PIE Tech

POLLACHI INSTITUTE OF ENGINEERING AND TECHNOLOGY

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Department of Civil and Mechanical Engineering

Regulation 2021

III Year – V Semester

MA3351- Transforms and Partial Differential Equations

UNIT - I

Partial Differential Equations.

An equation involving partial derivatives is known as partial differential equation. The order of a pde is the order of the highest derivative occuring in the equation. The degree of the pde is the degree of the highest order partial derivative occurring in the equation.

Ex: 1 $\quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$ —— ① order - 1, degree 1

$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial z}\right)^3$ —— ② order - 2 degree - 1

Formation of pdes: Pdes can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary funs. If the no of constants to be eliminated is equal to the no of independent variables, the pdes tha arise are of 1st order. If the no of arbitrary constants to be eliminated is more than the no of independent variables, the pdes obtained are of second or higher orde If the pde is obtained by elimination of arbitrary funs, then the order of the pde is equal to the no of arbitrary funs, eliminated.

We use the following notations in the place of partial derivatives.

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$$

Obtain a pde by eliminating a and b from the following.

1. $z = ax + by + ab$. — ①

Diff ① partially w.r.t. 'x' $\frac{\partial z}{\partial x} = a$ (ie) $\boxed{p = a}$

Diff ① " 'y', $\boxed{q = b}$

Sub for a & b in ①, we have $z = px + qy + pq$ which is the required pde.

2 $z = (x^2+a)(y^2+b)$. —— ①

$\frac{\partial z}{\partial x} = 2x(y^2+b)$ (ie) $p = 2x(y^2+b) \Rightarrow y^2+b = \frac{p}{2x}$

$q = 2y(x^2+a) \Rightarrow x^2+a = \frac{q}{2y}$

Sub in ①, $z = \frac{p}{2x}\frac{q}{2y} \Rightarrow \boxed{4xyz = pq}$

3. $\log(az-1) = x + ay + b$ — ①

Diff ① w.r.t. 'x'

$\frac{1}{az-1} ap = 1$ —— ②

Diff ① w.r.t. 'y'

$\frac{1}{az-1} aq = a$ $\frac{1}{az-1} q = 1$. — ③

$\frac{②}{③}$; $\frac{ap}{q} = 1 \Rightarrow$ $ap = q$

② $\Rightarrow q \cancel{ap} = az - 1$

$= \frac{q}{p} z - 1 = \frac{qz - p}{p}$

$pq - qz + p = 0$

$\boxed{p(q+1) = qz}$

4. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ —— (1)

Diff (1) w.r.t. 'x', $2p = \frac{2x}{a^2} \Rightarrow \frac{1}{a^2} = \frac{p}{x}$.

'y' $2q = \frac{2y}{b^2} \Rightarrow \frac{1}{b^2} = \frac{q}{y}$.

(1) $\Rightarrow$ $2z = x^2 \cdot \frac{p}{x} + y^2 \cdot \frac{q}{y}$ (ie) $\boxed{2z = px + qy}$

5. $z = ax^3 + by^3$

$p = 3ax^2 \Rightarrow a = \frac{p}{3x^2}$

$q = 3by^2 \Rightarrow b = \frac{q}{3y^2}$.

(1) $\Rightarrow$ $z = \frac{p}{3x^2} x^3 + \frac{q}{3y^2} y^3$

$3z = px + qy$

(6) $z = ax^n + by^n$

$nz = px + qy$.

MA132 M-III

I PDE

II Fourier Series

III Boundary value problems

IV Laplace Transform

V Fourier Transform.

(7) $\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1$

ie $(x^2+y^2)b^2 + z^2a^2 = a^2b^2$ —— (1)

$2xb^2 + 2za^2p = 0$

$za^2p = -b^2x$

$zp = \frac{-b^2x}{a^2}$ —— (2)

Diff (1) w.r.t. 'y'

$$2yb^2 + 2za^2q = 0$$

$$zq = \frac{-yb^2}{a^2} \quad \text{—} (3)$$

$\frac{(2)}{(3)}$; $\frac{zp}{zq} = \frac{-b^2x}{a^2} \cdot \left(\frac{-a^2}{yb^2}\right)$

$$\frac{p}{q} = \frac{x}{y} \qquad \boxed{py = qx}$$

8. $(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha$ —— (1)

Diff (1) partially w.r.t. 'x'

$$2(x-a) = 2zp\cot^2\alpha \Rightarrow x-a = zp\cot^2\alpha.$$

Diff (1) partially w.r.t. 'y'

$$2(y-b) = 2zq\cot^2\alpha. \Rightarrow y-b = zq\cot^2\alpha.$$

$$(1) \Rightarrow z^2p^2\cot^4\alpha + z^2q^2\cot^4\alpha = z^2\cot^2\alpha$$

$$p^2\cot^2\alpha + q^2\cot^2\alpha = 1.$$

$$p^2 + q^2 = \frac{1}{\cot^2\alpha} = \tan^2\alpha. \quad \boxed{p^2+q^2 = \tan^2\alpha}.$$

9. $z = (x+a)(y+b)$

$p = y+b, \quad q = x+a. \qquad \boxed{z = pq}$

10. $z = ax + by + \sqrt{a^2+b^2}$

$p = a, \quad q = b.$

$\therefore z = px + qy + \sqrt{p^2+q^2}.$

Derivation of pde by eliminating arbitrary functions.

1. $z = e^y f(x+y)$ — ①

Diff ① partially w.r.t. 'x', $p = e^y f'(x+y)$ — ②

Diff ① partially w.r.t. 'y' $q = e^y f'(x+y) + e^y f(x+y)$ — ③

Using ① & ② in ③, $\boxed{q = p+z}$.

2. $z = (x+y) f(x^2-y^2)$ — ①

$p = (x+y) f'(x^2-y^2).2x + 1.f(x^2-y^2)$ — ②

$q = (x+y) f'(x^2-y^2).(-2y) + 1.f(x^2-y^2)$ — ③

② × y; $py = (x+y) f'(x^2-y^2)(2xy) + y f(x^2-y^2)$ — ④

③ × x; $qx = (x+y) f'(x^2-y^2).(-2xy) + x f(x^2-y^2)$ — ⑤

④ + ⑤; $py + qx = f(x^2-y^2)[x+y]$

$\boxed{py + qx = z}$

3. $ax + by + cz = f(x^2+y^2+z^2)$ — ①

$a + cp = f'(x^2+y^2+z^2).(2x+2zp)$ — ②

$b + cq = f'(x^2+y^2+z^2).(2y+2zq)$ — ③

$\frac{②}{③}$; $\frac{a+cp}{b+cq} = \frac{x+zp}{y+zq}$

$(a+cp)(y+zq) = (b+cq)(x+zp)$.

4. $z = f(my-lx)$ — ①

$p = f'(my-lx)(-l)$ $\quad q = f'(my-lx).m$.

$\frac{p}{q} = -\frac{l}{m}$; $\boxed{pm + ql = 0}$

5. $z = F(x^2 - y^2)$

$p = F'(x^2 - y^2) \cdot (2x)$ $\quad \frac{p}{q} = \frac{x}{-y}$ $\quad \boxed{qx + py = 0}$

$q = F'(x^2 - y^2)(-2y)$

6. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ — (1)

$p = 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$

$= -\frac{2}{x^2} f'\left(\frac{1}{x} + \log y\right)$ — (2)

$q = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$

$= 2y + \frac{2}{y} f'\left(\frac{1}{x} + \log y\right)$ — (3)

(2) $\times x^2$; $px^2 = -2f'\left(\frac{1}{x} + \log y\right)$

(3) $\times y$; $qy = 2y^2 + 2f'\left(\frac{1}{x} + \log y\right)$

$\boxed{px^2 + qy = 2y^2}$

7. $z = (x+y)\phi(x^2 - y^2)$ — (1) $\quad \phi\left(x^2 - y^2, \frac{z}{x+y}\right)$

$p = \phi(x^2 - y^2) + (x+y)\phi'(x^2 - y^2)(2x)$

$q = \phi(x^2 - y^2) + (x+y)\phi'(x^2 - y^2)(-2y)$

$py = \cancel{y}\, y\phi(x^2 - y^2) + (x+y)\phi'(x^2 - y^2)\, 2xy$

$qx = x\phi(x^2-y^2) + (x+y)\phi'(x^2-y^2)(-2xy)$

$\frac{py}{qx}$ $\cancel{z}$. $py + qx = (x+y)\phi(x^2-y^2)$.

$\boxed{py + qx = z}$

8. $z = f\left(\frac{xy}{z}\right)$ or $f\left(\frac{xy}{z}, z\right) = 0$.

$p = f'\left(\frac{xy}{z}\right)\cdot\left(\frac{y}{z}\right)$ — ① $u = xyz^{-1}$, $v = z$

$q = f'\left(\frac{xy}{z}\right)\cdot\left(\frac{x}{z}\right)$ — ② $u_x = yz^{-1} + xy(-1)z^{-2}p$ $v_x = p$

$= \frac{y}{z} - \frac{xyp}{z^2}$ $v_y = q$.

$px = f'\left(\frac{xy}{z}\right)\left(\frac{xy}{z}\right)$ — ③ $u_y = \frac{x}{z} - \frac{xyq}{z^2}$

$qy = f'\left(\frac{xy}{z}\right)\left(\frac{xy}{z}\right)$ — ④ $px - qy = 0 \Rightarrow \boxed{px = qy}$

9. $xyz = f(x+y+z)$.

$yz + xyp = f'(x+y+z)(1+p)$ — ②

$xz + xyq = f'(x+y+z)(1+q)$ — ③

$$\begin{vmatrix} \frac{y}{z} - \frac{xyp}{z^2} & \frac{x}{z} - \frac{xyq}{z^2} \\ p & q \end{vmatrix} = 0$$

$\frac{②}{③}$; $\frac{yz + xyp}{xz + xyq} = \frac{1+p}{1+q}$

$\frac{yq}{z} - \frac{xypq}{z^2} - \frac{px}{z} + \frac{xypq}{z^2} = 0$

$\frac{yq - px}{z} = 0$

$\Rightarrow qy = px$.

$(yz + xyp)(1+q) = (xz + xyq)(1+p)$

$yz + xyp + yzq + xypq = xz + xyq + xzp + xypq$

$\cancel{yz} + xyp + yzq - \cancel{xz} - xyq - xzp = 0$ $xz - yz$

$\cancel{(y-x)z} + x(y-z)p + y(z-x)q = z(x-y)$.

If the given fun contains one arbitrary fun, then the pde is obtained by $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0$

By eliminating the arbitrary function ϕ from $\phi(u,v)=0$ where u & v are functions of x,y,z, we get the pde

$$Pp + Qq = R \quad \text{——} (1)$$

where $P = \dfrac{\partial(u,v)}{\partial(y,z)}$, $Q = \dfrac{\partial(u,v)}{\partial(z,x)}$, $R = \dfrac{\partial(u,v)}{\partial(x,y)}$

Eqn (1) is called Lagrange's linear equation.

Eliminate the arbitrary fun from the following

1. $f(x^2+y^2, z-xy)=0$.

$u = x^2+y^2$ $\quad v = z-xy$

$u_x = 2x$ $\quad v_x = p-y$.

$u_y = 2y$ $\quad v_y = q-x$

Given $f(x^2+y^2, z-xy) = 0$ —— (1)

Let $u = x^2+y^2$, $v = z-xy$

Then (1) becomes $f(u,v)=0$ —— (2)

Eliminating f from (2) we get the Lagrange's linear pde $Pp+Qq=R$ where $P = \dfrac{\partial(u,v)}{\partial(y,z)}$, $Q = \dfrac{\partial(u,v)}{\partial(z,x)}$, $R = \dfrac{\partial(u,v)}{\partial(x,y)}$.

Now $P = \begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 2y & 0 \\ -x & 1 \end{vmatrix} = 2y$

$Q = \begin{vmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 0 & 2x \\ 1 & -y \end{vmatrix} = -2x.$

$\begin{vmatrix} 2x & p-y \\ 2y & q-x \end{vmatrix} = 0$

$2xq - 2x^2 - 2py + 2y^2 = 0$

$$R = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ -y & -x \end{vmatrix} = -2x^2 + 2y^2$$

$$2yp - 2xq = -2x^2 + 2y^2$$

$$\Rightarrow yp - xq = y^2 - x^2$$

2. $\phi(x^2+y^2+z^2, x+y+z) = 0$ —— ①

Let $u = x^2+y^2+z^2$, $v = x+y+z$.

Then ① becomes $\phi(u,v) = 0$ —— ②

Eliminating ϕ from ② we get Lagrange's linear pde $Pp + Qq = R$

$$P = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 2y & 2z \\ 1 & 1 \end{vmatrix} = 2y - 2z$$

$$Q = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 2z & 2x \\ 1 & 1 \end{vmatrix} = 2z - 2x$$

$$R = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 1 & 1 \end{vmatrix} = 2x - 2y$$

$2x + 2z)$

$$(y-z)p + (z-x)q = (x-y)$$

3. $f(xy+z^2, x+y+z)=0$ — ① $xy+z^2 = f(x+y+z)$.

Let $u = xy+z^2$, $v = x+y+z$. $f(x+y+z, xy+z^2)=0$

$$P = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} x & 2z \\ 1 & 1 \end{vmatrix} = x-2z$$

$$Q = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 2z & y \\ 1 & 1 \end{vmatrix} = 2z-y$$

$$R = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 1 \end{vmatrix} = y-x$$

$(x-2z)p + (2z-y)q = y-x$.

4. $\phi(z^2-xy, \frac{x}{z})=0$ — ①

$u = z^2-xy$, $v = \frac{x}{z}$.

$$P = \begin{vmatrix} -x & 2z \\ 0 & \frac{-x}{z^2} \end{vmatrix} = \frac{x^2}{z^2} \qquad Q = \begin{vmatrix} 2z & -y \\ \frac{-x}{z^2} & \frac{1}{z} \end{vmatrix} = 2-\frac{xy}{z^2}$$

$$R = \begin{vmatrix} -y & -x \\ \frac{1}{z} & 0 \end{vmatrix} = \frac{x}{z} \qquad \frac{x^2}{z^2}p + \left(2-\frac{xy}{z^2}\right)q = \frac{x}{z}$$

$$\boxed{x^2p + (2z^2-xy)q = xz.}$$

5. $\phi(x^2+y^2,\ y^2+z^2)=0$ $\qquad yzp - xzq = xy$

6. $\phi(x^2+y^2+z^2,\ lx+my+nz)=0$ $\qquad (mz-ny)p+(nx-lz)q=(ly-mx)$

7. $xy+z^2 = f(x+y+z)$. we may assume the given problem as $f(x+y+z,\ xy+z^2)=0$.

$$P=\begin{vmatrix}1 & 1\\ x & 2z\end{vmatrix} = 2z-x \qquad R=\begin{vmatrix}1 & 1\\ y & x\end{vmatrix} = x-y$$

$$Q=\begin{vmatrix}1 & 1\\ 2z & y\end{vmatrix} = y-2z \qquad (2z-x)p+(y-2z)q = x-y.$$

8. $g\left(\frac{y}{x},\ x^2+y^2+z^2\right)=0$ $\qquad xzp+yzq+x^2+y^2=0.$

Elimination of arbitrary functions

i. $z = f(y+ax) + x\,\phi(y+ax)$ —①

Diff w.r.t 'x' alone,

$p = f'(y+ax).a + \phi(y+ax) + x.\phi'(y+ax).a$ —②

'y' $q = f'(y+ax) + x\,\phi'(y+ax)$ —③

② − a③; $p - aq = \phi(y+ax)$ —④

Diff ④ w.r.t 'x' $r - as = \phi'(y+ax).a$ —⑤

④ 'y' $s - at = \phi'(y+ax)$ —⑥

$\frac{⑤}{⑥}$; $\frac{r-as}{s-at} = a$

$r - as = as - a^2t$

$r - 2as + a^2t = 0$ which is the required pde.

2. $z = f(x+y)\,\phi(x-y)$ —— ①

Diff ① w.r.t. 'x' alone, $p = f'(x+y)\phi(x-y) + f(x+y)\phi'(x-y)$ —— ②

Diff ① w.r.t. 'y' alone, $q = f'(x+y)\phi(x-y) + f(x+y)\phi'(x-y)(-1)$ —— ③

②+③; $p+q = 2f'(x+y)\phi(x-y)$

$$= \frac{2z\,f'(x+y)}{f(x+y)} \quad [\text{by } ①]$$

$(p+q)f(x+y) = 2z\,f'(x+y)$ —— ④

Diff ④ partially w.r.t. 'x'

$(r+s)f(x+y) + (p+q)f'(x+y) = 2pf'(x+y) + 2zf''(x+y)$

$(r+s)f(x+y) + (q-p)f'(x+y) = 2zf''(x+y)$ —— ⑤

Diff ④ partially w.r.t. 'y',

$(s+t)f(x+y) + (p+q)f'(x+y) = 2qf'(x+y) + 2zf''(x+y)$

$(s+t)f(x+y) + (p-q)f'(x+y) = 2zf''(x+y)$ —— ⑥

From ⑤ & ⑥,

$(r+s)f(x+y) + (q-p)f'(x+y) - (p-q)f'(x+y) = (s+t)f(x+y)$

$(r-t)f(x+y) + 2(q-p)f'(x+y) = 0$

$(r-t)f(x+y) = 2(p-q)f'(x+y)$

$$\frac{f'(x+y)}{f(x+y)} = \frac{r-t}{2(p-q)}$$

$$\frac{p+q}{2z} = \frac{r-t}{2(p-q)}$$

$$\boxed{p^2 - q^2 = (r-t)z}$$

3. $z = x f(y) + y \phi(x)$ — ①

Diff ① partially w.r.t 'x', $p = f(y) + y\phi'(x)$ — ②

Diff ① 'y', $q = x f'(y) + \phi(x)$ — ③

$xp + yq = x f(y) + xy\phi'(x) + xy f'(y) + y\phi(x)$

$= z + xy f'(y) + xp - x f(y)$ by ②

$yq = z + xy f'(y) - x f(y)$ — ④

Diff w.r.t 'x' $y s = p + y f'(y) - f(y)$

$xy s = px + xy f'(y) - x f(y)$ — ⑤

④ - ⑤ ; $yq - xys = z - px$

$z - px = y(q - xs)$.

4. $z = f(x) + e^{y} g(x)$

$p = f'(x) + e^{y} g'(x)$

$q = e^{y} g(x)$

$t = e^{y} g(x)$ $\therefore$ $\boxed{q = t}$

5. $z = x f\left(\frac{y}{x}\right) + y\phi(x)$ — ①

Assign Diff ① partially w.r.t 'x'

$p = x f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + f(y/x) + y\phi'(x)$

$= -\frac{y}{x} f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right) + y\phi'(x)$ — ②

'y' $q = x f'\left(\frac{y}{x}\right)\cdot\frac{1}{x} + \phi(x)$

$= f'\left(\frac{y}{x}\right) + \phi(x)$ — ③

$③ \times \frac{y}{x}$; $\frac{qy}{x} = \frac{y}{x} f'\left(\frac{y}{x}\right) + \frac{y}{x}\phi(x)$ —④

$②+④$; $p + \frac{qy}{x} = f(y/x) + y\phi'(x) + \frac{y}{x}\phi(x)$

$$\frac{px+qy}{x} = f\left(\frac{y}{x}\right) + y\phi'(x) + \frac{y}{x}\phi(x).$$

$\times$ by x; $px + qy = xf\left(\frac{y}{x}\right) + xy\,\phi'(x) + y\phi(x)$

$$= z + xy\,\phi'(x) \quad —⑤$$

Diff ⑤ w.r.t. 'y' alone,

$$xs + yt + q = q + x\phi'(x)$$

$$xs + yt = x\phi'(x) \quad —⑥$$

Using ⑥ in ⑤; $px + qy = z + y(xs + yt)$

$$z = xp + yq - y(xs + yt)$$

$$z = xp + y(q - xs - yt)$$

1. Find the de of all plane thro' the origin.

Equation of all planes thro' the origin is given by $ax + by + cz = 0$ —① where a, b, c are constants.

