \; u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sin\omega n \; u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \sin\omega n \; z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j} \right) z^{-n}$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} - \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n} \right]$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} \left(e^{j\omega} z^{-1}\right)^n - \sum_{n=0}^{\infty} \left(e^{-j\omega} z^{-1}\right)^n \right]$$

$$= \frac{1}{2j} \left[\frac{1}{1 - e^{j\omega} z^{-1}} - \frac{1}{1 - e^{-j\omega} z^{-1}} \right]$$

$$= \frac{1}{2j} \left[\frac{z}{z - e^{j\omega}} - \frac{z}{z - e^{-j\omega}} \right]$$

$$= \frac{1}{2j} \left[\frac{z(z - e^{-j\omega}) - z(z - e^{j\omega})}{(z - e^{j\omega})(z - e^{-j\omega})} \right]$$

$$= \frac{1}{2j}\left[\frac{z(z-e^{-j\omega}) - z(z-e^{j\omega})}{z^2 - z(e^{j\omega}+e^{-j\omega})+1}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{z^2} - ze^{-j\omega} - \cancel{z^2} + ze^{j\omega}}{z^2 - 2z\cos\omega + 1}\right]$$

$$= \frac{1}{2j}\left[\frac{ze^{j\omega} - ze^{-j\omega}}{z^2 - 2z\cos\omega + 1}\right]$$

$$= \frac{z\left[\dfrac{e^{j\omega} - e^{-j\omega}}{2j}\right]}{z^2 - 2z\cos\omega + 1}$$

$$\boxed{X(z) = \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}}$$

Properties of z-transform

1. Linearity property :-

$$z\left[a x_1(n) + b x_2(n)\right] = a X_1(z) + b X_2(z)$$

Proof :-

$$\text{LHS} = z\left[a x_1(n) + b x_2(n)\right]$$

$$= \sum_{n=-\infty}^{\infty} \left[a x_1(n) + b x_2(n)\right] z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$R.H.S. = a X_1[z] + b X_2(z)$

2. <u>Time shifting property</u>

$$Z[x(n-m)] = z^{-m} X(z)$$
$$Z[x(n+m)] = z^{m} X(z).$$

<u>proof :-</u>

$$LHS = Z[x(n-m)]$$

$$= \sum_{n=-\alpha}^{\alpha} x(n-m) \, z^{-n}$$

$$n-m = p$$
$$n = p+m$$

$$= \sum_{n=-\alpha}^{\alpha} x(p) \, z^{-(p+m)}$$

$$= \sum_{n=-\alpha}^{\alpha} x(p) \, z^{-p} z^{-m}$$

$$\boxed{R.H.S = z^{-m} X(z)},$$

3. <u>Time delay</u> ~~Time~~ ~~Advance~~

$$Z[x(n-m)] = z^{-m} X(z) + z^{-m} \sum_{k=1}^{m} x(-k) z^{k}$$

<u>Time Advance</u>

$$Z[x(n+m)] = z^{m} X(z) - z^{m} \sum_{k=1}^{m} x(k) z^{-k}$$

prove :-

$$Z[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-m) z^{-n} z^{m} z^{-m}$$

$$= \sum_{n=-\infty}^{\infty} x(n-m) z^{-(n-m)} z^{-m}$$

$$= z^{-m} \sum_{n=-\infty}^{\infty} x(n-m) z^{-(n-m)}$$

$n - m = p$

$n = p + m$

$$= z^{-m} \left[\sum_{p=-m}^{\infty} x(p) z^{-p} \right]$$

$$= z^{-m} \left[\sum_{p=0}^{\infty} x(p) z^{-p} + \sum_{p=-m}^{-1} x(p) z^{-p} \right]$$

$p = -k$ in the 2^{nd} summation

$$\boxed{Z[x(n-m)] = z^{-m} X(z) + z^{-m} \sum_{k=1}^{m} x(-k) z^{k}}$$

Scaling property (or) multiplication property :-

$$Z[a^n x(n)] = \cancel{\sum_{n=}} \; X\left(\frac{z}{a}\right) = X(a^{-1} z)$$

Proof :-

$$Z\left[a^n x(n)\right] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{z^{-n}}{a^{-n}}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(z/a\right)^{-n}$$

$$= X\left[\frac{z}{a}\right]$$

$$R.H.S. = X\left[z a^{-1}\right]$$

Note :-

$$Z\left[e^{j\omega n} x(n)\right] = X\left[\frac{z}{e^{j\omega}}\right]$$

$$Z\left[e^{-j\omega n} x(n)\right] = X\left[\frac{z}{e^{-j\omega}}\right]$$

Time Reversal property :-

$$Z\left[x(n)\right] = X(z)$$

$$Z\left[x(-n)\right] = X(1/z)$$

Proof :-

$$Z\left[x(-n)\right] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$p = -n$ in the summation.