Diff ① partially w.r.t 'x', $a + cp = 0 \Rightarrow a = -cp$

Diff ① " 'y', $b + cq = 0 \Rightarrow b = -cq$

$① \Rightarrow -cpx - cqy + cz = 0$

$$\boxed{px + qy = z}$$

2. Find the de of all planes having equal x & y intercepts

Equation of the plane having equal x & y intercepts is

$\frac{x}{a} + \frac{y}{a} + \frac{z}{b} = 1.$

$bx + by + az = ab$ — ①

$b + ap = 0 \Rightarrow b = -ap$

$b + aq = 0 \Rightarrow b = -aq.$

Equating values of b, $\boxed{p = q}$

3. Obtain the DE of all planes which are at constant distant 'a' from the origin.

Let the equ of the planes be $a'x + b'y + c'z + d' = 0$ — ①

Since 'a' is the distance of the plane from the origin, we have $a = \frac{d'}{\sqrt{a'^2 + b'^2 + c'^2}}$ — ②

Diff ① w.r.t. 'x' alone, $a' + c'p = 0 \Rightarrow a' = -c'p$

'y' $b' + c'q = 0 \Rightarrow b' = -c'q.$

② $\Rightarrow$ $a = \frac{d'}{\sqrt{c'^2p^2 + c'^2q^2 + c'^2}}$

$= \frac{d'}{c'\sqrt{p^2+q^2+1}} \Rightarrow d' = ac'\sqrt{p^2+q^2+1}.$

① $\Rightarrow$ $-c'px - c'qy + c'z + ac'\sqrt{p^2+q^2+1} = 0$

$px + qy = z + a\sqrt{p^2+q^2+1}.$

4. Obtain the pde of all spheres whose centre lie on the plane $z=0$ & whose radius is constant & equal to r.

The eqn of the ~~plane~~ sphere whose centre lie on the plane $z=0$ & whose radius is equal to r is

$$(x-a)^2+(y-b)^2+z^2=r^2$$

$$2(x-a)+2pz=0 \qquad x-a=-pz$$

$$2(y-b)+2qz=0 \qquad y-b=-qz.$$

$$p^2z^2+q^2z^2+z^2=r^2$$

$$z^2(p^2+q^2+1)=r^2.$$

5. Find the DE of all spheres whose centre lie on the z-axis.

The equ of sphere whose centre lie on z-axis is $x^2+y^2+(z-c)^2=r^2$

$$2x+2(z-c)p=0 \Rightarrow z-c=\frac{-x}{p}$$

$$y+(z-c)q=0 \Rightarrow z-c=-y/q$$

$$\frac{x}{p}=\frac{y}{q} \Rightarrow \boxed{qx=py}$$

1. S.t. the Fourier transform of $f(x)=\begin{cases} a-|x|, & |x|<a \\ 0, & |x|>a>0 \end{cases}$

is $\sqrt{\frac{2}{\pi}}\,\frac{1-\cos as}{s^2}$. Hence show that $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 = \pi/3$.

2. Find the Fourier cosine transform of $f(x)=\begin{cases} 1-x^2, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

Hence p.t. $\int_0^\infty \frac{\sin x - x\cos x}{x^3}\cos\left(\frac{x}{2}\right)dx = \frac{3\pi}{16}$.

3. Find the ~~Fourier transform of $e^{-a|x|}$ if $a>0$. Deduce that $\int_0^\infty \frac{1}{(x^2+a^2)^2}dx = \frac{\pi}{4a^3}$, if $a>0$.~~

4. S.t. the F.T of $f(x) = \begin{cases} |x|, & |x|<a \\ 0, & |x|>a>0 \end{cases}$

is $F(s) = \sqrt{\frac{2}{\pi}}\, \frac{as\sin as + \cos as - 1}{s^2}$

5. S.t. the F.t. of $f(x) = \begin{cases} a^2 - x^2, & |x|<a \\ 0, & |x|>a>0 \end{cases}$ is

$2\sqrt{\frac{2}{\pi}}\, \frac{\sin as - as\cos as}{s^3}$. Hence deduce that

$\int_0^\infty \frac{\sin t - t\cos t}{t^3}dt = \frac{\pi}{4}$. using parseval's identity

S.t. $\int_0^\infty \frac{(\sin t - t\cos t)^2}{t^6}dt = \frac{\pi}{15}$.

6. Find the Fourier sine and Cosine transform of $ae^{-\alpha x} + be^{-\beta x}$, $\alpha, \beta > 0$

Solution of pdes by direct integration

Simple pdes can be solved by direct integration.

1. Solve $\frac{\partial z}{\partial y} = \sin x.$

$z = -\cos x + f(y).$ $\quad f(x)$

$\frac{\partial z}{\partial y} = \sin y$

$z = -\cos y + f(x)$

2. $\frac{\partial^2 z}{\partial x^2} = \sin y$

$\frac{\partial z}{\partial x} = \cancel{+\cos y +} x.\sin y + f(y)$

$z = \frac{x^2}{2}\sin y + x f(y) + \phi(y).$

3. $\frac{\partial^2 z}{\partial x \partial y} = \sin x$

Integrating w.r.t. 'x'

$\frac{\partial z}{\partial y} = -\cos x + f(y)$

$\int$ w.r.t. 'y'

$z = -y\cos x + F(y) + \phi(x)$

4. $\frac{\partial^2 z}{\partial x^2} = xy$

$\int$ w.r.t. 'x'

$\frac{\partial z}{\partial x} = \frac{x^2}{2}y + f(y)$

$\int$ w.r.t. 'x'

$z = \frac{x^3}{6}y + x f(y) + \phi(y)$

5. Solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$, given that $u=0$ when $t=0$ & $\frac{\partial u}{\partial t}=0$ when $x=0$. ~~Show also that as $t\to\infty$, $u\to\sin x$.~~

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x.$$

Integrating w.r.t. 'x'; $\frac{\partial u}{\partial t} = e^{-t}\sin x + f(t)$

when $x=0$, $\frac{\partial u}{\partial t}=0$.

$$0 = f(t).$$

Hence $\frac{\partial u}{\partial t} = e^{-t}\sin x$.

Integrating w.r.t. 't';

$$u(x,t) = -e^{-t}\sin x + \phi(x).$$

when $t=0$, $u=0$. $\therefore$ $0 = -\sin x + \phi(x) \Rightarrow \phi(x) = \sin x$.

$\therefore$ $u(x,t) = \sin x(1-e^{-t})$. $\quad -e^{-t}\sin x + \sin x$.

~~when $t\to\infty$, $u\to\sin x$.~~

6. Solve $\frac{\partial z}{\partial x} = 6x+3y$, $\frac{\partial z}{\partial y} = 3x-4y$.

$$\frac{\partial z}{\partial x} = 6x+3y$$

$$z = \frac{6x^2}{2} + 3xy + \phi(y)$$

$$\frac{\partial z}{\partial y} = 3x + \phi'(y)$$

$$3x-4y = 3x+\phi'(y)$$

$$\phi'(y) = -4y$$

$$\phi(y) = \frac{-4y^2}{2} + k.$$

$\therefore$ $z = 3x^2 + 3xy - 2y^2 + k$, where k is a Constant.

7. Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ and $z = e^y, \frac{\partial z}{\partial x} = 1$ when $x=0$.

$$\frac{\partial^2 z}{\partial x^2} + z = 0. \quad \text{—— (1)}$$

Here z is a function of 'x' alone. Then the given equation is an ODE $\frac{d^2 z}{dx^2} + z = 0$ $\quad m^2+1=0$

$z = A\cos x + B\sin x.$ $\quad m^2 = -1$

$m = \pm i$

The solution of (1) is

$$z = f(y)\cos x + \phi(y)\sin x. \quad \text{—— (2)}$$

Diff (2) w.r.t. 'x'; $\frac{\partial z}{\partial x} = -f(y)\sin x + \phi(y)\cos x.$ —— (3)

when

$x=0,\ z = e^{+y}.$ (2) $\Rightarrow$ $e^y = f(y)$

$x=0,\ \frac{\partial z}{\partial x} = 1$ (3) $\Rightarrow$ $1 = \phi(y)$

Sub the values of $f(y)$ & $\phi(y)$ in (2), we have

$$z = e^y\cos x + \sin x.$$

Different Solutions of pde.

Solution: A solution or integral of a pde is a relation between the dependent variables and independent variables that satisfies the DE.

Types of solutions: There are 4 types of solutions for a given pde. They are 1) Complete integral 2) P.I (3) Singular integral (4) General integral.

Complete integral: A solution containing as many arbitrary constants as the no of independent variables is called a complete integral.

Particular integral: A solution obtained by giving particular values to the arbitrary constants in a complete integral is known as P.I.

Singular integral: Let $F(x, y, z, p, q) = 0$ be the pde whose complete integral is $\phi(x, y, z, a, b) = 0$ —

The eliminant of a, b between the relations $\phi(x, y, z, a, b) = 0$, $\frac{\partial \phi}{\partial a} = 0$, $\frac{\partial \phi}{\partial b} = 0$ when it exists, is called the singular integral.

General integral: Suppose $\phi(x, y, z, a, b) = 0$ — ① is a complete integral of the p.de. $F(x, y, z, p, q) = 0$ — ②

We shall assume a relation between 'a' & 'b' in the form $b = f(a)$. Then ① becomes $\phi(x, y, z, a, f(a)) = 0$ — ③

The eliminant of a between these 2 equs ③ & $\frac{\partial \phi}{\partial a} = 0$ if it exists, is called the general integral of ①.

Note: A pde is said to be completely solved only when the complete integral, singular integral & general integral are found.

Solve the following equs.

1) $\frac{\partial z}{\partial x} = 0$

$\int$ w.r.t. 'x', $z = f(y)$ or $z = a$ is a Complete integral.

Diff $z = a$ w.r.t. 'a'; we get $0 = 1$ which is not true.

$\therefore$ Singular integral does not exist.

2) $\frac{\partial^2 z}{\partial x^2} = \cos x$.

$\int$ w.r.t. 'x'; $\frac{\partial z}{\partial x} = \sin x + f(y)$

$\int$ w.r.t 'x' $z = -\cos x + f(y)x + \phi(y)$

$= -\cos x + ax + b$ — (1) is a Complete integral

Diff (1) partially w.r.t. 'a', $0 = x$

(1) 'b' $0 = 1$ which is not true.

$\therefore$ Singular integral does not exist.

Standard form: I $f(p,q) = 0$.

In this case let $z = ax + by + c$ — (1) be a Solution of $f(p,q) = 0$ — (2).

Diff (1) partially w.r.t. 'x' $p = a$

(1) 'y', $q = b$.

Sub $p = a$ & $q = b$ in (2) we get $f(a,b) = 0$.

~~$\Rightarrow z = ax + by + c$ is a complete integral of (1) if $f(a,b) = 0$.~~

Solving this for 'b' we get $b = \phi(a)$.

Sub in ①, $z = ax + \phi(a)y + c$ —② which is a complete integral of ①

Diff ③ w.r.t. 'c' $0 = 1$, which is not true.

$\therefore$ Singular integral does not exist, in this case.

Putting $c = \psi(a)$, ③ becomes

$$z = ax + \phi(a)y + \psi(a) \quad —④$$

Diff ④ ~~part~~ w.r.t. 'a' $0 = x + \phi'(a)y + \psi'(a)$ —⑤

Eliminating 'a' between ④ & ⑤ we get the general ~~particular~~ integral

Note: For the eqn of the type $f(p,q)=0$, there is no singular integral

Solve

i) $p^2+q^2=4$.

The given eqn is $p^2+q^2=4$ —①

This is of the form $f(p,q)=0$

$z = ax+by+c$ is a solution of ① provided that

$a^2+b^2=4 \Rightarrow b^2 = 4-a^2$

$$b = \pm\sqrt{4-a^2}.$$

$\therefore$ Solution of ① is $z = ax \pm \sqrt{4-a^2}\, y + c$ —②

Since it contains 2 arbitrary constants 'a' & 'c', it is a complete integral of ①

For this eqn, SI does not exist.

To find GI, put $c = f(a)$ in ②.

$$z = ax \pm \sqrt{4-a^2}\, y + f(a) \quad —③$$

Diff w.r.t. 'a'; $0 = x \pm \frac{1}{2}(4-a^2)^{-1/2}(-2a)y + f'(a)$

$$x \pm \frac{a}{\sqrt{4-a^2}}\, y + f'(a) = 0 \quad (4)$$

Eliminant of 'a' between (3) & (4) is the general Solution of (1).

2) $p = q^2$ — (1)

This is of the form $f(p,q) = 0$

$z = ax + by + c$ is a solution of (1) provided that

$a = b^2 \Rightarrow b = \pm\sqrt{a}$

$\therefore$ solution of (1) is $z = ax \pm \sqrt{a}\, y + c$ — (2)

Since it contains 2 arbitrary constants 'a' & 'c', it is a Complete integral of (1).

For this eqn, Singular integral does not exist.

To find the general integral, put $c = f(a)$ in (2).

Then we have $z = ax \pm \sqrt{a}\, y + f(a)$ ← (3)

Diff w.r.t. 'a' we get

$$0 = x \pm \frac{1}{2} a^{-1/2} y + f'(a)$$

$$x \pm \frac{y}{2\sqrt{a}} + f'(a) = 0 \quad \text{—— (4)}$$

Eliminant of 'a' between (3) & (4) is the g.s. of (1).

(3) The given equ is $pq = 1$ —— (1)

This is of the form $f(p,q) = 0$.

$z = ax + by + c$ is a solution of (1) provided that $ab = 1 \Rightarrow b = 1/a$

$\therefore$ the solution of (1) is $z = ax + \frac{1}{a}y + c$ —— (2)

Since it contains 2 arbitrary constants 'a' & 'c', it is a complete integral of (1). For this equ, singular integral does not exist.

To find g.in put $c = f(a)$ in (2). Then we have

$$z = ax + \frac{1}{a}y + f(a) \text{ —— } (3)$$

Diff w.r.t. 'a' we get

$$0 = x + \left(\frac{-1}{a^2}\right)y + f'(a)$$

$$x - \frac{y}{a^2} + f'(a) = 0 \text{ —— } (4)$$

Eliminant of 'a' between (3) & (4) is the g. solution of (1).

(4) The equation is $pq + p + q = 0$ —— (1)

This is of the form $f(p,q) = 0$

$z = ax + by + c$ is a solution of (1) provided that

$ab + a + b = 0$

$$a + b(a+1) = 0 \Rightarrow b = \frac{-a}{a+1}$$

$\therefore$ Solution of (1) is $z = ax - \frac{a}{a+1}y + c$ —— (2)

Since it contains 2 arbitrary constants 'a' & 'c', it is a complete integral of (1). For this equation, singular integral does not exist.

To find the g. in put $c = f(a)$ in ②. Then we have

$$z = ax - \frac{a}{a+1}y + f(a) \quad \text{—③}$$

Diff w.r.t. 'a' we get

$$0 = x - \left[\frac{a+1-a}{(a+1)^2}\right]y + f'(a)$$

$$x - \frac{y}{(a+1)^2} + f'(a) = 0 \quad \text{—④}$$

Eliminant of 'a' between ③ & ④ is the g. sol of ①.

5. $p^2 + q^2 = npq.$ —①

This is of the form $f(p,q) = 0$.

$z = ax + by + c$ is a solution of ① provided that $a^2 + b^2 = nab$.

$b^2 - nab + a^2 = 0.$

$$b = \frac{na \pm \sqrt{n^2a^2 - 4a^2}}{2} = \frac{a(n \pm \sqrt{n^2 - 4})}{2}$$

Solution of ① is $z = ax + \frac{ay}{2}(n \pm \sqrt{n^2-4}) + c$ —②

Since it contains 2 arbitrary constants 'a' & 'c', it is a complete integral of ①. Singular integral does not exist.

To find general integral, put $c = f(a)$ in ②. Then

$$z = ax + \frac{ay}{2}(n \pm \sqrt{n^2-4}) + f(a) \quad \text{—③}$$

Diff partially w.r.t. 'a'; $0 = x + \frac{y}{2}(n \pm \sqrt{n^2-4}) + f'(a)$ —④

Eliminant of 'a' between ③ & ④ is the g. solution of ①.

Type II In this form only one of the variable x, y, z occurs explicitly in the equation i.e the equation will be in one of the three forms namely

i) $F(x,p,q)=0$; ii) $F(y,p,q)=0$ iii) $F(z,p,q)=0$.

1. $pq = x$. — ①

This is of the form $F(x,p,q)=0$.

Put $q=a$ in ①

$$pa = x \Rightarrow p = \frac{x}{a}$$

Now $dz = p\,dx + q\,dy$

$$= \frac{x}{a}\,dx + a\,dy$$

$$\Rightarrow z = \frac{x^2}{2a} + ay + b \quad \text{— ②}$$

This is a complete integral of ①.

In this case, singular integral does not exist.

To find general integral, put $b = f(a)$

$$z = \frac{x^2}{2a} + ay + f(a) \quad \text{— ③}$$

Diff w.r.t. 'a' we get

$$0 = \frac{-x^2}{2a^2} + y + f'(a).$$

$$\frac{-x^2}{2a^2} + y + f'(a) = 0 \quad \text{— ④}$$

Eliminant of 'a' between ③ & ④ is the general solution of ①.

2. $\sqrt{p}+\sqrt{q}=x$ —— ①

The given equation is of the form $F(x,p,q)=0$.

Put $q=a$ —— ②

$\sqrt{p}+\sqrt{a}=x \Rightarrow \sqrt{p}=x-\sqrt{a}$

$p=(x-\sqrt{a})^2 = x^2-2x\sqrt{a}+a$ —— ③

Sub ② & ③ in $dz=p\,dx+q\,dy$, we have

$$dz=(x^2-2x\sqrt{a}+a)\,dx+a\,dy$$

$$z=\frac{x^3}{3}-x^2\sqrt{a}+ax+ay+b.$$

This is a complete integral of ①.

Singular integral does not exist. $a^{1/2-1}$

To find general integral, put $b=f(a)$.

$$z=\frac{x^3}{3}-x^2\sqrt{a}+ax+ay+f(a) \quad \text{—— ④}$$

Diff w.r.t. 'a', $0=-x^2\frac{1}{2}a^{-1/2}+x+y+f'(a)$

$$\frac{-x^2}{2\sqrt{a}}+x+y+f'(a)=0 \quad \text{—— ⑤}$$

Eliminant of 'a' between ④ & ⑤ is the general integral of ①.

3. $\sqrt{p}+\sqrt{q}=\sqrt{y}$ —— ①

The given equ is of the form $F(y,p,q)=0$

Put $p=a$ in ①

$\sqrt{a}+\sqrt{q}=\sqrt{y}$

$\Rightarrow \sqrt{q}=\sqrt{y}-\sqrt{a}$

$q=(\sqrt{y}-\sqrt{a})^2 = y+a-2\sqrt{ay}$

$dz=p\,dx+q\,dy.$

$$\frac{y^{1/2+1}}{1/2+1}$$

$$= a\,dx + (y + a - 2\sqrt{ay})\,dy$$

$$z = ax + \frac{y^2}{2} + ay - 2\sqrt{a}\,\frac{y^{1/2+1}}{1/2+1} + b$$

$$= ax + \frac{y^2}{2} + ay - 2\sqrt{a}\,y^{3/2}\cdot\frac{2}{3} + b$$

$$= ax + \frac{y^2}{2} + ay - \frac{4}{3}\sqrt{a}\,y^{3/2} + b \quad \text{—— (2)}$$

This is a complete integral of (1).

Singular integral does not exist.

To find general integral put $b = f(a)$.

$$z = ax + \frac{y^2}{2} + ay - \frac{4}{3}\sqrt{a}\,y^{3/2} + f(a) \quad \text{—— (3)}$$

Diff w.r.t. 'a'; $0 = x + y - \frac{4}{3}\frac{1}{2\sqrt{a}}\,y^{3/2} + f'(a)$.

$$x + y - \frac{2y\sqrt{y}}{3\sqrt{a}} + f'(a) = 0 \quad \text{—— (4)}$$

Eliminant of 'a' between (3) & (4), we get the general integral of (1).