$$Z[x(-n)] = \sum_{p=-\infty}^{\infty} x(p)\, z^{p}$$

$$= \sum_{p=-\infty}^{\infty} x(p)\, (z^{-1})^{-p}$$

$$= \sum_{p=-\infty}^{\infty} x(p) \left(\frac{1}{z}\right)^{-p}$$

$$= X\left(\frac{1}{z}\right) = X(z^{-1})$$

<u>Time expansion property</u> :-

Multiplication by n (or) Differentiation in z-domain property :-

$$Z[n\,x(n)] = -z\frac{d}{dz}X(z)$$

proof :-

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Differentiating both sides w.r.t z

$$\frac{d}{dz}[X(z)] = \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n-1}(-n)$$

$$= -z^{-1} \sum_{n=-\infty}^{\infty} x(n)\, n\, z^{-n}$$

$$\frac{d}{dz}[X(z)] = -\frac{1}{z} \sum_{n=-\infty}^{\infty} n\,x(n) z^{-n}$$

$$-z\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} n\,x(n) z^{-n}$$

$$\boxed{-z\frac{d}{dz}X(z) = Z[n\,x(n)]}$$

Convolution property (or) theorem :-

$$Z[x_1(n) * x_2(n)] = X_1(z)\, X_2(z)$$

w.r.t $\quad x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$

Proof :-

$$Z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} [x_1(k)\, x_2(n-k)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} [x_1(k)\, x_2(n-k)]\, z^{-(n-k)} z^{-k}$$

Interchanging the order of summations.

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} \sum_{p=-\infty}^{\infty} x_2(p)\, z^{-p}$$

$n-k=p$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} \sum_{p=-\infty}^{\infty} x_2(p)\, z^{-p}$$

$$Z[x_1(n) * x_2(n)] = X_1(z)\, X_2(z).$$

Initial Value Theorem :-

$$\underset{n\to 0}{Lt}\; x(n) = x(0) = \underset{z\to\infty}{Lt}\; X(z)$$

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots$$

$$\underset{z\to\infty}{Lt}\; X(z) = x(0) + 0 + 0$$

$$\boxed{\underset{z\to\infty}{Lt}\; X(z) = x(0)}$$

Final Value Theorem :-

$$Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$Z[x(n+1)] = zX(z) - zx(0) = \sum_{n=0}^{\infty} x(n+1)\, z^{-n} - \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \{x(n+1) - x(n)\} z^{-n}$$

$$= [x(1) - x(0)] z^{-0} + [x(2) - x(1)] z^{-1} + [x(3) - x(2)] z^{-2} + \cdots$$

Taking limit $z \to 1$ on both sides

$$\lim_{z\to 1}\left[(z-1)X(z) - zx(0)\right] = \left[\cancel{x}(1) - x(0)\right] + \left[x(2) - \cancel{x(1)}\right]z^{-1} + \left[x(3) - \cancel{x(2)}\right]z^{-2} + \dots$$

$$\lim_{z\to 1}(z-1)X(z) - x(0) = x(\infty) - x(0)$$

$$= x(\infty)$$

$$\boxed{\lim_{n\to\infty} x(n) = x(\infty) = \lim_{z\to 1}(z-1)X(z)}$$

① Find the z-transform of the sequence

$x(n) = a^{n-2}u(n-2)$

$$x(n) = a^{n-2}\,u(n-2)$$

$x(n) = a^n u(n)$ is given by

$$X(z) = \frac{z}{z-a} \quad : \text{RO} : |z| > |a|$$

Using time shifting property

$$Z[x(n-m)] = z^{-m}X(z)$$

$$= z^{-2}\left[\frac{z}{z-a}\right]$$

$$= \frac{1}{z^2}\left[\frac{z}{z-a}\right]$$

$$= \frac{1}{z(z-a)} \quad ; \text{ ROC } |z| > |a|$$

(2) Find the Z-transform of the signal

$x(n) = 3(4)^n u(-n)$

$$Z[u(n)] = \frac{1}{1-z^{-1}} \quad ; \text{ ROC } |z| > 1$$

Using the time reversal property

$$Z[u(-n)] = Z[u(n)]\Big|_{z \to 1/z} = \frac{1}{1-z} \; ; \text{ ROC ? } |z|>1$$