4. $p = y^2q^2$ —— (1)

This is of the form $F(y, p, q) = 0$.

Put $p = a^2$ in (1).

$$a^2 = y^2q^2 \Rightarrow q = \frac{a}{y}.$$

Now $dz = p\,dx + q\,dy$ becomes

$$dz = a^2 dx + \frac{a}{y} dy$$

$$z = a^2 x + \frac{a \log y}{y^2} + b.$$

This is a Complete integral of ①

Singular integral does not exist in this case.

To find general integral, put $b = f(a)$.

$$z = a^2 x + \frac{a \log y}{y^2} + f(a) \quad \text{——— ②}$$

Diff with respect to 'a'

$$0 = 2ax + \frac{\log y}{y^2} + f'(a). \quad \text{——— ③}$$

Eliminant of 'a' between ② & ③ is the general integral of ①

5. $q^2 = y p^4$ —①

$$F(y, p, q) = 0$$

Put $p = a$. $\therefore q^2 = y a^4 \Rightarrow q = \pm \sqrt{y}\, a^2$

Now $dz = p\,dx + q\,dy$

$$= a\,dx \pm a^2 \sqrt{y}\, dy.$$

$$z = ax \pm a^2 y^{3/2} \cdot \frac{2}{3} + b$$

This is Complete integral of ①.

Singular integral does not exist in this case.

To find general integral, put $b = f(a)$

$$z = ax \pm \frac{2a^2}{3} y^{3/2} + f(a) \quad \text{——— ②}$$

Diff (2) w.r.t. 'a'; $0 = x + \frac{4a}{3} y^{3/2} + f'(a)$ — (3)

Eliminant of 'a' between (2) & (3) is the general solution of (1).

6. $z = 2yp^2$, $p = a$. $z = ax + a^2y^2 + c$.

7. $p(1+q) = qz$. — (1)

This is of the form $F(z, p, q) = 0$.

Put $q = ap$ in (1)

$$p(1+ap) = apz$$

$$p + ap^2 - apz = 0$$

$$1 + ap - az = 0$$

$$ap = az - 1$$

$$p = \frac{az-1}{a}$$

Now $dz = p\,dx + q\,dy$

$$= \left(\frac{az-1}{a}\right) dx + ap\,dy$$

$$= \left(\frac{az-1}{a}\right) dx + (az-1)\,dy$$

$$\frac{dz}{az-1} = \frac{1}{a} dx + dy$$

Integrating, $\frac{1}{a}\log(az-1) = \frac{x}{a} + y + b$.

$$\log(az-1) = x + ay + ab \quad \text{— (2)}$$

This is a Complete integral of ①.

Singular integral does not exist.

To find general integral, put $b = f(a)$.

$\log(az-1) = x + ay + a f(a)$ —— ③

Diff w.r.t. 'a', $\frac{1}{az-1} \cdot z = y + af'(a) + f(a)$

$$\frac{z}{az-1} = y + af'(a) + f(a) \quad \text{—— ④}$$

Eliminant of 'a' between ③ & ④ is the general Solution of ①

Find the Complete integral of

8. $z^2(p^2+q^2+1) = a^2$ —— ①

The given equ is of the form $F(z,p,q) = 0$.

Put $q = ap$.

$z^2(p^2+a^2p^2+1) = a^2$

$p^2 + a^2p^2 + 1 = \frac{a^2}{z^2}$, $p^2(1+a^2) = \frac{a^2}{z^2} - 1$

$$p^2 = \frac{a^2-z^2}{z^2(1+a^2)}$$

$$p = \frac{\sqrt{a^2-z^2}}{z\sqrt{1+a^2}}$$

Now $dz = p\,dx + q\,dy$ becomes

$$= \frac{\sqrt{a^2-z^2}}{z\sqrt{1+a^2}}\,dx + a\,\frac{\sqrt{a^2-z^2}}{z\sqrt{1+a^2}}\,dy$$

$$= \frac{\sqrt{a^2-z^2}}{z\sqrt{1+a^2}} (dx + a\,dy)$$

$$\frac{z\sqrt{1+a^2}}{\sqrt{a^2-z^2}}\,dz = dx + a\,dy$$

$$\sqrt{1+a^2}\int \frac{z}{\sqrt{a^2-z^2}}\,dz = \int dx + a\int dy$$

$$= x + ay + b.$$

put $t = a^2 - z^2$

$dt = -2z\,dz.$

$$-\sqrt{1+a^2}\int \frac{dt}{2}\frac{1}{\sqrt{t}} = x + ay + b$$

$$-\frac{\sqrt{1+a^2}}{2}\int \frac{dt}{\sqrt{t}} = x + ay + b.$$

$$-\frac{(\sqrt{1+a^2})}{2}\cdot 2\sqrt{t} = x + ay + b.$$

$$(1+a^2)(a^2-z^2) = (x+ay+b)^2$$

Type III variable separable. In this type, the variable z does not occur explicitly and the equ will be of the form $F(x,y,p,q)=0$. This equ can be written in the form $f_1(x,p) = f_2(y,q)$. This form is known as variable separable form.

Solve the following equations.

The given eqn is

1. $q-p+x-y=0.$ — ①

$$q-y=p-x$$

This is in variable separable form.

$\therefore$ put $q-y=a \Rightarrow q=a+y$

$p-x=a \Rightarrow p=a+x.$

Now $dz=p\,dx+q\,dy$ becomes

$$dz=(a+x)\,dx+(a+y)\,dy$$

Integrating $z=\int (a+x)\,dx+\int (a+y)\,dy$

$\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c, \quad n\neq -1.$

$$=\frac{1}{1}\frac{(a+x)^2}{2}+\frac{(a+y)^2}{2}+b. \quad \text{— ②}$$

This is a complete integral of ①. Singular integral does not exist. To find general integral, put $b=f(a)$ in ②.

$$z=\frac{(a+x)^2}{2}+\frac{(a+y)^2}{2}+f(a) \quad \text{— ③}$$

Diff w.r.t. 'a', $0=2\frac{(a+x)}{2}+2.\frac{(a+y)}{2}+f'(a)$

$$a+x+a+y+f'(a)=0$$

$$\text{(i.e)} \quad 2a+x+y+f'(a)=0 \quad \text{— ④}$$

Eliminating 'a' between ③ & ④, we get the general integral of ①.

2. $p^2+q^2=x+y$ — ①

$p^2-x=\cancel{q^2}\;y-q^2$

This is in variable separable form.

$\therefore$ put $p^2 - x = a \Rightarrow p^2 = a + x \Rightarrow p = \pm\sqrt{a+x}$

$y - q^2 = a \Rightarrow q^2 = -a + y \Rightarrow q = \pm\sqrt{-a+y}$.

Now $dz = p\,dx + q\,dy$ becomes

$$= (\pm\sqrt{a+x})\,dx \pm (\sqrt{y-a})\,dy$$

Integrating we get

$$z = \pm \frac{2}{3}(a+x)^{3/2} \pm \frac{2}{3}(y-a)^{3/2} + b$$

$$= \pm \frac{2}{3}\left[(a+x)^{3/2} + (y-a)^{3/2}\right] + b \quad \text{—— (2)}$$

This is a Complete integral of (1)

Singular integral does not exist.

To find general integral, put $b = f(a)$

$$z = \pm \frac{2}{3}\left[(a+x)^{3/2} + (y-a)^{3/2}\right] + f(a) \quad \text{—— (3)}$$

Diff w.r.t. 'a',

$$0 = \pm \frac{2}{3}\cdot\frac{3}{2}\left[(a+x)^{1/2} + (y-a)^{1/2}(-1)\right] + f'(a)$$

$$\pm\left[(a+x)^{1/2} - (y-a)^{1/2}\right] + f'(a) = 0 \quad \text{—— (4)}$$

Eliminating 'a' between (3) & (4), we get the general integral of (1).

3. $\sqrt{p} + \sqrt{q} = 2x$ —— (1)

$\sqrt{p} - 2x = -\sqrt{q}$

This is in variable separable form.

put $\sqrt{p} - 2x = a \Rightarrow \sqrt{p} = a + 2x$ $\quad p = (a+2x)^2$

$-\sqrt{q} = a \Rightarrow q = a^2$

Now $dz = p\,dx + q\,dy$

$= (a+2x)^2 + a^2 dy$

$$z = \frac{1}{2} \cdot \frac{(a+2x)^3}{3} + a^2 y + b$$

$$z = \frac{(a+2x)^3}{6} + a^2 y + b \quad \text{—— (2)}$$

This is a Complete integral of (1).

Singular integral does not exist.

To find the general integral put $b = f(a)$

$$z = \frac{(a+2x)^3}{6} + a^2 y + f(a) \quad \text{—— (3)}$$

Diff w.r.t. a,

$$0 = \frac{3(a+2x)^2}{6} + 2ay + f'(a)$$

$$\frac{(a+2x)^2}{2} + 2ay + f'(a) = 0 \quad \text{—— (4)}$$

Eliminating 'a' between (3) & (4), we get the general integral of (1).

4. $p + q = \sin x + \sin y$ —— (1)

$p - \sin x = \sin y - q$

This is in variable separable form.

Put $p - \sin x = a \Rightarrow p = a + \sin x$

$\sin y - q = a \Rightarrow q = \sin y - a$

$dz = p\,dx + q\,dy$

$= (a + \sin x)\,dx + (\sin y - a)\,dy$

$z = ax - \cos x - \cos y - ay + b$ — (2)

This is a complete integral of (1)

Singular integral does not exist.

To find general integral, put $b = f(a)$

$z = ax - \cos x - \cos y - ay + f(a)$ — (3)

Diff w.r.t. 'a'

$0 = x - y + f'(a)$

$f'(a) = y - x.$

$f(a) = (y-x).a$ — (4)

Using (4) in (3), $z = ax - \cos x - \cos y - ay + yx - ax$

$z = -(\cos x + \cos y).$

5. $p + q = px + qy.$

$p - px = qy - q$

$p(1-x) = q(y-1).$

This is in variable separable form.

put $p(1-x) = a \Rightarrow p = \frac{a}{1-x}$

$q(y-1) = a \Rightarrow q = \frac{a}{y-1}.$

Now $dz = p\,dx + q\,dy$

$$= \left(\frac{a}{1-x}\right)dx + \left(\frac{a}{y-1}\right)dy$$

Integrating we get

$$z = -a\log(1-x) + a\log(y-1) + b \quad \text{--- (2)}$$

This is a Complete integral of (1).

Singular integral does not exist.

To find general integral, put $b = f(a)$ in (2)

$$z = -a\log(1-x) + a\log(y-1) + f(a) \quad \text{--- (3)}$$

Diff w.r.t. 'a' we get

$$0 = -\log(1-x) + \log(y-1) + f'(a)$$

$$\log\left(\frac{y-1}{1-x}\right) + f'(a) = 0 \quad \text{--- (4)}$$

Eliminating 'a' between (3) & (4), we get the general integral of (1).

Type IV Clairaut's form — The pde is of the form $z = px + qy + f(p, q)$.

1. $z = px + qy + pq$ --- (1)

This is Clairaut's form. Its Complete integral is

$z = ax + by + ab$ --- (2)

Diff (2) w.r.t. 'a', $0 = x + b \Rightarrow b = -x$.

(2) 'b', $0 = y + a \Rightarrow a = -y$

Sub the values of a & b in (2) we get

$$z = -xy - xy + xy,$$

$z = -xy \Rightarrow z + xy = 0.$

This is the singular integral of (1).

To find g. integral, put $b = f(a)$ in (2)

$$z = ax + f(a)y + af(a) \quad \text{--- (3)}$$

Diff (3) w.r.t. 'a', $0 = x + f'(a)y + f(a) + af'(a)$ --- (4)

Eliminating 'a' between (3) & (4), we get the general integral of (1).

(2) $(1-x)p + (2-y)q = 3 - z.$

$$p - px + 2q - qy - 3 = -z$$

$$z = px + qy + (3 - p - 2q) \quad \text{--- (1)}$$

This is Clairaut's form.

$\therefore z = ax + by + (3 - a - 2b)$ is a complete integral of (1) --- (2)

Diff (2) w.r.t. 'a', $0 = x - 1 \Rightarrow \boxed{x = 1}$

(2) 'b', $0 = y - 2 \Rightarrow \boxed{y = 2}$

Since we cannot eliminate 'a' & 'b' from (2), Singular integral does not exist. To find g.I put $b = f(a)$

$$z = ax + f(a)y + (3 - a - 2f(a)) \quad \text{--- (3)}$$

Diff w.r.t. 'a', $0 = x + f'(a)y - 1 - 2f'(a)$

$$1 - x = f'(a)[y - 2]$$

$$f'(a) = \frac{1-x}{y-2}$$

$$f(a) = \int \frac{1-x}{y-2}\, da$$

$$= \left(\frac{1-x}{y-2}\right) a$$

Sub this in (3)

$$z = ax + \left(\frac{1-x}{y-2}\right) ay + \left[3 - a - 2\left(\frac{1-x}{y-2}\right) a\right].$$

$$(y-2) z = ax(y-2) + (1-x)ay + (3-a)(y-2) - 2a(1-x)$$

$$= \cancel{axy} - \cancel{2ax} + \cancel{ay} - \cancel{axy} + 3y - 6 - \cancel{ay} + \cancel{2a} - \cancel{2a} + \cancel{2ax}$$

$$= 3(y-2)$$

$$\boxed{z = 3}$$

This is the general solution of (1).

3. $\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$

$$z = px + qy + pq\sqrt{pq} \quad \text{—(1)}$$

This is Clairaut's form.

$\therefore$ $z = ax + by + ab\sqrt{ab}$ (2) is the Complete integral of (1)

Diff (2) w.r.t. 'a', $0 = x + \frac{3}{2}(a.b)^{1/2}.b$ $\quad a^{3/2}.b^{3/2}$

$$x = -\frac{3}{2} b\sqrt{ab} \qquad x = -\frac{3}{2} a^{1/2} b^{3/2}$$

$$x^2 = \frac{9}{4} ab^3$$

Diff (2) w.r.t. 'b', $0 = y + \frac{3}{2}(ab)^{1/2}.a$

$$y = -\frac{3a}{2}\sqrt{ab}$$

$$y^2 = \frac{9}{4} a^3 b$$

$$\frac{x^2}{y^2} = \frac{b^2}{a^2} \Rightarrow b^2 = \frac{a^2 x^2}{y^2}.$$

$\Rightarrow$ b is a fun of 'a' which is not true, since 'a' & 'b' are independent. To find G.I put $b = f(a)$ in (2)

$$z = ax + f(a)y + (a f(a))^{3/2} \quad \text{--- (3)}.$$

Diff (3) w.r.t. 'a',

$$0 = x + f'(a)y + \frac{3}{2}\left[a f(a)\right]^{1/2}\left[f(a) + a f'(a)\right] \quad \text{--- (4)}$$

Eliminating 'a' from (3) & (4), we get the general integral of (1)

Solve Find the complete solution and singular integral of (1)

4. $z = px + qy + \sqrt{1+p^2+q^2}$ --- (1)

This is Clairaut's form.

$z = ax + by + \sqrt{1+a^2+b^2}$ is the C.I. of (1). --- (2)

Diff (2) partially w.r.t. 'a'

$$0 = x + \frac{1}{2}\frac{1}{\sqrt{1+a^2+b^2}} \cdot 2a$$

$$x = \frac{-a}{\sqrt{1+a^2+b^2}} \quad \text{--- (3)}, \quad x^2 = \frac{a^2}{1+a^2+b^2} \quad \text{--- (4)}$$

Diff (2) w.r.t. partially w.r.t. 'b',

$$0 = y + \frac{1}{2}\frac{1}{\sqrt{1+a^2+b^2}} \cdot 2b$$

$$y = \frac{-b}{\sqrt{1+a^2+b^2}} \quad \text{--- (5)}, \quad y^2 = \frac{b^2}{1+a^2+b^2} \quad \text{--- (6)}$$

④ + ⑥
③ + ⑥ ; $x^2+y^2 = \dfrac{a^2+b^2}{1+a^2+b^2}$

$$1-(x^2+y^2) = 1-\frac{a^2+b^2}{1+a^2+b^2}$$

$$= \frac{1}{1+a^2+b^2}$$

$$1-x^2-y^2 = \frac{1}{1+a^2+b^2}$$

$$\therefore \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}.$$

Sub this in ③ & ⑥, we have

$$a = -x\sqrt{1+a^2+b^2} = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$b = \frac{-y}{\sqrt{1-x^2-y^2}}.$$

Sub the values of 'a' & 'b' in ②

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$x^2+y^2+z^2=1$. This is the singular integral of ①

To find general integral, put $b = f(a)$ in (2)

$$z = ax + f(a)y + \sqrt{1+a^2+[f(a)]^2} \quad \text{---} (7)$$

Diff (7) w.r.t. 'a';

$$0 = x + f'(a)y + \frac{1}{2}\frac{1}{\sqrt{1+a^2+[f(a)]^2}}\, 2a + 2f(a)f'(a)$$

$$x + f'(a)y + \frac{a + f(a)\,f'(a)}{\sqrt{1+a^2+[f(a)]^2}} = 0. \quad \text{---} (8)$$

Type V Equations of the form $F(x^m p, y^n q) = 0$ &
$F(z, x^m p, y^n q) = 0$

Equations reducible to standard form:

Some nonlinear pde of first order do not fall under any of the four standard types. However in some cases, it is possible to transform the pde into one of the std. types by changing the variables. (ie) by proper Substitution.

Case i) If $m \neq 1$ & $n \neq 1$, then put $x^{1-m} = X$ & $y^{1-n} = Y$.

Substituting, $p = \frac{\partial z}{\partial x}$

$$= \frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$$

$$= P(1-m)x^{-m}$$

$$x^m p = P(1-m) \quad \text{---} (1)$$

$|||^{rly}$ $\quad y^n q = Q(1-n). \quad \text{---} (2)$

Sub (1) & (2) in the given eqn, we get a DE of the form $F(P,Q) = 0$ & $F(z,P,Q) = 0$ which can be solved easily.

Case ii) If $m = n = 1$, then we use the substitution,

$\log x = X$, $\log y = Y$.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$$

$$= P \cdot \frac{1}{x}$$

$$px = P \quad -(1)$$

IIIrly $qy = Q \quad -(2)$

Sub (1) & (2) in the given equ, we get DE of the form $F(P, Q) = 0$ & or $F(z, P, Q) = 0$ which can be solved easily.

1. $x^4p^2 + y^2qz = 2z^2$.

$$(x^2p)^2 + y^2qz = 2z^2 \quad -(1)$$

Here $m = 2$, $n = 2$.

Put $X = x^{1-m}$, $Y = y^{1-n}$

$= x^{1-2}$, $Y = y^{1-2}$

$X = x^{-1}$, $Y = y^{-1}$

$x^m p = (1-m)P$ & $y^n q = (1-n)Q$

$x^2 p = (1-2)P$, $y^2 q = (1-2)Q$

$= -P$ $\qquad$ $= -Q$ $\}$ $-(2)$

Sub (2) in the given equ (1),

$$(-P)^2 + (-Q)z = 2z^2$$

$$P^2 - Qz = 2z^2 \quad -(3)$$

This is of the form $F(z, P, Q) = 0$.

Put $Q = aP$ in ③

$P^2 - aPz = 2z^2$

$P^2 - aPz - 2z^2 = 0$

$P = \dfrac{az \pm \sqrt{a^2z^2 + 8z^2}}{2} = \dfrac{z}{2}\left[a \pm \sqrt{a^2+8}\right]$

$dz = P\,dx + Q\,dy$

$dz = \dfrac{z}{2}\left[a \pm \sqrt{a^2+8}\right] dx + \dfrac{az}{2}\left[a \pm \sqrt{a^2+8}\right] dy$

$\dfrac{dz}{z} = \dfrac{a \pm \sqrt{a^2+8}}{2}\left[dx + a\,dy\right]$

$\log z = \dfrac{a \pm \sqrt{a^2+8}}{2}\left[x + ay\right] + b$

$= \dfrac{a \pm \sqrt{a^2+8}}{2}\left(x^{-1} + ay^{-1}\right) + b.$ — ④

This is the complete integral of ①

Singular integral does not exist.