Using scaling property

$$Z[(4)^n u(-n)] = Z[u(-n)]\Big|_{z \to \frac{z}{4}} = \frac{1}{1-z}\Big|_{z = z/4}$$

Using Linearity property

$$= \frac{1}{1 - z/4}$$

$$Z[3(4)^n u(-n)] = 3\,Z[4^n u(-n)] = \frac{3}{1 - z/4}$$

$$= \frac{12}{4-z}$$

(3) Find the Z-transform of $x(n) = n\,u(n)$

$$Z[u(n)] = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

using multiplication property.

$$Z[n\,u(n)] = -z \frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$= -z \frac{(z-1)1 - z(1)}{(z-1)^2}$$

$$= -z \left(\frac{\cancel{z}-1-\cancel{z}}{(z-1)^2}\right)$$

$$= \frac{z}{(z-1)^2} \qquad \text{ROC} : |z| > 1$$

(4) Find the Z-transform of the sequence

$x(n) = a^n \cos \frac{n\pi}{2}\, u(n)$

$$Z[\cos \omega n\, u(n)] = \frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}$$

$\omega = \pi/2$

$$Z\left[\cos n\pi/2\, u(n)\right] = \frac{z^2 - z\cos\pi/2}{z^2 - 2z\cos\pi/2 + 1}$$

$$= \frac{z^2}{z^2+1}$$

using scaling property

$$z[a^n x(n)] = X[a^{-1}z]$$

$$z\left[a^n \cos\frac{n\pi}{2} u(n)\right] = \frac{(a^{-1}z)^2}{(a^{-1}z)^2+1}$$

$$= \frac{a^{-2}z^2}{a^{-2}z^2+1}$$

$$= \frac{\frac{z^2}{a^2}}{\frac{z^2}{a^2}+1}$$

$$= \frac{z^2/\cancel{a^2}}{\frac{z^2+a^2}{\cancel{a^2}}}$$

$$= \frac{z^2}{z^2+a^2}$$

(5) Find the z - transform of the sequence $x(n) = a^n \sin \omega n\, u(n)$

$$z[\sin \omega n\, u(n)] = \frac{z \sin \omega}{z^2 - 2z\cos\omega + 1}$$

using scaling property

$$z[a^n x(n)] = X[a^{-1}z]$$

$$z[a^n \sin \omega n\, u(n)] = \frac{a^{-1}z \sin \omega}{(a^{-1}z)^2 - 2a^{-1}z\cos\omega + 1}$$

$$= \frac{\dfrac{\cancel{a}\, z \sin w}{a}}{a^{-2} z^2 - 2(a^{-1} z)\cos w + \cancel{a^{2}}\, 1}$$

$$= \frac{z \sin w / a}{\dfrac{z^2}{a^2} - \dfrac{2z}{a}\cos w + 1}$$

$$= \frac{z \sin w / a}{\dfrac{z^2 - 2za\cos w + a^2}{a^{\cancel{2}}}}$$

$$= \frac{a z \sin w}{z^2 - 2za\cos w + a^2}$$

6. Find $X(z)$ using convolution theorem $x_1(n) = \left(\frac{1}{3}\right)^n u(n)$ and $x_2(n) = \left(\frac{1}{4}\right)^n u(n)$