To find g.in, put $b = f(a)$. in ④

$\log z = \dfrac{a \pm \sqrt{a^2+8}}{2}\left(x^{-1} + ay^{-1}\right) + f(a)$ — ⑤

Diff ⑤ w.r.t. 'a'; $0 = \dfrac{1}{2}\left[\left(a \pm \sqrt{a^2+8}\right)\left(y^{-1}\right) + \left(x^{-1} + ay^{-1}\right)\left(1 \pm \tfrac{1}{\cancel{2}}(a^2+8)^{-1/2}\cancel{2}a\right)\right] + f'(a)$ — ⑥

Eliminating 'a' between ⑤ & ⑥, we get the general solution of ①.

2. $x^2p^2+y^2q^2=z^2$.

$(xp)^2+(yq)^2=z^2$. — (1)

Here $m=1,\ n=1$.

Put $X=\log x,\quad Y=\log y$

$px=P,\quad qy=Q$.

Then (1) becomes $P^2+Q^2=z^2$. — (2)

This is of the form $F(z,p,q)=0$.

Put $Q=aP$ in (1).

$P^2+a^2P^2=z^2$

$P^2(1+a^2)=z^2$

$$P=\pm\frac{z}{\sqrt{1+a^2}}$$

$$dz=\pm\frac{z}{\sqrt{1+a^2}}dX\pm\frac{az}{\sqrt{1+a^2}}dY.$$

$$\frac{dz}{z}=\pm\frac{1}{\sqrt{1+a^2}}[dX+a\,dY]$$

$$\log z=\pm\frac{1}{\sqrt{1+a^2}}[X+aY]+b$$

$$=\pm\frac{1}{\sqrt{1+a^2}}[\log x+a\log y]+b \quad \text{— (3)}$$

This is the complete integral of (1).

Singular integral does not exist.

To find G.I, put $b=f(a)$ in (3)

$$\log z = \pm \frac{1}{\sqrt{1+a^2}} [\log x + a \log y] + f(a) \quad -④$$

Diff w.r.t. 'a'

$$0 = \pm \frac{1}{\sqrt{1+a^2}} [\log y] \pm \frac{1}{2} (1+a^2)^{-3/2} \cdot 2a (\log x + a \log y) + f'(a).$$

Eliminating 'a' from ④ & ⑤, we get G.I. of ①. —⑤

3. $x^2p^2 + xpq = z^2$.

$(xp)^2 + (xp)q = z^2$. —①

Here $m=1$. $X = \log x$, $xp = P$.

① ⇒ $P^2 + Pq = z^2$ —②

This is of the form $F(z, p, q) = 0$.

Put $q = aP$

② ⇒ $P^2 + aP^2 = z^2$

$$P^2 (1+a) = z^2 \Rightarrow P = \pm \frac{z}{\sqrt{1+a}}$$

$$dz = \pm \frac{z}{\sqrt{1+a}} dX \pm \frac{az}{\sqrt{1+a}} dy$$

$$\frac{dz}{z} = \pm \frac{1}{\sqrt{1+a}} [dX \pm a\, dy]$$

$$\log z = \pm \frac{1}{\sqrt{1+a}} [X + ay] + b.$$

$$= \pm \frac{1}{\sqrt{1+a}} [\log x + ay] + b \quad —③$$

This is the complete integral of ①.

Singular integral does not exist.

To find G.I, put $b = f(a)$ in (1)

$$\log z = \pm \frac{1}{\sqrt{1+a}} (\log x + ay) + f(a). \quad \text{--- (4)}$$

Diff w.r.t. 'a' $0 = \pm \frac{1}{2} (1+a)^{-3/2} (\log x + ay)$

$$\pm \frac{1}{\sqrt{1+a}} (y) + f'(a) \quad \text{--- (5)}$$

Eliminating 'a' from (4) & (5), we get G.S. of (1).

4. $2x^4p^2 - yzq - 3z^2 = 0.$

$$2(x^2p)^2 - (yq)z = 3z^2. \quad \text{--- (1)}$$

Here $m = 2, n = 1.$

$X = x^{1-m}$, $Y = \log y$

$= x^{-1}$

$x^m p = P(1-m)$ $\quad qy = Q.$

$x^2 p = -P$

Sub in (1), $2P^2 - Qz = 3z^2. \quad \text{--- (2)}$

This is of the form $F(z, p, q) = 0.$

$Q = aP$ in (2)

$2P^2 - aPz - 3z^2 = 0$

$$P = \frac{az \pm \sqrt{a^2z^2 + 24z^2}}{4} = \frac{z}{4}\left[a \pm \sqrt{a + 24}\right]$$

$$dz = \frac{z}{4}\left(a \pm \sqrt{a+24}\right)dx + \frac{az}{4}\left(a \pm \sqrt{a+24}\right)dy$$

$$\frac{dz}{z} = \frac{1}{4}\left(a \pm \sqrt{a+24}\right)(dx + a\,dy)$$

$$\log z = \frac{1}{4}\left(a \pm \sqrt{a+24}\right)(x + ay) + b \quad \text{--- (3)}$$

$(x^{-1} + a\log y)$

This is the complete integral of (1)

Singular integral does not exist.

To find G.I., put $b = f(a)$ in (3),

$$\log z = \frac{1}{4}\left(a \pm \sqrt{a+24}\right)(x^{-1} + a\log y) + f(a) \quad \text{--- (4)}$$

Diff w.r.t. 'a'

$$0 = \frac{1}{4}\left(a \pm \sqrt{a+24}\right)(\log y) + \frac{1}{4}\left(1 \pm \frac{1}{2}(a+24)^{-1/2}\right)(x^{-1} + a\log y) + f'(a) \quad \text{--- (5)}$$

Eliminating 'a' from (4) & (5), we get the G.S. of (1).

<u>Type VI</u> Eqns of the form $F(z^m p, z^m q) = 0$ &

$F_1(x, z^m p) = F_2(y, z^m q)$.

<u>case i)</u> If $m \neq -1$, then ~~put z^{m+1}~~

$$Z = z^{m+1} \Rightarrow \frac{\partial Z}{\partial z} = (m+1)z^m$$

$$P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x}$$

$$= (m+1)z^m p$$

$$\boxed{\frac{P}{m+1} = z^m p}$$

$$|||^{ly} \quad \boxed{\frac{Q}{m+1} = z^m q} \quad \text{--- (2)}$$

<u>case ii)</u> If $m = -1$, put

$$Z = \log z \Rightarrow \frac{\partial Z}{\partial z} = 1/z$$

$$P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x}$$

$$= \frac{1}{z}p$$

$$P = z^{-1}p \quad \text{--- (1)}$$

$$|||^{ly} \quad Q = z^{-1}q \quad \text{--- (2)}$$

subs (1) & (2) in

$F(P, Q) \Rightarrow$ or $f_1(x, P) = f_2(y, Q)$

Sub (1) & (2) in the given eqn, we get pde of the form

$F(p,q)=0$ or $F_1(x,p) = f_2(y,q)$ which can be solved easily.

1. $4z^2q^2 = y + 2z^2p - x$.

$4(zq)^2 - 2(z^2p) = y - x$ — ①

Here $m = 1$

$$Z = z^{m+1} \; \& \; \frac{P}{m+1} = z^m p, \quad \frac{Q}{m+1} = z^m q$$

$$Z = z^2 \; \& \; \frac{P}{2} = z p, \quad \frac{Q}{2} = z q.$$

Sub pz & qz in ①

$$4\frac{Q^2}{4} - 2\frac{P}{2} = y - x$$

$$Q^2 - P = y - x.$$

$$Q^2 - y = P - x \quad — ②$$

This is in variable separable form.

$Q^2 - y = a \Rightarrow Q = \pm\sqrt{a+y}$

$P - x = a \Rightarrow P = a + x.$

$\frac{(a+y)^{1/2+1}}{1/2+1/2}$ $\frac{2}{3}(a+y)^{3/2}$

$dZ = (a+x)\,dx \pm \sqrt{a+y}\,dy$

$$Z = \frac{(a+x)^2}{2} \pm \frac{2}{3}(a+y)^{3/2} + b.$$

$$z^2 = \frac{(a+x)^2}{2} \pm \frac{2}{3}(a+y)^{3/2} + b \quad ③$$

Complete integral

S.I. No

G.I put $b = f(a)$ ④

$0 = \frac{2(a+x)}{2} \pm (a+y)^{1/2} + f'(a)$ ⑤

③ & ⑤ → G.I.

2. $(zp+x)^2 + (zq+y)^2 = 1$ —(1)

$\cancel{(zp)^2 + x^2 + 2zpx + (zq)^2 + y^2 + 2zqy = 1}$

$\cancel{(zp)^2 + 2(zp)x + (zq)^2 + 2(zq)y = 1 - x^2 - y^2}$ —(1)

$m = 1$.

$$Z = z^{m+1} = z^2 \qquad \frac{P}{m+1} = z^m p, \quad \frac{Q}{m+1} = z^m q$$

$$\frac{P}{2} = zp \qquad \frac{Q}{2} = zq$$

(1) $\Rightarrow$ $\left(\frac{P}{2}+x\right)^2 + \left(\frac{Q}{2}+y\right)^2 = 1$.

$$\left(\frac{P}{2}+x\right)^2 = 1 - \left(\frac{Q}{2}+y\right)^2 = a^2 \text{ (say)}.$$

$$\left(\frac{P}{2}+x\right)^2 = a^2$$

$$\frac{P}{2}+x = a \Rightarrow \frac{P}{2} = a-x \Rightarrow P = 2(a-x).$$

$$1 - \left(\frac{Q}{2}+y\right)^2 = a^2$$

$$\left(\frac{Q}{2}+y\right)^2 = 1-a^2$$

$$\frac{Q}{2}+y = \sqrt{1-a^2}$$

$$\frac{Q}{2} = \sqrt{1-a^2} - y$$

$$Q = 2\left(\sqrt{1-a^2} - y\right).$$

$$dZ = 2(a-x)\,dx + 2\left(\sqrt{1-a^2}-y\right)dy$$

$$Z = 2\left(\frac{(a-x)^2}{-2}\right) + 2\left(\sqrt{1-a^2}\,y - \frac{y^2}{2}\right) + b.$$

$= -(a-x)^2 + 2\sqrt{1-a^2}\, y - y^2 + b$

$z^2 = -(a-x)^2 + 2y\sqrt{1-a^2} - y^2 + b$ — (2)

S.I. This is C.I of (1). S.I. does not exist.

To find G.I, put $b = f(a)$ in (2)

$z^2 = -(a-x)^2 + 2y\sqrt{1-a^2} - y^2 + f(a)$. — (3)

Diff w.r.t. 'a' — (4).

Eliminating 'a' from (3) & (4), we get G.I. of (1).

Lagrange's Linear equations

The equ of the form $Pp + Qq = R$ is known as Lagrange's equation, where P, Q, R are functions of x, y and z. To solve this equation, we solve the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

If the solution of the subsidiary equ is of the form $u(x,y) = c_1$ and $v(x,y) = c_2$, then the solution of the given Lagrange's equation is $\phi(u,v) = 0$.

1. Solve $px + qy = z$. — (1)

This is a Lagrange's linear equation.

Here $P = x$, $Q = y$, $R = z$.

The subsidiary equations are

Method of grouping In the auxiliary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ if the variables can be separated in any pair of equations, then we get a solution of the form $u(x,y) = c_1$ & $v(x,y) = c_2$.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(ie) $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}.$$

Taking 1st two ratios, $\frac{dx}{x} = \frac{dy}{y}$.

Integrating, $\log x = \log y + \log c_1$

$\log\left(\frac{x}{y}\right) = \log c_1 \Rightarrow \boxed{c_1 = \frac{x}{y}}$

Taking 1st & 3rd ratios, $\frac{dx}{x} = \frac{dz}{z}$ $\log(x/z) = \log c_2$

$\boxed{c_2 = \frac{x}{z}}$

Solution of ① is $f\left(\frac{x}{y}, \frac{x}{z}\right) = 0$.

2. $pyz + qzx = xy$

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

1st & 3rd $\frac{dx}{yz} = \frac{dz}{xy}$

$\frac{dx}{z} = \frac{dz}{x}$

$x\,dx = z\,dz$

$\frac{x^2}{2} = \frac{z^2}{2} + c_1$

$x^2 - z^2 = c_1$

$\frac{dy}{z} = \frac{dz}{y}$

$y^2 - z^2 = c_2$

$\phi(x^2 - z^2, y^2 - z^2) = 0.$

x^{-2} $\frac{x^{-2+1}}{-2+1}$ $\frac{x^{-1}}{-1}$

3. $x^2p + y^2q = z^2$ — ①

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

$$\frac{1}{y} - \frac{1}{x} = c_1$$

$$-\frac{1}{y} = -\frac{1}{z} + c_2$$

$$\frac{1}{z} - \frac{1}{y} = c_2$$

g.s of ①

$$\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0.$$

4. Find the G.S. of $p\tan x + q\tan y = \tan z$.

The auxiliary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

1^{st} & 2^{nd} $\frac{dx}{\tan x} = \frac{dy}{\tan y}$

$$\int \cot x\, dx = \int \cancel{\tan} \cot y\, dy$$

$$\log \sin x = \log \sin y + \log c_1$$

$$\log\left(\frac{\sin x}{\sin y}\right) = \log c_1 \Rightarrow c_1 = \frac{\sin x}{\sin y}$$

Taking $\frac{dy}{\tan y} = \frac{dz}{\tan z}$

$$\int \cot y\, dy = \int \cot z\, dz$$

$$\log \sin y = \log \sin z + \log c_2$$

$$c_2 = \frac{\sin y}{\sin z}.$$

$$f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

$\frac{\sin x}{\cos x}$ $\log \cos x$

5. $p \cot x + q \cot y = \cot z.$

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

$\int \tan x \, dx = \int \tan y \, dy =$

$\int \tan y \, dy = \int \tan z \, dz$

$=$

6. $\frac{y^2 z}{x} p + xzq = y^2.$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}.$$

Taking 1^{st} ② $\frac{x\,dx}{y^2 z} = \frac{dy}{xz}$

$$\frac{x\,dx}{y^2} = \frac{dy}{x}$$

$x^2 dx = y^2 dy$

$x^3 - y^3 = c_1$

1^{st} & Last

$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2}$

$x\,dx = z\,dz$

$x^2 - z^2 = c_2$

Solution is $f(x^3 - y^3, x^2 - z^2) = 0.$

Mtd of multipliers Choose any 3 multipliers l, m, n which may be Constants or function of x, y, z. we have $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$ $\longrightarrow$

1. $(y-z)p + (z-x)q = x-y$.

This is a Lagrange's equation.

The A.E.'s are $\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$

Choosing 1,1,1 as multipliers,

Each ratio $= \frac{dx+dy+dz}{y-z+z-x+x-y}$

$= \frac{dx+dy+dz}{0}$

$\Rightarrow dx+dy+dz = 0$

Integrating, we get $\boxed{x+y+z = c_1}$

Choosing x, y, z as multipliers, we get

Each ratio $= \frac{x\,dx + y\,dy + z\,dz}{xy - xz + yz - yx + xz - yz}$

$= \frac{x\,dx + y\,dy + z\,dz}{0}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

Integrating, $x^2+y^2+z^2 = c_2$

$\therefore$ Solution of (1) is $f(x+y+z, x^2+y^2+z^2) = 0$.

2. $x(y-z)p + y(z-x)q = z(x-y)$

$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

Choosing 1,1,1 as multipliers,

Each ratio $= \frac{dx+dy+dz}{xy - xz + yz - xy + xz - ~~xy~~ yz}$

If it is possible to choose l, m, n $\ni$ $lP + mQ + nR = 0$, then $l\,dx + m\,dy + n\,dz = 0$. If $l\,dx + m\,dy + n\,dz$ is an exact differential, then on integration we get a solution $u = c_1$. Similarly if we can find another set of independent multipliers l', m', n', we get another independent solution $v = c_2$. The multipliers l, m, n are called Lagrangian multipliers

$$= \frac{dx + dy + dz}{0}$$

$\Rightarrow dx + dy + dz = 0.$

Integrating we get, $x + y + z = c_1$.

Again choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

$$\text{Each ratio} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y - z + z - x + x - y}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating we get $\log x + \log y + \log z = \log c_2$

$\log (xyz) = \log c_2$ $\boxed{c_2 = xyz}$

Solution of (1) is $f(x + y + z, xyz) = 0$.

3. $(mz - ny)p + (nx - lz)q = ly - mx$.

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}.$$

Choosing l, m, n as multipliers,

$$\text{Each ratio} = \frac{l\,dx + m\,dy + n\,dz}{l(mz - ny) + m(nx - lz) + n(ly - mx)}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

$\Rightarrow \quad lx + my + nz = c_1$

Choosing x, y, z as multipliers,

Each ratio $= \dfrac{x\,dx + y\,dy + n\,dz}{x(mz - ny) + y(nx - lz) + z(ly - mx)}$

$= \dfrac{x\,dx + y\,dy + z\,dz}{0}$

$\Rightarrow \quad x^2 + y^2 + z^2 = c_2$

$f(lx + my + nz,\ x^2 + y^2 + z^2) = 0.$

4. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2).$

~~Cho~~ $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{y(z^2 - x^2)} = \dfrac{dz}{z(x^2 - y^2)}.$

Choosing x, y, z as multipliers,

Each ratio $= \dfrac{x\,dx + y\,dy + z\,dz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)}$

$= \dfrac{x\,dx + y\,dy + z\,dz}{0}$

$\Rightarrow \quad x^2 + y^2 + z^2 = c_1$

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

Each ratio $= \dfrac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$

$= \dfrac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

$$xyz = c_2.$$

Solution is $f(x^2+y^2+z^2, xyz) = 0.$

5. $(a-x)p + (b-y)q = c-z.$

$$\frac{dx}{a-x} = \frac{dy}{b-y} = \frac{dz}{c-z}.$$

$$\frac{dx}{a-x} = \frac{dy}{b-y}$$

$$-\log(a-x) = -\log(b-y) + \log c_1$$

$$\log\left(\frac{b-y}{a-x}\right) = \log c_1 \qquad c_1 = \frac{b-y}{a-x}$$

$$\frac{dx}{a-x} = \frac{dz}{c-z} \qquad f\left(\frac{b-y}{a-x}, \frac{c-z}{a-x}\right) = 0.$$

$$-\log(a-x) = -\log(c-z) + \log c_2$$

$$\frac{c-z}{a-x} = c_2$$

6. $(y^2+z^2-x^2)p - 2xyq + 2xz = 0.$

$$+xy^2 + xz^2 - x^3$$
$$-2xy^2 - 2xz^2$$
$$-x^3 - xy^2 - xz^2$$

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

$$-x(y^2+z^2+x^2) - 2xy^2 - 2xz^2$$

$$\frac{dy}{y} = \frac{dz}{z} \qquad -x(x^2$$

$$\boxed{y/z = c_1}$$

Choosing x, y, z as multipliers,

Each ratio $= \dfrac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}$

$$\therefore \cancel{\frac{dy}{-2xy} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}}$$

$$\frac{dy}{-2xy} = \frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}$$

$$\frac{dy}{y} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

$$\log y = \log(x^2+y^2+z^2) + \log c_2$$

$$c_2 = \frac{y}{x^2+y^2+z^2}$$

7. $(3z-4y)p + (4x-2z)q = 2y-3x.$

$$\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x}.$$

Choosing x, y, z as multipliers,

Each ratio $= \dfrac{x\,dx + y\,dy + z\,dz}{3xz - 4xy + 4xy - 2yz + 2yz - 3xz}$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\Rightarrow \quad x^2+y^2+z^2 = c_1.$$

Equations in which the partial derivatives occurring are all of the same order (with degree 1 each) and the coefficients are constants are called homogeneous linear pde's with constant coefficients.

Choosing 2, 3, 4 as multipliers

Each ratio = $\dfrac{2dx+3dy+4dz}{6z-8y+12x-6z+8y-12x}$

$= \dfrac{2dx+3dy+4dz}{0}$

$\Rightarrow 2x+3y+4z = c_2$

$f(x^2+y^2+z^2, 2x+3y+4z) = 0$ //.

Homogeneous linear equation

PDE of higher order with constant coefficients.

we divide this into 2 groups

i) Homogeneous linear equations

ii) Nonhomogeneous linear equations.

Ex i) $2\dfrac{\partial^3 z}{\partial x^3} + 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} + 5\dfrac{\partial^3 z}{\partial y^3} = x^2+y$

partial derivatives occuring are all of the same order and the coefficients are constants.

Ex: 2) $\dfrac{\partial^3 z}{\partial x^3} + 2\dfrac{\partial^2 z}{\partial y^2} - 4\dfrac{\partial z}{\partial x} + z = x^2+y^2$

possesses derivatives which are not all of the same order but with constant ~~Const~~ coefficients.

Notations: $D = \dfrac{\partial}{\partial x}$; $D' = \dfrac{\partial}{\partial y}$

Equations in which the partial derivatives occurring are not of the same order and the coefficients are constants are called non-homogeneous linear p.d.e's with constant coefficients.

A homogeneous linear pde of nth order with Constants Coefficients is of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y).$$

To find C.F.:

The C.F. is the solution of the equ

$$\left(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n \right) z = 0.$$

Put $D = m$ & $D' = 1$.

The auxiliary equation is

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0$$

Let the roots of this equ be $m_1, m_2, \ldots, m_n$.

Case i) If the roots are distinct, the C.F is

$$z = f_1(y + m_1 x) + f_2(y + m_2 x) + \cdots + f_n(y + m_n x)$$

Case ii) If any 2 roots are equal, $m_1 = m_2 = m$ & others are different, then C.F. is

$$z = f_1(y + mx) + x f_2(y + mx) + f_3(y + m_3 x) + \cdots + f_n(y + m_n x)$$

Case iii) If three roots are equal, then C.F is

$$z = f_1(y + mx) + x f_2(y + mx) + x^2 f_3(y + mx) + \cdots + f_n(y + m_n x)$$

1. $(D^2 + 6DD' + 9D'^2)\, z = 0.$

The auxiliary equ is

$m^2 + 6m + 9 = 0$

$(m+3)^2 = 0 \Rightarrow m = -3, -3.$

$\therefore\ z = f_1(y \cancel{+} \underline{-}3x) + x\, f_2(y \cancel{+} \underline{-} 3x).$

2. $(D^4 - D'^4)\, z = 0$

The auxiliary equ is $m^4 - 1 = 0$

$(m^2)^2 - 1 = 0$

$(m^2+1)(m^2-1) = 0$

$m^2+1 = 0, \quad m^2 - 1 = 0$

$m^2 = -1 \quad m^2 = 1$

$m = \pm\sqrt{-1} \quad m = \pm 1$

$= \pm i, \ \pm 1$

$z = f_1(y + ix) + f_2(y - ix) + f_3(y+x) + f_4(y-x)$

3. $(2D^2 + 5DD' + 2D'^2)\, z = 0$

$2m^2 + 5m + 2 = 0$

$m = \dfrac{-5 \pm \sqrt{25-16}}{4}$

$= \dfrac{-5 \pm \sqrt{9}}{4} = \dfrac{-5 \pm 3}{4}$

$= \dfrac{-8}{4}, \dfrac{-2}{4}$

$= -2, -1/2$

$z = f_1(y - 2x) + f_2(y - 1/2\, x)$

4. $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial x^2 \partial y} - 8\frac{\partial^3 z}{\partial x \partial y^2} + 12\frac{\partial^3 z}{\partial y^3} = 0.$

$(D^3 - D^2D' - 8DD'^2 + 12D'^3)z = 0$

$m^3 - m^2 - 8m + 12 = 0.$

$m = 2, 2, -3.$

$z = f_1(y+2x) + x f_2(y+2x) + f_3(y-3x)$

$$\begin{array}{c|cccc} 2 & 1 & -1 & -8 & 12 \\ & 0 & 2 & 2 & -12 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

$x^2 + x - 6 = 0$

$\cancel{+6x+2x}$

$3x - 2 = -6$

$3 - 2 = 1$

$x = 2, 2, -3.$

5. $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

$(a^2D^2 - D'^2)y = 0$

$a^2m^2 - 1 = 0$

$a^2m^2 = 1$

$y = f_1(x-at) + f_2(x+at)$

$m^2 = \frac{1}{a^2} \quad m = +\frac{1}{a}, -1/a.$

$(D^2 + DD' - 2D'^2)z = 0$

$m^2 + m - 2 = 0$

$(m+2)(m-1) = 0 \Rightarrow m = 1, -2$

$z = f_1(y+x) + f_2(y - 2x)$

Type I $F(x,y) = e^{ax+by}$.

$$P.I = \frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}, \text{ if } a,b \neq 0.$$

Note: 1. $\frac{e^{ax+by}}{D - \frac{a}{b}D'} = x e^{ax+by}$ [where Denominator vanishes where D is replaced by a & D' by b]

2. $\frac{e^{ax+by}}{\left(D - \frac{a}{b}D'\right)^2} = \frac{x^2}{2!} e^{ax+by}$

1. $$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}.$$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & 0 & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$m = 2, 2.$

$(D^3 - 3D^2D' + 4D'^3)z = e^{x+2y}.$

The auxiliary equation is $m^3 - 3m^2 + 4 = 0.$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$$C.F = f_1(y-x) + f_2(y+2x) + x f_3(y+2x).$$

$$P.I. = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

$$= \frac{e^{x+2y}}{(1)^3 - 3(1)^2(2) + 4(2)^3}$$

$$= \frac{e^{x+2y}}{1-6+32} = \frac{e^{x+2y}}{27}$$

The complete solution of ① is

$z = C.F + P.I$

$$= f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{e^{x+2y}}{27}.$$

2. $(D^2 - 2DD')z = e^{2x}$.

The auxiliary equation is $m^2 - 2m = 0$ $\quad m(m-2)=0$
$\Rightarrow m = 0, 2$.

$C.F. = f_1(y) + f_2(y+2x)$.

$$P.I. = \frac{1}{D^2 - 2DD'} e^{2x}$$

$$= \frac{e^{2x}}{4 - 2(2)(0)} = \frac{e^{2x}}{4}.$$

$$z = f_1(y) + f_2(y+2x) + \frac{e^{2x}}{4}.$$

3. $(D^2 - 2DD' + D'^2)z = e^{x+2y}$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1, 1.$

$C.F. = f_1(y+x) + x f_2(y+x)$

$$P.I. = \frac{1}{D^2 - 2DD' + D'^2} e^{x+2y}$$

$$= \frac{e^{x+2y}}{1 - 2(1)(2) + (2)^2} = \frac{e^{x+2y}}{1-4+4} = e^{x+2y}$$

$$z = f_1(y+x) + x f_2(y+x) + e^{x+2y}.$$

4. $(D^4 - D'^4)z = e^{x+y}$

$m^4 - 1 = 0$

$(m^2)^2 - 1 = 0$

$(m^2-1)(m^2+1) = 0 \quad m = -1, 1, i, -i$

$$C.F. = f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix).$$

$$P.I. = \frac{1}{D^4 - D'^4} e^{x+y}$$

$$\frac{1}{4D^3} x e^{x+y} \quad \frac{1}{4(D^3)} \quad \frac{x}{4(1)} e^{x+y}$$

$$= \frac{1}{(D^2)^2 - (D'^2)^2} e^{x+y}$$

$$= \frac{1}{(D^2-D'^2)(D^2+D'^2)} e^{x+y}$$

$$= \frac{1}{(D-D')(D+D')(D^2+D'^2)} e^{x+y}$$

$$= \frac{1}{(D-D')(1+1)(1+1)} e^{x+y}$$

$$= \frac{1}{4}\frac{1}{(D-D')} e^{x+y} = \frac{x}{4} e^{x+y}$$

5. $(D^2 - 4DD' + 4D'^2) z = e^{2x+y}$.

The auxiliary eqn is $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0$

$m = 2, 2.$

$$C.F. = f_1(y+2x) + x f_2(y+2x)$$

$$P.I. = \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$$

$$\frac{x}{2D - 4D'} \quad 2$$

$$2(2) - 4(1) \quad 4 - 8 + 16$$

$$4 - 4(2) + 4(4)$$

$$4 - 8 + 16$$

$$= \frac{1}{(D-2D')^2} e^{2x+y}$$

$$\frac{x^2}{2}$$

$$= \frac{x^2}{2!} e^{2x+y} = \frac{x^2}{2} e^{2x+y}$$

$$\therefore z = f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}.$$

Type II $F(x,y) = x^r y^s$.

$$P.I. = \frac{1}{f(D,D')} x^r y^s = [f(D,D')]^{-1} x^r y^s.$$

Note: ~~Ref~~ In $x^r y^s$, if $r<s$, write $f(D,D')$ as $f\left(\frac{D}{D'}\right)$ & if ~~r~~ $s<r$, write $f(D,D')$ as $f\left(\frac{D'}{D}\right)$.

1. $(D^2 - 2DD')z = \cancel{e^{2x}} + x^3 y + e^{2x}$.

$m^2 - 2m = 0 \quad m(m-2) = 0 \Rightarrow m = 0, 2.$

C.F. $= f_1(y) + f_2(y+2x)$.

$$P.I. = \frac{1}{D^2 - 2DD'} x^3 y$$

$$= \frac{1}{D^2\left(1 - \frac{2D'}{D}\right)} x^3 y$$

$$= \frac{1}{D^2}\left[1 - \frac{2D'}{D}\right]^{-1}(x^3 y)$$

$$= \frac{1}{D^2}\left[1 + \frac{2D'}{D} + \left(\frac{2D'}{D}\right)^2 + \cdots\right](x^3 y)$$

$$= \frac{1}{D^2}\left[x^3 y + \frac{2x^3}{D}\right]$$

$$= \frac{x^5}{4.5} y + \frac{2x^6}{4.5.6} = \frac{x^5}{20} y + \frac{x^6}{60}$$

$$P.I_2 = \frac{1}{D^2 - 2DD'} e^{2x} = \frac{e^{2x}}{4}$$

$$z = f_1(y) + f_2(y+2x) + \frac{x^5 y}{20} + \frac{x^6}{60}.$$

2. $(D^2 + 4DD' - 5D'^2) z = x + y^2 + \pi$

$m^2 + 4m - 5 = 0$

$(m+5)(m-1) = 0 \Rightarrow m = 1, -5.$

$C.F. = f_1(y+x) + f_2(y-5x).$

$$P.I. = \frac{1}{D^2 + 4DD' - 5D'^2} x + y^2 + \pi$$

$$= \frac{1}{D^2\left[1 + \left(\frac{4D'}{D} - \frac{5D'^2}{D^2}\right)\right]}(x + y^2 + \pi)$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{4D'}{D} - \frac{5D'^2}{D^2}\right)\right]^{-1}(x + y^2 + \pi)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{4D'}{D} - \frac{5D'^2}{D^2}\right) + \left(\frac{4D'}{D} - \frac{5D'^2}{D^2}\right)^2 - \cdots\right](x + y^2 + \pi)$$

$$= \frac{1}{D^2}\left[1 - \frac{4D'}{D} + \frac{5D'^2}{D^2} + \frac{16D'^2}{D^2} - \cdots\right]$$

$$= \frac{1}{D^2}\left[(x + y^2 + \pi) - \frac{4}{D}(2y) + \frac{10}{D^2} + \frac{16 \cdot 2}{D^2}\right]$$

$$= \frac{1}{D^2}(x + y^2 + \pi) - \frac{4}{D^3}(2y) + \frac{42}{D^4}$$

$$= \frac{x^3}{6} + (y^2 + \pi)\frac{x^2}{2} - 8y\frac{x^3}{6} + 42 \cdot \frac{x^4}{24}$$

$$= \frac{x^3}{6} + \frac{x^2}{2}(y^2+7) - \frac{4}{3}x^3y + \frac{7}{4}x^4.$$

$$z = f_1(y+x) + f_2(y-5x) + \frac{x^3}{6} + \frac{x^2}{2}(y^2+7) - \frac{4}{3}x^3y + \frac{7}{4}x^4.$$

3. $(D^2 + 3DD' + 2D'^2)z = x+y$

$m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0 \Rightarrow m = -1, -2.$

C.F. $= f_1(y-x) + f_2(y-2x).$

$$\text{P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2}\, x+y$$

$$= \frac{1}{D^2\left(1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right)}\, x+y$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right]^{-1}(x+y)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right) + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)^2 - \cdots\right](x+y)$$

$$= \frac{1}{D^2}\left[(x+y) - \frac{3}{D}(1) - 0\right]$$

$$= \frac{x^3}{6} + \frac{yx^2}{2} - \frac{3x^3}{6} = \frac{x^3}{6} + \frac{x^2y}{2} - \frac{x^3}{2}\ \ {-\frac{2x^3}{6}}$$

$$= \frac{x^2y}{2} - \frac{x^3}{3}$$

$$z = f_1(y-x) + f_2(y-2x) + \frac{x^2y}{2} - \frac{x^3}{3}.$$

4. $(D^2 - 6DD' + 9D'^2)z = 6x + 2y.$

$m^2 - 6m + 9 = 0 \quad (m-3)^2 = 0 \quad m = 3, 3.$

C.F. $= f_1(y+3x) + x f_2(y+3x)$

P.I. $= \dfrac{1}{D^2 - 6DD' + 9D'^2} \, 6x + 2y$

$$= \frac{1}{D^2\left[1 - \left(\frac{6D'}{D} - \frac{9D'^2}{D^2}\right)\right]} \, 6x + 2y$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{6D'}{D} - \frac{9D'^2}{D^2}\right)\right]^{-1} (6x+2y)$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{6D'}{D} - \frac{9D'^2}{D^2}\right) + \left(\frac{6D'}{D} - \frac{9D'^2}{D^2}\right)^2 + \cdots\right](6x+2y)$$

$$= \frac{1}{D^2}\left[6x + 2y + \frac{6}{D}(2)\right]$$

$$= 6 \cdot \frac{x^3}{6} + 2y\,\frac{x^2}{2} + 12 \cdot \frac{x^3}{6}$$

$$= x^2y + 3x^3$$

5. $(D^2 + 2DD' + D'^2)z = x^2y$

6. $(D^2 - DD' - 6D'^2)z = xy$

7. $(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$

$f_1(y) + x f_2(y) + f_3(y+2x) + \frac{1}{4}e^{2x} + \frac{1}{20}x^5y + \frac{1}{60}x^6$

$z = f_1(y-x) + x f_2(y-x) + \dfrac{x^4y}{12} - \dfrac{x^5}{30}$

(6) $z = f_1(y-2x) + f_2(y+3x) + \dfrac{x^4}{24} + \dfrac{x^3y}{6}$

Note: 1. $\dfrac{1}{D^2-\frac{a^2}{b^2}D'^2}\cos(ax+by) = \dfrac{x}{2a}\sin(ax+by)$

2. $\dfrac{1}{D^2-\frac{a^2}{b^2}D'^2}\sin(ax+by) = \dfrac{-x}{2a}\cos(ax+by)$

Type III

$$P.I. = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

$$= \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

1. $(D^3 - 7DD'^2 - 6D'^3)z = x^2y + \sin(x+2y)$.

$m^3 - 7m - 6 = 0$

$\cancel{-6x}$ $m = -1, -2, 3.$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & 0 & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$x^2 - x - 6 = 0$

$-3 \times 2 = -3$

$-3+2 = -1$

$(x-3)(x+2)$

C.F. $= f_1(y-x) + f_2(y-2x) + f_3(y+3x)$.

$$P.I._1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}(x^2y)$$

$$= \frac{1}{D^3\left[1 - \left(\frac{7D'^2}{D^2} - \frac{6D'^3}{D^3}\right)\right]}(x^2y)$$

$$= \frac{1}{D^3}\left[1 - \left(\frac{7D'^2}{D^2} - \frac{6D'^3}{D^3}\right)\right]^{-1}(x^2y)$$

$$= \frac{1}{D^3}\left[\cancel{x^2y - \frac{1}{D^2}(0)}\ 1 + \left(\frac{7D'^2}{D^2} - \frac{6D'^3}{D^3}\right) + \cdots\right](x^2y)$$

$$= \frac{1}{D^3}\left[x^2y\right] = \frac{yx^5}{3.4.5} = \frac{x^5y}{60}$$

$$P.I_2 = \frac{1}{D^3 - 4DD'^2 - 6D'^3} \sin(x+2y)$$

Replacing $D^2 = -a^2 = -1$, $D'^2 = -b^2 = -4$, $DD' = -ab$

$$= \frac{1}{-D - 4D(-4) - 6D'(-4)} \sin(x+2y)$$

$$= \frac{1}{-D + 28D + 24D'} \sin(x+2y)$$

$$= \frac{1}{27D + 24D'} \sin(x+2y)$$

$$= \frac{1}{3} \frac{1}{9D + 8D'} \sin(x+2y)$$

$$= \frac{1}{3} \frac{9D - 8D'}{81D^2 - 64D'^2} \sin(x+2y)$$

$$\begin{array}{r} 256 \\ -81 \\ \hline 175 \times 3 \\ \hline 525 \end{array}$$

$$= \frac{1}{3} \frac{9D - 8D'}{81(-1) - 64(-4)} \sin(x+2y)$$

$$= \frac{1}{3} \frac{1}{-81 + 256} (9D - 8D') \sin(x+2y)$$

$$= \frac{1}{525} \left[9.D[\sin(x+2y)] - 8D'[\sin(x+2y)] \right]$$

$$= \frac{1}{525} \left[9.\cos(x+2y).1 - 8\cos(x+2y).2 \right]$$

$$= \frac{1}{525} \left(-7\cos(x+2y) \right)$$

$$= -\frac{1}{75} \cos(x+2y)$$

(1) $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

(3) $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

(4) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

(2) $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

2. $(D^2 - DD')z = \sin x \cos 2y$

$m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m = 0, 1$

$C.F. = f_1(y) + f_2(y+x)$

$P.I. = \frac{1}{D^2 - DD'} \sin x \cos 2y$

$= \frac{1}{D^2\left[1 - \frac{D'D}{D}\right]} \frac{1}{2}[\sin(x+2y) + \sin(x-2y)]$

$= \frac{1}{2D^2}\left[1 - \frac{D'}{D}\right]^{-1}\{\sin(x+2y) + \sin(x-2y)\}$

$= \frac{1}{2D^2}\left\{1 + \frac{D'}{D} + \left(\frac{D'}{D}\right)^2 + \cdots\right\}\sin(x+2y)$

$+ \left\{1 + \frac{D'}{D} + \left(\frac{D'^2}{D^2}\right) + \cdots\right\}(\sin(x-2y))$

$\frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$

$a = 1 \quad b = 2$

$-a^2 = -1, \; -b^2 = -4$

$DD' = -(ab) = -(1.2) = -2$

$= \frac{1}{2D^2}\left\{\sin(x+2y) + \frac{1}{D}\cos(x+2y).2\right.$

$a = 1, b = 2$

$D^2 = -1 \qquad -1-(-2) \quad -1+2 \quad -1-(+2) \quad -1-3$

$= \frac{1}{2} \cdot \frac{1}{D^2 - DD'}\sin(x+2y) \quad -ab$

$a = 1, \; b = -2$

$+ \frac{1}{2}\frac{1}{D^2 - DD'}\sin(x-2y)$

$D^2 = -a^2 = -1$

$D'^2 = -b^2 = -(-2)^2$

$= -4$

$= \frac{1}{2}\frac{1}{-1-(-2)}\sin(x+2y) + \frac{1}{2}\frac{1}{-1-(+2)}\sin(x$

$DD' = -ab$

$= -(1.(-2))$

$= 2$

$= \frac{1}{2}\frac{1}{(-1+2)}\sin(x+2y) + \frac{1}{2}\cdot\frac{1}{-3}\sin(x-2y)$

$\frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$

3. $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x-y) + x^2 + xy^2 + y^3$.