$$X_1(z) = \frac{z}{z - 1/3} \quad : \text{ROC } |z| > 1/3$$

$$X_2(z) = \frac{z}{z - 1/4} \quad : \text{ROC } |z| > 1/4$$

$x(n) = x_1(n) * x_2(n)$

Convolution theorem,

$$Z\left[x_1(n) * x_2(n)\right] = X_1(z)\, X_2(z)$$

$$X(z) = \frac{z}{(z-\frac{1}{3})} \, \frac{z}{(z-\frac{1}{4})}$$

$$= \frac{z^2}{(z-\frac{1}{3})(z-\frac{1}{4})}$$

7. Find the convolution of two sequences

$x_1(n) = \{2, 1, 0, -1\}$

$x_2(n) = \{1, -1, 2\}$

$x_1(n)$

$x_2(n)$	2	1	0	-1
1	2	1	0	-1
-1	-2	-1	0	+1
2	4	2	0	-2

$x(n) = \{2, -1, 3, 1, 1, -2\}$

8. Find the initial and final value if

$$X(z) = 2 + 3z^{-1} + 4z^{-2}$$

using Initial value theorem

$$x(0) = \lim_{z \to \infty} X(z)$$

$$= \lim_{z \to \infty} \left[2 + 3z^{-1} + 4z^{-2} \right]$$

$$= 2$$

$$x(\infty) = \lim_{z \to 1} (z-1) X(z)$$

$$= \lim_{z \to 1} (z-1) \left[2 + 3z^{-1} + 4z^{-2} \right]$$

$$= 0$$

9. Find $x(0)$ if $X(z) = \dfrac{z+2}{(z+1)(z+3)}$

Using initial value theorem

$$X(z) = \frac{z+2}{(z+1)(z+3)}$$

$$= \frac{\cancel{z}\left[1 + 2/z\right)}{\cancel{z}(1+1/z)\cancel{z}(1+3/z)}$$

$$= \frac{1 + 2/z}{z(1+1/z)(1+3/z)}$$

$$x(0) = \lim_{z\to\infty} \frac{1+2/z}{z(1+1/z)(1+3/z)}$$

$$\boxed{x(0) = 0}$$

10. Find $x(\infty)$ if $X(z)$ is given by

(a) $\dfrac{z+3}{(z-0.5)^2}$. (b) $\dfrac{z+1}{4(z-1)(z+0.6)}$

(a) $$X(z) = \frac{z+3}{(z-0.5)^2}$$

$$x(\infty) = \lim_{z\to 1} (z-1)\,X(z)$$

$$= \lim_{z\to 1} z-1 \; \frac{z+3}{(z-0.5)^2}$$

$$= 0$$

(b) $$X(z) = \frac{z+1}{4(z-1)(z+0.6)}$$

$$x(\alpha) = \lim_{z\to 1} \frac{\cancel{(z-1)}(z+1)}{4\cancel{(z-1)}(z+0.6)}$$

$$= \lim_{z\to 1} \frac{z+1}{4(z+0.6)} = \frac{2}{4(1+0.6)}$$

$$= \frac{2}{4(1.6)}$$

$$= 0.3125$$

Unit V LTI - Discrete time systems.

⟹ Impulse response
⟹ Difference eqn.
⟹ Convolution sum
⟹ DFT and z transform,
⟹ Analysis of recursive & non recursive system.
⟹ DT system connected in series & parallel.

Difference eqn :-

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$

Two responses are
(i) natural response
(ii) Forced "

(1) Find the natural response of the system described by difference eqn.

$$y(n) - 5y(n-1) + 6y(n-2) = x(n)$$

whose initial conditions $y(-1)=1$ $y(-2)=0$

$$y(n) - 5y(n-1) + 6y(n-2) = 0 \quad \text{—} (1)$$

$$y_h(n) = \lambda^n$$

$$\lambda^n - 5\lambda^{n-1} + 6\lambda^{n-2} = 0$$

$$\lambda^{n-2}[\lambda^2 - 5\lambda + 6] = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$\times$ $+$

6 -5

$$\lambda_1 = 2 \quad \lambda_2 = 3$$

-2×-3

$$y_h(n) = c_1(3)^n + c_2(2)^n$$

$n = 0$

$$y(0) = c_1 + c_2 \quad \text{--- (2)}$$

$n = 1$ $$y(1) = 3c_1 + 2c_2 \quad \text{--- (3)}$$

From the homogeneons ~~eqn~~ n.