$z = f_1(y-x) + f_2(y+3x) + f_3(y-2x) + \frac{x}{4}\cos(y-x)$
$+ \frac{x^5}{60} + \frac{x^4y^2}{24} + \frac{x^3y^3}{6} + \frac{7x^5y}{20} + \frac{5x^6}{72}$.

4. $(D^2 - 2DD' + D'^2)z = \cos(x-3y)$

$m^2 - 2m + 1 = 0$

$m = 1, 1.$

C.F. $= f_1(y+x) + f_2(y+x)$.

P.I. $= \frac{1}{D^2 - 2DD' + D'^2}\cos(x-3y)$

$= \frac{1}{-1-2(+1.3)-9}\cos(x-3y)$

$= \frac{1}{-1\cancel{+}6-9}\cos(x-3y)$

$= \frac{-1}{16}\cos(x-3y)$.

$DD' = -(ab) = -(1.(-3)) = 3$

$a = 1, \quad b = -3$

$D^2 = -a^2 = -(1)^2 = 1$

$D'^2 = -(b^2) = -(-3)^2 = -9$

5. $(D^2 - 6DD' + 5D'^2)z = e^x \sinh y + xy$ Type II

$m^2 - 6m + 5 = 0$

$(m-5)(m-1) = 0$

$m = 1, 5$

C.F $= f_1(y+x) + f_2(y+5x)$

$2y)$

$D^2 - 6DD' + 5D'^2$
$D(D-6D') + 5D'^2$
$(D-\frac{3}{2}D')($
$\to X$

$$P.I. = \frac{1}{D^2-6DD'+5D'^2} e^{x} \sinh y$$

$$= \frac{1}{D^2-6DD'+5D'^2}\left[e^{x}\left(\frac{e^{y}-e^{-y}}{2}\right)\right]$$

$$= \left[\frac{e^{x+y}}{2} - \frac{e^{x-y}}{2}\right]$$

$$= \frac{1}{2}\left\{\frac{1}{D^2-6DD'+5D'^2}e^{x+y} - \frac{1}{D^2-6DD'+5D'^2}e^{x-y}\right\}$$

$$= \frac{1}{2}\left\{\frac{x\,e^{x+y}}{2D-6D'} - \frac{1}{1-6(1)(-1)+5(-1)^2}\right\}$$

$1+6+5$

$$= \frac{1}{2}\left\{\frac{x}{-4}e^{x+y} - \frac{1}{12}e^{x-y}\right\}.$$

$$P.I. = \frac{1}{D^2-6DD'+5D'^2}\,xy$$

$$= \frac{1}{D^2\left(1-\left(\frac{6D'}{D}-\frac{5D'^2}{D^2}\right)\right)}xy$$

$\frac{yx^3}{6} + \frac{x^4}{4}.$

$$= \frac{1}{D^2}\left[1-\left(\frac{6D'}{D}-\frac{5D'^2}{D^2}\right)\right]^{-1}(xy)$$

$$= \frac{1}{D^2}\left[1+\left(\frac{6D'}{D}-\frac{5D'^2}{D^2}\right)+\left(\frac{6D'}{D}-\frac{5D'^2}{D^2}\right)^2+\cdots\right](xy)$$

$$= \frac{1}{D^2}\left[xy + \frac{6x}{D}\right]$$

$$= \frac{1}{D^2}(xy) + \frac{1}{D^3}6x$$

$$= \frac{yx^3}{6} + \frac{6\cdot x^4}{24} = \frac{yx^3}{6} + \frac{x^4}{4} \,//.$$

$$\frac{1}{2}\left[\frac{1}{3}\cos x\cos 2y + \sin x \sin 2y - \cos x \cos 2y + \sin x \sin 2y\right]$$

5. $(D^2 - DD')z = \sin x \sin 2y.$

Ans $m^2 - m = 0 \Rightarrow m = 0, 1.$

$C.F. = f_1(y) + f_2(y+x).$

$$P.I. = \frac{1}{D^2 - DD'}\frac{1}{2}\left[\cos(x-2y) - \cos(x+2y)\right]$$

$D^2 = -a^2 = -1$

$D'^2 = -b^2 = -(-2)^2 = -4$

$DD' = -(ab) = -(-2) = 2$

$$= \frac{1}{2}\left\{\frac{1}{D^2 - DD'}\cos(x-2y) - \frac{1}{D^2 - DD'}\cos(x+2y)\right]$$

$a = 1, b = 2,$

$DD' = -(ab) = -(1.2)$

$$= \frac{1}{2}\left\{\frac{1}{-1-2}\cos(x-2y) - \frac{1}{-1-(-2)}\cos(x+2y)\right]$$

$DD' = -(1.2) = -2$

$-1+2$

$$= \frac{1}{2}\left\{\frac{1}{-3}\cos(x-2y) - \frac{1}{1}\cos(x+2y)\right]$$

$$= -\frac{1}{6}\cos(x-2y) - \frac{1}{2}\cos(x+2y)$$

$(D^2+3DD'-4D'^2)z = \sin y$.

The A.E. is $m^2+3m-4=0$

$(m+4)(m-1)=0 \Rightarrow m=1,-4$.

C.F. $= f_1(y+x)+f_2(y-4x)$.

P.I. $= \dfrac{1}{D^2+3DD'-4D'^2}\sin y$

$= \dfrac{1}{D^2+3DD'-4D'^2}\sin(0x+y)$.

Replacing $D^2=0$, $DD'=0$, $D'^2=-b^2=-1$

P.I. $= \dfrac{1}{-4(-1)}\sin(0x+y)$

$= \dfrac{1}{4}\sin y$.

$z = f_1(y+x)+f_2(y-4x)+\dfrac{1}{4}\sin y$.

Ans $(D^3-4D^2D'+4DD'^2)z = 6\sin(3x+6y)$.

The A.E is $m^3-4m^2+4m=0$

$m(m^2-4m+4)=0$

$\Rightarrow m=0,\ m=2,2$.

C.F. $= f_1(y)+f_2(y+2x)+x f_3(y+2x)$

P.I. $= \dfrac{1}{D^3-4D^2D'+4DD'^2}\, 6\sin(3x+6y)$.

$D^2 = -a^2 = -(3^2) = -9$. $D'^2 = -b^2 = -(6^2) = -36$.

$\cancel{DD' = -(ab) = -(3.6) = -18}$

$$\frac{2}{45}\left[\cos(3x+6y)\right]$$

$$P.I. = \cancel{\frac{1}{-9D-4D(-18)+4D(-36)}} \cancel{\cos(3x+6y)}$$

$$= \frac{1}{-9D+36D'\,\cancel{\phi}-144D}$$

1. Find P.I.

$(D^2-4D'^2)z = \sin(2x+y)$

$$P.I. = \frac{1}{D^2-4D'^2}\sin(2x+y) \qquad D^2=-4, \quad D'^2=-1$$

$$= \frac{-x}{2(2)}\cos(2x+y)$$

2. $(16D^4-D'^4)z = \cos(x+2y)$ $\qquad 4m^2+1=0 \Rightarrow 4m^2=-1$

The A.E. is $16m^4-1=0$ $\quad (4m^2)^2-1=0 \Rightarrow 4m^2=1 \Rightarrow m=\pm 1/2, \; m=\pm i/2$

$$C.F. = f_1(y+\tfrac{1}{2}x)+f_2(y-\tfrac{1}{2}x)+f_3(y+\tfrac{i}{2}x)+f_4(y-\tfrac{i}{2})x.$$

$$P.I. = \frac{1}{(4D^2)^2-(D'^2)^2}\cos(x+2y)$$

$$= \frac{1}{(4D^2-D'^2)(4D^2+D'^2)}\cos(x+2y)$$

$$= \frac{1}{(4D^2-D'^2)(4(-1)+(-4))}\cos(x+2y)$$

$$= \frac{1}{-8}\,\frac{1}{4D^2-D'^2}\cos(x+2y)$$

$$= \frac{1}{-8(4)}\,\frac{1}{D^2-\frac{D'^2}{4}}\cos(x+2y) \qquad \frac{1}{D^2-\frac{a^2}{b^2}D'^2} = \cos(ax+by)$$

$$= \frac{x}{2a}\sin(ax+by)$$

$$= -\frac{1}{32}\,\frac{x}{2}\sin(x+2y)$$

Solve $(D^3+D^2D'-DD'^2-D'^3)z = 3\sin(x+y)$

The A.E. is $m^3+m^2-m-1=0$.

$$\begin{array}{c|cccc} 1 & 1 & 1 & -1 & -1 \\ & 0 & 1 & 2 & +1 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$m = 1, -1, -1$.

$x^2+2x+1=0$

$1\times1=1$

$1+1=2$

C.F. $= f_1(y+x) + f_2(y-x) + x f_3(y-x)$.

P.I. $= \dfrac{1}{D^3+D^2D'-DD'^2-D'^3} 3\sin(x+y)$

Replacing $D^2=-1$, $D'^2=-1$,

P.I. $= \dfrac{1}{-D-D'+D+D'} 3\sin(x+y)$

Here Denominator is zero.

$Dr = D^3+D^2D'-DD'^2-D'^3$

$= D^2(D+D') - D'^2(D+D')$

$= (D+D')[D^2-D'^2] = (D+D)(D-D')(D+D')$.

P.I. $= \dfrac{1}{(D-D')(D+D')(D+D')} 3.\sin(x+y)$ $\qquad y=$

Here $m=1$

$= 3\dfrac{1}{(D-D')(D+D')}\int \sin(x+a+x)\,dx.$ $\qquad y = c+x$

$= 3\dfrac{1}{(D-D')(D+D')}\int \sin(2x+a)\,dx.$

$= 3\dfrac{1}{(D-D')(D+D')}\left(-\dfrac{\cos(2x+a)}{2}\right)$

$y=c_1+x$

$= -\dfrac{3}{2}\dfrac{1}{(D-D')(D+D')}\cos(y+x)$ $\qquad y = c_1+x$

$= -\dfrac{3}{2}\dfrac{1}{D-D'}\int \cos(c_1+2x)\,dx = \dfrac{-3}{4}\dfrac{1}{(D-D')}\left(\dfrac{\sin(2x+c_1)}{2}\right)$

$y = c_2-x$ $\qquad = -\dfrac{3}{4}\int \sin(x+c_2-x)\,dx = -\dfrac{3}{4}\int \sin c_2\,dx$

Type I $\quad \dfrac{1}{f(D,D')} e^{ax+by}$

$$P.I. = \frac{1}{f(a,b)} e^{ax+by}, \text{ if } f(a,b) \neq 0.$$

Type 2: $\quad F(x,y) = x^r y^s.$

$$P.I. = \frac{1}{f(D,D')} x^r y^s = [f(D,D')]^{-1} x^r y^s.$$

where $[f(D,D')]^{-1}$ is to be expanded in powers of D, D'.

Type 3: $\quad \dfrac{1}{f(D^2, DD', D'^2)} \sin(ax+by)$ or $\cos(ax+by)$

$$P.I. = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

Type ~~4~~ 5: $\quad F(x,y) = e^{ax+by} \phi(x,y)$

$$P.I. = e^{ax+by} \frac{1}{f(D+a, D'+b)} \phi(x,y).$$

Type ~~5~~ 4: $\quad \dfrac{1}{f(D^2, D'^2)} \sin ax \sin by$ or $\cos ax \cos by$

$$P.I. = \frac{1}{f(-a^2, -b^2)} \sin ax \sin by \text{ if } Dr \neq 0.$$

~~Type 6 4: $\dfrac{1}{f(D^2, D'^2)} \cos ax \cos by$~~

~~$$P.I. = \frac{1}{f(-a^2, -b^2)} \cos ax \cos by. \text{ if } Dr \neq 0.$$~~

$$= -\frac{3}{\cancel{8}\,4} \frac{1}{(D-D')} \sin(2x+y-x) = -\frac{3}{\cancel{8}\,4} \frac{1}{(D-D')} \sin(y+x)$$

$$= -\frac{3}{\cancel{8}} \sin c_2 \cdot x = -\frac{3x}{\cancel{8}\,4} \sin(x+y).$$

Type 6 next page

General then rule to find $\frac{1}{D-mD'} F(x,y)$.

First change y to $c - mx$ in $F(x,y)$. Integrate it with respect x treating y as a constant and then in the resulting integral change c to $y+mx$.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x.$$

1\. Solve: $(D^2+DD'-6D'^2)z = y\cos x$. $\qquad D(D+D') - 6D'^2$

A.E. is m^2+m-6

$(m+3)(m-2)=0 \Rightarrow m = 2, -3.$

3×-2

$3-2$

C.F. $= f_1(y+2x) + f_2(y-3x)$

P.I. $= \frac{1}{D^2+DD'-6D'^2} y\cos x$. $\qquad (D-2D')(D+3D')$

$= \frac{1}{(D-2D')(D+3D')} y\cos x$ $\qquad y = a_1 + 3x$

$\rightarrow D-(-3)D' \rightarrow m=-3$, $y = a - mx = a-(-3)x = a+3x$

$= \frac{1}{(D-2D')} \int (a_1+3x)\cos x\,dx$, $\quad y = a_1+3x$

$= \frac{1}{D-2D'}\left[(a_1+3x)\sin x - 3(-\cos x)\right]$

$= \frac{1}{D-2D'}\left[y\sin x + 3\cos x\right]$ $\qquad a_2 - 2x$

$= \int [(a_2-2x)\sin x + 3\cos x]\,dx$

$= (a-2x)(-\cos x) - (-2)(-\sin x) + 3\sin x$

$= \cancel{-a\cos x + 2x\cos x} - y\cos x - 2\sin x + 3\sin x$

$= -y\cos x + \sin x.$

2. $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y.$ $\quad |\times 1 = 1$ $\quad 1+1$

A.E. is $\quad m^2 + 2m + 1 = 0$

$(m+1)^2 = 0 \Rightarrow m = -1, -1$

C.F. $= f_1(y-x) + x f_2(y-x).$

$y = a + x$

P.I. $= \dfrac{1}{(D+D')(D+D') \to D-(-1)D'} 2\cos y - x\sin y.$

$y = a - mx$
$= a - (-1)x$
$= a + x.$

$= \dfrac{1}{D+D'} \int [2\cos(a+x) - x\sin(a+x)]\,dx$

$= \dfrac{1}{D+D'} \left\{ 2\sin(a+x) - \left[x.\left(-\cos(a+x)\right) - 1\left(-\sin(a+x)\right)\right]\right\}$

$= \dfrac{1}{D+D'} \left\{ 2\sin(a+x) + x\cos(a+x) - \sin(a+x) \right]$

$= \dfrac{1}{D+D'} \left\{ x\cos(a+x) + \sin(a+x) \right\}$

$= \dfrac{1}{D+D'} \left\{ x\cos(y-x+x) + \sin(\overset{y-x}{a}+x) \right\}$

$= \dfrac{1}{D+D'} \left\{ x\cos y + \sin y \right\},$

$= \int [x\cos(a+x) + \sin(a+x)]\,dx$

$y = a + x.$

$= x \sin(a+x) - 1\cdot(-\cos(a+x)) - \cos(a+x)$

$= x \sin(a+x) + \cos(a+x) - \cos(a+x)$

where $y = a+x$

$= x \sin y$

$\therefore$ G.S. of (1) is $z = f_1(y-x) + x f_2(y-x) + x \sin y$.

e^{ax+by} $\phi(x,y)$

Type 5

1. Solve: $(D^2 - 3DD' + 2D'^2) z = (2+4x) e^{x+2y}$.

The A.E. is $m^2 - 3m + 2 = 0$

$\Rightarrow m = 1, 2.$

$1 - 3(1)(2) + 2(4)$
$1 - 6 + 8 = 2$
$-1 \times -2 = 2$

C.F. $= f_1(y+x) + f_2(y+2x)$

P.I. $= \dfrac{1}{D^2 - 3DD' + 2D'^2} e^{x+2y} (2+4x)$

$= \cancel{e^{x+2y}} \dfrac{1}{(D-D')(D-2D')} e^{x+2y} (2+4x)$

$\dfrac{1}{(D+1)^2 - 3(D+1)(D'+2) + 2(D'+2)^2}$

Replace D by $D+a$, D' by $D'+b$

$= e^{x+2y} \dfrac{1}{[(D+1) - (D'+2)][(D+1) - 2(D'+2)]} (2+4x)$

$a=1$ $b=2$

then solve (or)

$D^2 + 5D' + 2D'^2 - 3DD' - (1) + 3$

$= e^{x+2y} \dfrac{2+4x}{(D - D' - 1)(D - 2D' - 3)}$

$\dfrac{-1}{(2+4x)}$

$3\left[1 + \left(\dfrac{\quad}{3}\right)\right.$

$= e^{x+2y} \dfrac{1}{-1(1-(D-D'))(-3)\left(1 + \dfrac{D-2D'}{-3}\right)} (2+4x)$

$\left[4x + \dfrac{22}{3}\right]$

$= \dfrac{e^{x+2y}}{3}$

$$= \frac{e^{x+2y}}{3}\left\{\left[1-(D-D')\right]^{-1}\left[1-\left(\frac{D-2D'}{3}\right)\right]^{-1}(2+4x)\right\}$$

$$= \frac{e^{x+2y}}{3}\left\{\left[1+(D-D')+(D-D')^2+\cdots\right]\left[1+\left(\frac{D-2D'}{3}\right)+\left(\frac{D-2D'}{3}\right)^2+\cdots\cdots\right](2+4x)\right\}$$

$$= \frac{e^{x+2y}}{3}\left[1+(D-D')+\left(\frac{D-2D'}{3}\right)+\cdots\cdots\cdots\right](2+4x)$$

$$= \frac{e^{x+2y}}{3}\left[1+\frac{3D-3D'+D-2D'}{3}+\cdots\cdots\cdots\right](2+4x)$$

$$= \frac{e^{x+2y}}{3}\left[1+\frac{4D-5D'}{3}+\cdots\cdots\right](2+4x)$$

$$= \frac{e^{x+2y}}{3}\left[(2+4x)+\frac{4}{3}D(2+4x)-\frac{5}{3}D'(2+4x)+\cdots\right]$$

$$= \frac{e^{x+2y}}{3}\left[2+4x+\frac{4}{3}(4)\right]$$

$$= \frac{e^{x+2y}}{3}\left[2+4x+\frac{16}{3}\right] = \frac{e^{x+2y}}{3}\left(4x+\frac{22}{3}\right)$$

$$z = f_1(y+x)+f_2(y+2x)+\frac{e^{x+2y}}{3}\left(4x+\frac{22}{3}\right)$$

Type A

(1) $(D^2 + D'^2) z = \cos 2x \cos 2y$

$f_1(y+ix) + f_2(y-ix) \quad -\frac{1}{8} \cos 2x \cos 2y$

Solve: $(D^2 - 4D'^2) z = \cos 2x \cos 3y$

The auxiliary eqn is $m^2 - 4 = 0 \Rightarrow m^2 = 4 \quad m = \pm 2$.

$C.F. = f_1(y+2x) + f_2(y-2x)$

$$P.I. = \frac{1}{D^2 - 4D'^2} \cos 2x \cos 3y.$$

$D^2 = -a^2, \; D'^2 = -b^2$

$= -4 \qquad = -9$

$$= \frac{1}{-4-4(-9)} \cos 2x \cos 3y$$

$$= \frac{1}{-4+36} \cos 2x \cos 3y$$

$$= \frac{1}{32} \cos 2x \cos 3y.$$

$$z = f_1(y+2x) + f_2(y-2x) + \frac{1}{32} \cos 2x \cos 3y$$

1. Form the pde by eliminating f from $z = f\left(\frac{y}{x}\right)$.
2. " " arbitrary functions from $z = x f(y/x) + y \phi(x)$
3. Solve $\frac{\partial z}{\partial x} = 6x + 3y, \; \frac{\partial z}{\partial y} = 3x - 4y$
4. Eliminate the arbitrary fun from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

$px^2 + qy = 2y^2$

5. Solve: $z = px + qy + \frac{p}{q} - p$
6. Solve: $p \cot x + q \cot y = \cot z$
7. Solve: $yp = 2yx + \log q$.

$p = 2x + \frac{1}{y}\log q$

$p - 2x = \frac{1}{y}\log q = a$

$p = 2x + a \qquad \frac{1}{y}\log q = a \quad . \quad \log q = ay \Rightarrow q = e^{ay}$

$dz = (2x+a)\,dx + e^{ay}\,dy$

$z = x^2 + ax + \frac{e^{ay}}{a} + b$

8. Non homogeneous linear factors.

The Complete solution = C.F. + P.I.

The P.I is found by the same methods as in the case of homogeneous linear equations.

To find C.F.

If $f(D, D')$ contains nonlinear factors in D, D'.

Assume a trial solution, $z = Ce^{hx+ky}$ where c, h, k are Constants

have to ... Solution $f(D,D')z=0$

put $D = h$ & $D' = k$. $\qquad Ce^{hx+ky} f(h,k) = 0$

~~If so~~ find h or k. $\qquad \Rightarrow c \neq 0,\ e^{hx+ky} \neq 0$

Hence $f(h,k) = 0$.

~~Consider $(D - mD' - c)z = Q$.~~

If $f(D, D')$ is of degree r in D', then $f(h,k) = 0$ will be of rth degree in k. $\qquad CF = \sum C_s e^{hx + f_s(h)y}$

$C.F. = \sum C_1 e^{hx + f_1(h)y} + \sum C_2 e^{hx + f_2(h)y} + \ldots\ldots + \sum C_r e^{hx + f_r(h)y}$ $\quad k = f_s(h)$.

k in terms of h

$C.F = \sum C_1 e^{f_1(k)x + ky} + \sum C_2 e^{f_2(k)x + ky} + \ldots\ldots + \sum C_r e^{f_r(k)x + ky}$

h in terms of k.

If $f(D,D')$ is factorisable, then factorise $f(D,D')$ & write this in the form $(D-m_1D'-c_1)(D-m_2D'-c_2)\cdots\cdots(D-m_nD'-c_n)z=0$

roots $m_1, m_2, \ldots, m_n$ are distincts then

Case ① If $(D-m_1D'-c_1)(D-m_2D'-c_2)\cdots$

$(D-m_nD'-c_n)z=0$ is ~~the PDE~~

then C.F $z = e^{c_1x}\phi_1(y+m_1x) + e^{c_2x}\phi_2(y+m_2x) + \cdots + e^{c_nx}\phi_n(y+m_nx)$

Case②

In case of repeated factors,

$(D-mD'-c)^r z=0,$

CF $z = e^{cx}\phi_1(y+mx) + xe^{cx}\phi_2(y+mx) + x^2e^{cx}\phi_3(y+mx) + \cdots + x^{r-1}e^{cx}\phi_r(y+mx).$

1. ⑥ $(D^2+DD'+D'-1)z=5e^x$.

Assume $z = ce^{hx+ky}$ to be a trial solution of $(D^2+DD'+D'-1)z=0$.

Put $D=h$ $D'=k$

$\therefore h^2+hk+(k-1)=0.$

$h(h+k)+$

$(h+1)(h+k-1)=0$

$\Rightarrow h=-1$ or $h=1-k$.

$h = \dfrac{-k \pm \sqrt{k^2-4(k-1)}}{2}$

$= \dfrac{-k\pm(k-2)}{2}$ $x+k(y-x)$ e

$= -1, 1-k.$

C.F. $= \sum c_1e^{-x+ky} + \sum c_2e^{(1-k)x+ky}$

$= e^{-x}\sum c_1e^{ky} + e^x\sum c_2e^{k(y-x)}$

$= e^{-x}\phi_1(y) + e^x\phi_2(y-x).$

P.I. $= \dfrac{1}{D^2-DD'+D'-1}5e^x = 5\dfrac{e^x}{1-1}$

$= \dfrac{x}{2D-D'}5e^x = \dfrac{5xe^x}{2}$ //.

2. Solve $(D-D'-1)(D-D'-2)z = e^{2x-y}$

Here $m_1 = 1,\ c_1 = 1,\quad m_2 = 1,\ c_2 = 2$

$C.F = e^{x}\phi_1(y+x) + e^{2x}\phi_2(y+x).$

$$P.I. = \frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y}$$

$$= \frac{e^{2x-y}}{(2+1-1)(2+1-2)}$$ Re D by 2, D' by -1

$$= \frac{e^{2x-y}}{2\cdot 1} = \frac{e^{2x-y}}{2}$$

$$z = e^{x}\phi_1(y+x) + e^{2x}\phi_2(y+x) + \frac{1}{2}e^{2x-y}.$$

2. $(D+D'-1)(D+2D'-3)z = 4+3x+6y.$

$(D-(-1)D'-1)(D-(-2)D'-3)$
$\downarrow$
$m_1 = -1,\ c_1 = 1$
$m_2 = -2,\ c_2 = 3$

Here $m_1 = -1,\ c_1 = +1.\quad m_2 = -2,\ c_2 = +3.$

$C.F. = e^{x}\phi_1(y-x) + e^{3x}\phi_2(y-2x).$

$$P.I. = \frac{1}{(D+D'-1)(D+2D'-3)}(4+3x+6y)$$

$$= \frac{1}{-1\left(1-(D+D')\right)(-3)\left(1+\left(\frac{D+2D'}{-3}\right)\right)}(4+3x+6y)$$

$$= \frac{1}{3}\left[1-(D+D')\right]^{-1}\left[1-\left(\frac{D+2D'}{3}\right)\right]^{-1}(4+3x+6y)$$

$$= \frac{1}{3}\left[1 + (D+D') + (D+2D')^2 + \cdots\right]$$

$$\left[1 + \frac{1}{3}(D+2D') + \frac{1}{9}(D+2D')^2 + \cdots\right](4+3x+6y)$$

$$= \frac{1}{3}\left[1 + \frac{1}{3}(D+D') + \frac{D+2D'}{3}\right](4+3x+6y)$$

$$= \frac{1}{3}\left[1 + \frac{4D}{3} + \frac{5D'}{3}\right](4+3x+6y)$$

$$= \frac{1}{3}\left[4+3x+6y + \frac{4}{3}(3) + \frac{5}{3}(6)\right]$$

$$= \frac{4}{3} + x + 2y + \frac{4}{3} + \frac{10}{3} \qquad \frac{8}{3} + 10$$

$$= x + 2y + 6 \;//$$

3. $(D^2 - D'^2 - 3D + 3D')\, z = xy + 7.$

$$\left[(D-D')(D+D') - 3(D-D')\right] z = xy + 7.$$

$$(D-D')\left[D+D'-3\right]$$

$$z = \phi_1(y+x) + e^{3x} f(y-x) - \frac{1}{3}\left[\frac{x^2 y}{2} + \frac{xy}{3} + 7x + \frac{x^2}{3} + \frac{x}{3}\right.$$

$$\left. + \frac{y}{9} + \frac{x^3}{6} + \frac{67}{27}\right]$$

4. $(D-D'-1)(D-D'-2)z = e^{2x+y}$

$m_1=1,\ c_1=1,\ m_2=1,\ c_2=2$

C.F. $= e^{x}\phi_1(y+x) + e^{2x}\phi_2(y+x)$

P.I. $= \dfrac{1}{(D-D'-1)(D-D'-2)} e^{2x+y}$

$= \dfrac{1}{D-D'-1} \dfrac{e^{2x+y}}{2-1-2}$

$= -\dfrac{e^{2x+y}}{D-D'-1}$

$= -x e^{2x+y}$.

$z = e^{x}\phi_1(y+x) + e^{2x}\phi_2(y+x) - xe^{2x+y}$.

Type IV

1. $(D^3+D^2D'-DD'^2-D'^3)z = e^{x}\cos 2y$.

$m^3+m^2-m-1=0$.

$(m-1)(m+1)^2=0$

$\Rightarrow m=1,-1,-1$.

C.F $= f_1(y+x) + f_2(y-x) + x f_3(y-x)$.

P.I $= \dfrac{1}{D^3+D^2D'-DD'^2-D'^3} e^{x}\cos 2y$

$= e^{x} \dfrac{1}{(D+1)^3+(D+1)^2D'-(D+1)D'^2-D'^3} \cos 2y$

$$= e^x \frac{1}{D^3 + 3D^2 + 3D + 1 + D^2D' + 2DD' + D' - DD' - D'^2 - D'^3} \cos 2y$$

$$= e^x \frac{1}{1 + D' - D'^2 - D'^3 + \cancel{2D'^3}} \cos 2y$$

$$= e^x \frac{1}{+3D\,1 + 4D' + 4 + 4D'} \cos 2y$$

$$= e^x \frac{1}{5 + 5D' + 3D} \cos 2y$$

$$= \frac{e^x}{\cancel{5}} \frac{1}{5 + 5D' + 3D} \cos 2y = e^x \frac{5 - (5D' + 3D)}{[5 + (5D' + 3D)][5 - (5D' + 3D)]}$$

$$= \frac{e^x}{5} \frac{1 - D'}{1 - D'^2} \cos 2y = e^x \cdot \frac{5 - (5D' + 3D)}{25 - (5D' + 3D)^2} \cos 2y$$

$$= \frac{e^x}{5} \frac{(1 - D')\cos 2y}{1 + 4} = e^x \frac{5 - (5D' + 3D)}{25 - (25D'^2 + 30DD' + 9D^2)}$$

$$= e^x \cdot \frac{[5\cos 2y - 5(2)\sin 2y]}{25 - (25(-4))}$$

$$= \frac{e^x}{25}(\cos 2y + 2\sin 2y)$$

$$= \frac{e^x [5\cos 2y + 10\sin 2y]}{125}$$

$$= 5 \cdot e^x \frac{[\cos 2y + 2\sin 2y]}{125}$$

$$= \frac{e^x[\cos 2y + 2\sin 2y]}{25} \,//.$$

UNIT - II FOURIER SERIES.

Periodic functions: A function $f(x)$ which satisfies the relation $f(x+T) = f(x)$ for all x is called a periodic function. The smallest positive number T, is called the period of $f(x)$.

Ex: $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$ are periodic functions with period 2π. $\tan x$, $\cot x$ are periodic with period π. $\sin nx$, $\cos nx$ are periodic with period $\frac{2\pi}{n}$.

Fourier Series:

Periodic functions are of common occurrence in many physical and engineering problems; for example, in conduction of heat and mechanical vibrations. It is useful to express the functions in a series of sines and cosines. Most of the single valued functions which occur in applied mathematics can be expressed in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

within a desired range of values of the variable. Such a series is known as Fourier Series.

For ex,

Euler's formulae for finding Fourier Coefficients.

The Fourier Series for the function f(x) in the interval $c \leq x \leq c+2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

These values of a_0, a_n, b_n are known as Euler's formulae.

:1 When $c=0$, the interval becomes $0 \leq x \leq 2\pi$ and the expressions for a_0, a_n & b_n are given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

2. When $c = -\pi$, the interval becomes $-\pi \leq x \leq \pi$ and the expres[sions] for a_0, a_n & b_n are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Even and odd functions

A function $f(x)$ is said to be an even function if $f(-x) = f(x)$. Ex: $\cos x, x^2, |x|, x^{2n} (n=1,2,3...)$ are even functions

A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$. Ex: $x, \tan x, x^{2n+1} (n=0,1,2,...)$ are odd functions of x.

i) Even function × Even function = Even function

ii) Even function × odd function = odd function

iii) odd function × odd function = even function.

Continuous function

A function $f(x)$ is said to be continuous at $x=a$ if given $\epsilon > 0$, however small, we can find a no $\delta > 0$ ∋ $|f(x) - f(a)| < \epsilon$ when $|x-a| < \delta$ and is denoted by $\lim_{x \to a} f(x) = f(a)$

Discontinuous function

A function $f(x)$ is said to be discontinuous at a pt if it is not continuous at that point.

Piecewise Continuous function

A function $f(x)$ is said to be piecewise continuous in an interval if

i) the interval can be divided into a finite no of subintervals in each of which $f(x)$ is continuous and

ii) the limits of $f(x)$ as x approaches the end pts of each subinterval are finite.

Dirichlet's Conditions

If a function $f(x)$ is defined in $c \le x \le c+2\pi$, it can be expanded as a Fourier series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, provided the following DIRICHLET'S conditions are satisfied.

i) $f(x)$ is single valued and finite in $(c, c+2\pi)$

ii) $f(x)$ is continuous or piece-wise continuous with finite no of finite discontinuities in $(c, c+2\pi)$.

iii) $f(x)$ has a finite no of maxima or minima in $(c, c+2\pi)$

Convergence of Fourier Series

If $x=a$ is a pt of discontinuity of $f(x)$, then the value of the Fourier series at $x=a$ is $\frac{1}{2}[f(a+) + f(a-)]$.

The Fourier series of $f(x)$ converges to $f(x)$ at all pts where $f(x)$ is continuous. If $f(x)$ is continuous at $x=a$, the sum of the Fourier series when $x=a$ is $f(a)$.

Value of Fourier Series at end pts of the interval:

The value of the Fourier series of $f(x)$ at $x=c$ or $x=c+2\pi$ is $\frac{1}{2}[f(c) + f(c+2\pi)]$

(ie) the average of values of $f(x)$ at $x=c$ & $x=c+2\pi$.

1. Expand $f(x) = x^2$, $0 < x < 2\pi$ in a Fourier Series if the period is 2π.

The Fourier Series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$u = x^2 \qquad dv = \cos nx\,dx$

$v = \frac{\sin nx}{n}$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{8\pi^2}{3}$$

$x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right)$

$+ 2\left(-\frac{\cos nx}{n^3}\right)$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$u' = 2x \qquad v_1 = -\frac{\cos nx}{n^2}$

$u'' = 2 \qquad v_2 = -\frac{\cos nx}{n^3}$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{2(2\pi)\cos 2n\pi}{n^2} - 0\right]$$

$$= \frac{4\pi}{\pi n^2}(-1)^{2n}$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(+\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2\cos 2n\pi}{n} + \frac{4\pi\sin 2n\pi}{n^2} + \frac{2}{n^3}\cos 2n\pi - \frac{2}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2}{n}(-1)^{2n} + 0 + \frac{2}{n^3}(-1)^{2n} - \frac{2}{n^3}\right]$$

$$= \frac{-4\pi^2}{\pi n} = \frac{-4\pi}{n}$$

$$\therefore f(x) = \frac{1}{2}\left(\frac{8\pi^2}{3}\right) + \sum \frac{4}{n^2}\cos nx + \sum \frac{-4\pi}{n}\sin nx$$

$$= \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

2. If $f(x) = x(2\pi - x)$ in $0 < x < 2\pi$, p.t.

$$f(x) = \frac{2\pi^2}{3} - 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots\right]$$

<u>Solution</u> Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{2\pi x^2}{2} - \frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[4\pi^3 - \frac{8\pi^3}{3}\right]$$

$$= \frac{\pi^3}{3\pi}[12-8] = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (2\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + (2\pi - 4\pi)\left(\frac{\cos 2n\pi}{n^2}\right) + 0 - \frac{2\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi}{n^2}(-1)^{2n} - \frac{2\pi}{n^2}\right]$$

$$= \frac{-2\pi}{\pi n^2}[1+1] = -\frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2) \sin nx \, dx$$

$$= \frac{1}{\pi}\Big[(2\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (2\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right) \Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + 0 - \frac{2}{n^3}(-1)^{2n} + \frac{2}{n^3} \right]$$

$$= 0.$$

$$\therefore f(x) = \frac{1}{2}\left(\frac{4\pi^2}{3}\right) + \sum_{n=1}^{\infty} \frac{-4}{n^2} \cos nx$$

$$= \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

$$= \frac{2\pi^2}{3} - 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots \right]$$

5. Expand $f(x) = \frac{(\pi - x)^2}{4}$, $0 \le x \le 2\pi$ in a Fourier Series and hence deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$.

H.W

The Fourier Series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4} dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2\,dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi}$$

$$= -\frac{1}{12\pi}\left[-\pi^3 - \pi^3\right] = \frac{\pi^2}{6} \qquad \boxed{a_0 = \frac{\pi^2}{6}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{4\pi}{4\pi n^2} = \frac{1}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2(-1)(-1)\left(+\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right] = 0$$

$\therefore\ f(x) = \frac{1}{2}\left(\frac{\pi^2}{6}\right) + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx + 0$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

(ie) $\frac{(\pi-x)^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ ——— ①

Deduction:

$x=0$ lies in $[0, 2\pi]$ and is a pt of Continuity of $f(x) = \frac{(\pi-x)^2}{4}$.

Putting $x=0$ in ①, we get

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12}$$

$$= \frac{3\pi^2 - \pi^2}{12}$$

$$= \frac{2\pi^2}{12}$$

$$= \frac{\pi^2}{6}$$

(ie) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots = \frac{\pi^2}{6}$.

4. Express $f(x) = x \sin x$ as a Fourier Series in $0 \le x \le 2\pi$. Also Evaluate $\frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \frac{1}{7\cdot9} + \cdots$

The fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx.$$

Putting $x = \pi/2$ $\frac{\pi-2}{4}$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$= \frac{1}{\pi}\left[x(-\cos x) - 1(-\sin x)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-2\pi\right] = -2.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\sin x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$\cos A\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - \left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$$

$(\because \sin 2(n+1)\pi = 0,\ \sin 2(n-1)\pi = 0,\ \cos 2(n+1)\pi = 1,\ \cos 2(n-1)\pi = 1,$ whether n is even or odd)

$$a_n = \frac{-1}{n+1} + \frac{1}{n-1} = \frac{-n+1+n+1}{n^2-1} = \frac{2}{n^2-1}, \text{ provided } n \neq 1$$

When $n=1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx \qquad \sin 2x = 2\sin x \cos x$$

$$= \frac{1}{2\pi}\left\{x\left(\frac{-\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{\frac{-2\pi}{2}\cos 4\pi\right\} = \frac{-1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx \qquad 2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin nx \sin x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - 1.\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$= 0$, provided $n \neq 1$

Deduction: $x=\pi/2$ is a pt of Continuity.

$$\frac{\pi}{2}\sin\frac{\pi}{2} = -1-\frac{1}{2}(0)+\pi+2\left[\frac{1}{1.3}\cos\pi+\frac{1}{2.4}\cos\frac{3\pi}{2}+\frac{1}{3.5}\cos 2\pi+\dots\right]$$

$$\frac{\pi}{2} = -1+\pi+2\left[\frac{-1}{1.3}+\frac{1}{3.5}-\dots\right]$$

$$= -1+\pi-2\left[\frac{1}{1.3}-\frac{1}{3.5}+\dots\right]$$

$$2\left[\frac{1}{1.3}-\frac{1}{3.5}+\dots\right] = -1+\pi-\frac{\pi}{2}$$

$$= -1+\frac{\pi}{2}$$

$$\frac{1}{1.3}-\frac{1}{3.5}+\dots = \frac{\pi-2}{4}$$

when $n=1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x\left(x-\frac{\sin 2x}{2}\right)-1.\left(\frac{x^2}{2}+\frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi)-2\pi^2-\frac{1}{4}+\frac{1}{4}\right]$$

$$= \frac{2\pi^2}{2\pi} = \pi.$$

$$f(x) = \frac{a_0}{2}+a_1\cos x+b_1\sin x+\sum_{n=2}^{\infty}a_n\cos nx+\sum_{n=2}^{\infty}b_n\sin nx$$

$$= -1-\frac{1}{2}\cos x+\pi\sin x+\sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx+0$$

$$x\sin x = -1-\frac{1}{2}\cos x+\pi\sin x+\frac{2}{2^2-1}\cos 2x+\frac{2}{3^2-1}\cos 3x+\dots$$

5. Express $f(x)=(\pi-x)^2$ as a Fourier Series of period 2π in the interval $0<x<2\pi$. Hence deduce the sum of the series $1+\frac{1}{2^2}+\frac{1}{3^2}+\dots$

The Fourier Series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2}+\sum a_n\cos nx+\sum b_n\sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} (\pi-x)^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi} = \frac{-1}{3\pi}\left[-\pi^3-\pi^3\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi-x)^2\cos nx\,dx$$

$$= \frac{1}{\pi}\Big[(\pi-x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi-x)(-1)\left(-\frac{\cos nx}{n^2}\right)$$
$$+ 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n^2}+\frac{2\pi}{n^2}\right] = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi-x)^2\sin nx\,dx$$

$$= \frac{1}{\pi}\Big[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi-x)(-1)\left(-\frac{\sin nx}{n^2}\right)$$
$$+ 2(-1)(-1)\left(\frac{\cos nx}{n^3}\right)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2}{n}+\frac{2}{n^3}+\frac{\pi^2}{n}-\frac{2}{n^3}\right] = 0.$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

(ii) $$(\pi-x)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

$x=0$ is an end point in the range.