$$y(n) - 5y(n-1) + 6y(n-2) = 0$$

$n = 0$

$$y(0) - 5y(0-1) + 6y(0-2) = 0$$

$$y(0) - 5y(-1) + 6y(-2) = 0$$

$$y(0) - 5(+1) + 6(0) = 0$$

$$y(0) - 5 = 0$$

$$\boxed{y(0) = 5}$$

$n = 1$

$$y(1) - 5y(1-1) + 6y(1-2) = 0$$

$$y(1) - 5y(0) + 6y(-1) = 0$$

$$y(1) - 5(5) + 6(-1) = 0$$

$$y(1) = 25 - 6 = 19$$

$$c_1 + c_2 = 5$$

$$3c_1 + 2c_2 = 19$$

$$3c_1 + 3c_2 = 15$$

$$\cancel{3c_1} + 2c_2 = 19$$

$$(-) \quad (-) \quad (-)$$

$$\boxed{c_2 = -4}$$

$$c_1 + c_2 = 5$$

$$c_1 - 4 = 5$$

$$c_1 = 5 + 4$$

$$\boxed{c_2 = 9.}$$

The natural response

$$y_n(n) = 9(3)^n - 4(2)^n \quad \text{for } n>0$$

$$\boxed{y_n(n) = 9(3)^n u(n) - 4(2)^n u(n)}$$

(2) Find the forced response of the system described by the difference eqn $y(n) - 1.5y(n-1) + 0.5y(n-2) = x(n)$ for an input signal $x(n) = 3^n u(n)$

$$y(n) - 1.5y(n-1) + 0.5y(n-2) = x(n)$$

Forced response = homogeneous solution + particular solution

$y(n) - 1.5y(n-1) + 0.5y(n-2) = 0$

$y_h(n) = \lambda^n$

$\lambda^n - 1.5\lambda^{n-1} + 0.5\lambda^{n-2} = 0$

$\lambda^{n-2}[\lambda^2 - 1.5\lambda + 0.5] = 0$

$\lambda^2 - 1.5\lambda + 0.5 = 0$

$(\lambda - 1)(\lambda - 0.5) = 0$

$\times \quad +$

$0.5 \quad -1.5$

-1×-0.5

$\lambda_1 = 1$

$\lambda_2 = 0.5$

$y_h(n) = c_1(1)^n + c_2(0.5)^n$

$x(n) = 3^n u(n)$

$y_p(n) = K\,3^n u(n)$

$y_p(n)$ & $x(n)$ in the difference eqn

$K\,3^n u(n) - 1.5K\,3^{n-1}u(n-1) + 0.5K\,3^{n-2}u(n-2) = 3^n u(n)$

$n = 2$

$K\,3^2 - 1.5K\,3^1 + 0.5K = 3^2$

$9K - 4.5K + 0.5K = 9$

$9K - 4K = 9$

$5K = 9$

$\boxed{K = 9/5}$

$$y_p(n) = \frac{9}{5} 3^n u(n)$$

The forced response

$$y_f(n) = y_h(n) + y_p(n)$$

$$y_f(n) = c_1 (1)^n + c_2 (0.5)^n + \frac{9}{5} 3^n u(n)$$

Let $n=0$

$$y_f(0) = c_1 + c_2 + \frac{9}{5} \quad \text{— } (1)$$

$n = 1$

$$y_f(1) = c_1 + 0.5 c_2 + \frac{27}{5} \quad \text{— } (2)$$

$$y(n) - 1.5 y(n-1) + 0.5 y(n-2) = x(n)$$

$n = 0$

$$y(0) - 1.5 y(-1) + 0.5 y(-2) = x(0)$$

$$\boxed{y(0) = 1}$$

$y(-1) = y(-2) = 0$

$x(n) = 3^n u(n)$

$x(0) = 3^0 u(n)$

$\boxed{x(0) = 1}$

$x(n) = 3^n u(n)$

$x(1) = 3^1 u(1)$

$\boxed{x(1) = 3}$

Let $n=1$

$$y(1) - 1.5 y(0) + 0.5 y(-1) = x(1)$$

$$y(1) - 1.5 = 3$$

$$\boxed{y(1) = 4.5}$$

From eqn (1) & (2)

$$c_1 + c_2 + \frac{9}{5} = 1$$

$$(-)\ c_1 + 0.5 c_2 + \frac{27}{5} = 4.5$$

$$0.5C_2 - 18/5 = -3.5$$

$$0.5C_2 = -3.5 + 18/5$$

$$0.5C_2 = \frac{-9.25 + 18}{15}$$

$$= \frac{8.75}{2.9.15}$$

$$C_2 = \frac{\cancel{8.75}}{\cancel{15}_{3}} \times \cancel{0.5}^{0.1}$$

$$C_2 = 0.2$$

$$C_1 = -1$$

The forced response

$$y_f(n) = -1\,(1)^n + 0.2\,(0.5)^n + 9/5\; 3^n u(n)$$

$$= -1\,u(n) + 0.2\,(0.5)^n u(n) + 9/5\; 3^n u(n)$$

3) Find the response of the system described by the difference eqn $y(n) - \frac{1}{2}y(n-1) - \frac{1}{2}y(n-2) = x(n)$ for $n \geq 0$ when the i/p signals $x(n) = \left(\frac{1}{2}\right)^n u(n)$ with initial conditions $y(-1) = 1$ $y(-2) = 0$

$$y(n) - \frac{1}{2} y(n-1) - \frac{1}{2} y(n-2) = x(n) \quad \text{--- (1)}$$

The homo geneous solution

$$y_h(n) = \lambda^n$$

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$$

$$(\lambda - 1)(\lambda + \frac{1}{2}) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = -\frac{1}{2}$$

$$y_h(n) = C_1 (1)^n + C_2 (-\frac{1}{2})^n$$

If the i/p signal $x(n) = (\frac{1}{2})^n u(n)$

the particular solution is

$$y_p(n) = k (\frac{1}{2})^n u(n) \quad \text{--- (2)}$$

(2) in (1)

$$k (\frac{1}{2})^n u(n) - \frac{1}{2} k (\frac{1}{2})^{n-1} u(n-1)$$

$$- \frac{1}{2} k (\frac{1}{2})^{n-2} u(n-2) = (\frac{1}{2})^n u(n)$$

For $n = 2$

$$\frac{1}{4}k - \frac{1}{4}k - \frac{1}{2}k = \frac{1}{4}$$

$$k = \frac{\frac{1}{4}}{-\frac{1}{2}} = -\frac{1}{2} \qquad \boxed{k = -\frac{1}{2}}$$

The particular solution

$$y_p(n) = -\frac{1}{2}\left(\frac{1}{2}\right)^n u(n)$$

The Total response

$$y(n) = y_h(n) + y_p(n)$$

$$= c_1(1)^n + c_2\left(-\frac{1}{2}\right)^n - \frac{1}{2}\left(\frac{1}{2}\right)^n u(n)$$

For $n=0$

$$y(0) = c_1 + c_2 - \frac{1}{2} \quad \text{——} \ 3$$

For $n=1$

$$y(1) = c_1 - \frac{1}{2}c_2 - \frac{1}{4} \quad \text{——} \ 4$$

From the given eqn.

$$y(n) - \frac{1}{2}y(n-1) - \frac{1}{2}y(n-2) = \left(\frac{1}{2}\right)^n u(n)$$

$y(-1) = 1$ and $y(-2) = 0$

For $n=0$

$$y(0) - \frac{1}{2}y(-1) - \frac{1}{2}y(-2) = 1$$

$$y(0) - \frac{1}{2}(1) = 1$$

$$y(0) = 1 + \frac{1}{2} = \frac{3}{2}$$

$$y(0) = \frac{3}{2}$$

$n = 1$

$y(1) - \frac{1}{2} y(0) - \frac{1}{2} y(-1) = \frac{1}{2}$

$y(1) - \frac{1}{2}\left(\frac{3}{2}\right) - \frac{1}{2}(1) = \frac{1}{2}$

$y(1) = \frac{1}{2} + \frac{1}{2} + \frac{3}{4}$

$= 1 + \frac{3}{4}$

$\boxed{y(1) = \frac{7}{4}}$

$\frac{3}{2} - \frac{7}{4}$

$\frac{6-7}{4} = -\frac{1}{4}$

$\frac{1}{4} - \frac{1}{2}$

4

From eqn (3) & (4)

$c_1 + c_2 - \frac{1}{2} = \frac{3}{2}$

$\cancel{c_1} - \frac{1}{2} c_2 - \frac{1}{4} = \frac{7}{4}$

$(-) \quad (+) \quad (+) \quad (-)$

$\frac{3}{2} c_2 - \frac{1}{4} = -\frac{1}{4}$

$\frac{3}{2} c_2 = 0$

$\boxed{c_2 = 0}$

$c_1 = \frac{1}{2} + \frac{3}{2} = \frac{4}{2}$

$\boxed{c_1 = 2}$

Total response

$$y(n) = 2(1)^n - \frac{1}{2}\left(\frac{1}{2}\right)^n u(n)$$

$$y(n) = \left[2 - \frac{1}{2}\left(\frac{1}{2}\right)^n\right] u(n).$$