The value of the Fourier series at $x=0$ is $\dfrac{f(0)+f(2\pi)}{2}$.

Putting $x=0$ in the Fourier series

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2+\pi^2}{2}$$

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2$$

$$\sum\frac{1}{n^2} = \frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right)$$

$$\sum\frac{1}{n^2} = \frac{\pi^2}{6}$$

6. If $f(x) = \begin{cases} \sin x, & \text{in } 0 \le x \le \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$

find a Fourier series of periodicity 2π and hence evaluate $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$ to ∞.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin x\,dx + 0\right] = \frac{1}{\pi}(-\cos x)_0^{\pi} = \frac{1}{\pi}(1+1) = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} \sin x \cos nx\,dx + 0\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos nx \sin x\,dx.$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}, \text{ if } n \neq 1$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n+1}\left\{(-1)^{n+1} - 1\right\} + \frac{1}{n-1}\left\{(-1)^{n-1} - 1\right\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{n+1}\left\{(-1)^n + 1\right\} + \frac{1}{n-1}\left\{\frac{(-1)^n}{-1} - 1\right\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{n+1}\left\{(-1)^n + 1\right\} + \frac{1}{n-1}\left\{\frac{(-1)^n + 1}{-1}\right\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{n+1}\left\{(-1)^n+1\right\} - \frac{1}{n-1}\left\{(-1)^n+1\right\}\right]$$

$$= \frac{\left\{(-1)^n+1\right\}}{2\pi}\left[\frac{n-1-n-1}{n^2-1}\right]$$

$$= \frac{1}{2\pi(n^2-1)}(-2)\left\{1+(-1)^n\right\}$$

$$= \frac{-1}{\pi(n^2-1)}\left\{1+(-1)^n\right\}$$

$$= \begin{cases} 0, \text{ if } n \text{ is odd} \\ \dfrac{-2}{\pi(n^2-1)} \text{ if } n \text{ is even}, \cancel{n\neq 1} \end{cases}$$

$$a_1 = \frac{1}{\pi}\int_0^\pi \sin x \cos x \, dx$$

$$= \frac{1}{2\pi}\int_0^\pi \sin 2x \, dx$$

$$= \frac{1}{2\pi}\left(-\frac{\cos 2x}{2}\right)_0^\pi$$

$$= \frac{-1}{4\pi}(1-1) = 0.$$

$$b_n = \frac{1}{\pi}\int_0^\pi \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^\pi \left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^\pi, \text{ if } n \neq 1$$

0, if $n \neq 1$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\pi\right] = \frac{1}{2}$$

$$f(x) = \frac{1}{\pi} + \sum_{n=2,4,6\ldots} -\frac{2}{\pi}\frac{1}{(n^2-1)}\cos nx + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{\cos nx}{n^2-1}$$

Putting $x=0$ in the Fourier series

$$\frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,6\ldots}^{\infty}\frac{1}{(n-1)(n+1)} = 0$$

$$\frac{2}{\pi}\sum_{n=2,4,\ldots}^{\infty}\frac{1}{(n-1)(n+1)} = \frac{1}{\pi}$$

$$\sum_{n=2,4\ldots}^{\infty}\frac{1}{(n-1)(n+1)} = \frac{1}{2}$$

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots \text{ to } \infty = \frac{1}{2}.$$

1. Find the Fourier Series of periodicity 2π for

$$f(x) = \begin{cases} x & \text{in } (0,\pi) \\ 2\pi - x & \text{in } (\pi, 2\pi) \end{cases}$$

and hence deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$

$a_0 = \pi$, $a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$, $b_n = 0.$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right].$$

$x=0$ and pt→ for deduction

2. Find the Fourier Series of $f(x) = e^x$ in $(-\pi, \pi)$ of periodicity 2π

$a_0 = \frac{2}{\pi} \sinh \pi$, $a_n = \frac{2(-1)^n}{\pi(1+n^2)} \sinh \pi$

$$b_n = \frac{-2(-1)^n \cdot n \sinh \pi}{\pi(1+n^2)}$$

$$e^x = \frac{\sinh \pi}{\pi}\left[1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2}(\cos nx - n \sin nx)\right] \text{(Ans)}$$

Find the Fourier Series of $f(x) = x + x^2$ in $(-\pi, \pi)$ of periodicity 2π. Hence deduce $\sum \frac{1}{n^2} = \frac{\pi^2}{6}.$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\,dx + \int_{-\pi}^{\pi} x^2\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + 2\int_0^{\pi} x^2\,dx\right]$$

$$= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\,dx + \int_{-\pi}^{\pi} x^2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + 2\int_0^{\pi} x^2\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right] = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{\pi} x\sin nx\,dx}_{\text{even}} + \underbrace{\int_{-\pi}^{\pi} x^2\sin nx\,dx}_{\text{odd}}\right]$$

$$= \frac{2}{2\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{2\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\cancel{2}\pi}\left[\frac{-\pi(-1)^n}{n}\right] = -\frac{2}{n}(-1)^n$$

$$\therefore \quad f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

Deduction:

$x = \pi$ is an end pt.

The value of the Fourier series at $x=\pi$ is the average of the values of $f(x)$ at $x=\pi$ & $x=-\pi$.

Put $x = \pi$ in Fourier Series,

$$\frac{\pi^2}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos n\pi = \frac{f(\pi)+f(-\pi)}{2}$$

$$\frac{\pi^2}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{\pi+\pi^2 + (-\pi)+\pi^2}{2}$$

$$\frac{\pi^2}{3} + \sum \frac{4}{n^2} = \frac{\pi^2}{\cancel{6}}$$

$$\sum \frac{4}{n^2} = \pi^2 - \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{3\cdot 4} = \frac{\pi^2}{6}$$

— x —

K.T.G Find the Fourier Series of $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$

Evaluate $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$ to ∞.

$\boxed{x = \pi/2}$ a_n a_1 b_n b_1

↓

pt of continuity

Expand $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$ in a Fourier Series and hence Show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \text{ to } \infty = \frac{\pi^2}{8}$$

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\,(x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{-\pi^2}{2\pi} = -\frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left\{x\left(\frac{\sin nx}{n}\right) - 1\cdot\left(\frac{-\cos nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi}\left[\frac{-1 + (-1)^n}{n^2}\right]$$

$$= \begin{cases} \frac{-2}{\pi n^2}, & \text{when } n \text{ is } \cancel{\text{even}} \text{ odd} \\ 0, & \text{when } n \text{ is } \cancel{\text{odd}} \text{ even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0}\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + \left\{x\left(\frac{-\cos nx}{n}\right) - 1\cdot\left(\frac{-\sin nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n}\right]$$

$$= \frac{1}{n}\left[1 - 2(-1)^n\right]$$

$$\therefore\ f(x) = \frac{-\pi}{4} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{-2}{\pi n^2}\cos nx + \sum_{n=1}^{\infty}\left(\frac{1-2(-1)^n}{n}\right)\sin nx$$

$$= \frac{-\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

$$+ \sum_{n=1}^{\infty}\frac{1}{n}\left[1-2(-1)^n\right]\sin nx$$

By putting $x=0$ we get the required result.
$x=0$ is a pt of discontinuity

value of Fourier Series at $x=0$ $\Big\} = \dfrac{f(0-)+f(0+)}{2}$

$$= \frac{-\pi+0}{2}$$

$$= -\frac{\pi}{2}.$$

$f(0-) = -\pi$
$f(0+) = 0$
$\uparrow x=0$

B Put $x=0$ in the Fourier series,

$$-\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = -\frac{\pi}{2}$$

$$\frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi}{2}$$

$$\frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi}{2}\left(\frac{\pi}{4}\right)$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Even and odd functions

A function $f(x)$ is said to be even if $f(-x) = f(x)$

Ex: x^2, $\cos x$, $|x|$, x^{2n} $(n = 1, 2, 3, \dots)$

A function $f(x)$ is said to be odd if $f(-x) = -f(x)$

Ex: x, $\tan x$, $\sin x$.

Note: 1. The product of 2 even functions or 2 odd functions is an even function.

2. The product of an even function and an odd function is an odd function.

3. When $f(x)$ is an even function, the Euler's Coefficients becomes

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

$\therefore$ if a function $f(x)$ is even, its Fourier expansion contains only cosine terms.

(ie) $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$, where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

4. If $f(x)$ is an odd function, then its Fourier expansion Contains only sine terms.

(ie) $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

1. Find the Fourier Series of the function $f(x) = x^2$, $-\pi < x < \pi$. Deduce the relations

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \text{ to } \infty = \frac{\pi^2}{6}$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \text{ to } \infty = \frac{\pi^2}{12}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \text{ to } \infty = \frac{\pi^2}{8}$$

Given $f(x) = x^2$

$$f(-x) = (-x)^2 = x^2 = f(x).$$

Hence $f(x)$ is an even function. $\therefore$ the Fourier Coefficient $b_n = 0$.

The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^2}{3}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{ x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\frac{2\pi\cos n\pi}{n^2}\right\} = \frac{4}{n^2}(-1)^n$$

$$\therefore f(x) = \frac{1}{2}\left(\frac{2\pi^2}{3}\right) + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\left\{\frac{-1}{1^2}\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right\}$$

$$x^2 = \frac{\pi^2}{3} - 4\left\{\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right\}$$

$x=0$ is a pt of continuity

Putting $x=0$ in the Fourier Series,

$$0 = \frac{\pi^2}{3} - 4\left\{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\cdots\right\}$$

$$4\left\{1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right\} = \frac{\pi^2}{3}$$

$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12} \quad \text{——} (1)$$

$x = \pi$ is an end pt

The value of the Fourier series at $x = \pi$ $\Bigg\} = \dfrac{f(\pi) + f(-\pi)}{2}$ | $f(\pi) = \pi^2$, $f(-\pi) = (\pi)^2 = \pi^2$

$$= \frac{\pi^2 + \pi^2}{2} = \pi^2$$

Putting $x = \pi$ in the Fourier Series,

$$\pi^2 = \frac{\pi^2}{3} - 4\left\{ -1 - \frac{1}{2^2} - \frac{1}{3^2} - \cdots \right\}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left\{ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right\}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left\{ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right\}$$

$$\frac{2\pi^2}{3 \cdot 4} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \quad \text{—— (2)}$$

$$\begin{array}{r|l} 2 & 12, 6 \\ \hline 3 & 6, 3 \\ \hline & 2, 1 \end{array}$$

Adding (1) & (2);

$$2\left\{ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right\} = \frac{\pi^2}{12} + \frac{\pi^2}{6}$$

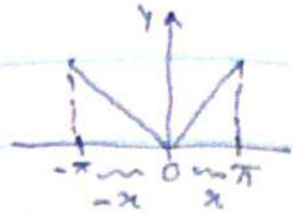
$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{1}{2}\left(\frac{\pi^2 + 2\pi^2}{12} \right) = \frac{3\pi^2}{24} = \frac{\pi^2}{8}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

2. Obtain the Fourier series of periodicity 2π for

i) $f(x) = -x$, when $-\pi < x \le 0$ and $f(x) = x$, when $0 < x < \pi$,

ii) $f(x) = |x|$, when $-\pi < x < \pi$ & deduce $\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$



Given $f(x) = |x|$

$f(-x) = |-x| = |x| = f(x)$

$\therefore f(x) = |x|$ is an even function.

Hence $b_n = 0$.

The Fourier Series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{ x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right\}$$

$$= \frac{2}{\pi n^2}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{-4}{\pi n^2}, & \text{when } n \text{ is odd} \end{cases}$$

~~The~~ $\therefore$
$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,\dots} \frac{-4}{\pi n^2} \cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left\{ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right\}.$$

$x = 0$ is a pt of continuity.

Putting $x = 0$ in the Fourier series,

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left\{ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right\}$$

$$\frac{4}{\pi}\left\{ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots \right\} = \frac{\pi}{2}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots = \frac{\pi}{2} \cdot \frac{\pi}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots\cdots = \frac{\pi^2}{8}$$

3. Obtain the Fourier series of periodicity 2π for $f(x) = x$, in $-\pi < x < \pi$. Deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

Given $f(x) = x$

$$f(-x) = -x = -f(x).$$

$\therefore f(x)$ is an odd function

~~Hence~~ The Fourier series of $f(x)$ is given by

$$f(x) = \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty} b_n \sin nx$$

$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$

$= \frac{2}{\pi}\left\{ x\left(\frac{-\cos nx}{n}\right) - 1.\left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\pi}$

$= \frac{2}{\pi}\left\{ -\pi\frac{(-1)^n}{n}\right\} = -\frac{2(-1)^n}{n}$

$f(x) = -2\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx\right\}$

$= -2\left\{ -1\sin x + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \dots\right\}$

$= 2\left\{ \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\dots\right\}$

Putting $x = \pi/2$ we get the required result

$x = \pi/2$ is a pt of continuity

Putting $x = \pi/2$ in the Fourier series,

$\frac{3\times 180}{2}$ 90°

$2\pi + \pi/2$

$90 + 2\pi$

$\frac{\pi}{2} = 2\left\{1 + \frac{\sin\frac{3\pi}{2}}{3} + \frac{\sin\frac{5\pi}{2}}{5} + \dots\dots\right\}$

$\frac{\pi}{4} = \left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right\}$

$\sin\frac{3\pi}{2}$

$= \sin(\pi/2 + \pi)$

$= \cos\pi = -1$

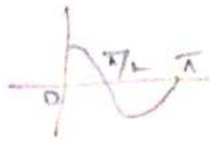
$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\dots = \frac{\pi}{4}$

$\sin\frac{5\pi}{2}$

$\sin\frac{7\pi}{2} = \sin(3\pi + \pi/2)$ $= \sin(\pi/2 + 2\pi)$

$= \cos 3\pi$ $= \cos 2\pi$

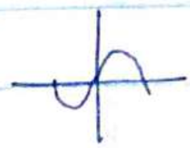
$= -1$ $\sin\frac{5\pi}{2} = 1$

$|\cos x|$ 

4. Find the Fourier Series of $f(x) = |\sin x|$, $-\pi < x < \pi$. Hence deduce the sum of the series $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots \infty$.

$x = \pi$ and pt $\frac{1}{2}[f(-\pi)+f(\pi)]$ (OR) $x=0$

$\Downarrow$ $\frac{1}{2}$.

$f(x) = \begin{cases} -\sin x & -\pi < x \le 0 \\ \sin x & 0 \le x < \pi \end{cases}$ 

5. If $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

S.t. $f(x) = \frac{8}{\pi^2}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$

Given $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

$$f(-x) = \begin{cases} 1 - \frac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1 + \frac{2x}{\pi}, & 0 \le -x \le \pi \end{cases}$$

$$= \begin{cases} 1 - \frac{2x}{\pi}, & \pi \ge x \ge 0 \\ 1 + \frac{2x}{\pi}, & 0 \ge x \ge -\pi \end{cases}$$

$$= \begin{cases} 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \\ 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \end{cases}$$

$$= f(x)$$

$\therefore$ $f(x)$ is an even function.

The Fourier Series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{2x^2}{2\pi}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi - \pi\right] = 0.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{-\frac{2}{\pi n^2}(-1)^n + \frac{2}{\pi n^2}\right\}$$

$$= \frac{4}{\pi^2 n^2}\left\{1-(-1)^n\right\}$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{8}{\pi^2 n^2}, & \text{" " odd.} \end{cases}$$

Write down the even & odd extensions of $f(x)$ in $(-l,0)$, if $f(x) = x^2+x$ in $(l,0)$ $\quad (l,2l)$

even extension $\rightarrow f(-x) = x^2 - x$. $\quad f(2l-x)$

odd $\rightarrow -f(-x) = -[x^2-x] = x - x^2$. $\quad -[f(2l-x)]$

$$\therefore \; f(x) = \frac{8}{\pi^2} \sum_{n=1,3,\ldots}^{\infty} \frac{\cos nx}{n^2}$$

$$= \frac{8}{\pi^2}\left\{ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right\}$$

6. In $(-\pi,\pi)$ find the Fourier Series of periodicity 2π for $f(x) = \begin{cases} 1+x & \text{in } \; 0<x<\pi \\ -1+x & \text{in } \; -\pi<x<0. \end{cases}$

<u>Ans</u>: $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[1-(1+\pi)(-1)^n\right] \sin nx.$

7. If $f(x)$ is defined in $(-\pi,\pi)$ & if $f(x) = x+1$ in $(0,\pi)$ find $f(x)$ in $(-\pi,0)$ if

i) $f(x)$ is odd $\quad$ ii) $f(x)$ is even.

odd extension

$-f(-x) = -x+1$

i) If $f(x)$ is odd, then $f(x) = x-1$ $(-\pi,0)$

even extension

ii) If $f(x)$ is even, then $f(x) = -x+1$ $(-\pi,0)$ $= f(-x) = -f($

$= -[-x+1]$

$= x-1$

Find the Fourier Series of period 2π for the function $f(x) = |\cos x|$ in $-\pi \le x \le \pi$.

<u>Note</u>: The values $f(-x)$ & $-f(-x)$ assigned to $f(x)$ in $(-l,0)$ in order to make $f(x)$ even & odd respectively in $(-l,l)$ are called the even & odd extensions of $f(x)$ in $(-l,0)$.

The values $f(2l-x)$ & $-f(2l-x)$ are called the periodic extensions of $f(x)$ in $(l,2l)$.

Half range Series:

When $f(x)$ is defined in $(0,\pi)$, then we can represent $f(x)$ in a series of sines only or Cosines Only. These are called Half range series.

Half range sine series: The Half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$

Half range Cosine series: The Half range Cosine Series is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx.$$

1) Find Half range Sine series and Cosine series for $f(x) = x - x^2$ in $0 < x < \pi$.

<u>Half range Sine Series:</u>

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (x - x^2) \sin nx \, dx$$

$$= \frac{2}{\pi}\left\{ (x - x^2)\left(\frac{-\cos nx}{n}\right) - (1 - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right) \right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ \frac{(\pi^2 - \pi)(-1)^n}{n} - (1 - 2\pi).0 - \frac{2}{n^3}(-1)^n + \frac{2}{n^3} \right\}$$

$$= \frac{2}{\pi}\left\{ \frac{(\pi^2 - \pi)(-1)^n}{n} + \frac{2}{n^3}\left[1 - (-1)^n\right] \right\}$$

$$\therefore f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{ \frac{(\pi^2 - \pi)(-1)^n}{n} + \frac{2}{n^3}\left[1 - (-1)^n\right] \right\} \sin nx.$$

Half range Cosine Series

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(x-x^2)\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3}\right] = \frac{2\pi^2}{6\pi}\left[3 - 2\pi\right]$$

$$= \frac{\pi}{3}(3 - 2\pi)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{(x-x^2)\left(\frac{\sin nx}{n}\right) - (1-2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\frac{(1-2\pi)(-1)^n}{n} - \frac{1}{n^2}\right\}$$

$$\therefore f(x) = \frac{\pi}{6}(3 - 2\pi) + \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(1-2\pi)(-1)^n}{n} - \frac{1}{n^2}\right\}\cos nx$$

2. If $f(x) = \frac{\pi x}{4}, \quad 0 < x < \frac{\pi}{2}$

$= \frac{\pi}{4}(\pi - x), \quad \frac{\pi}{2} < x < \pi$

Express $f(x)$ in a series of Cosines only.

<u>Solution</u>: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{\pi x}{4}\,dx + \int_{\pi/2}^{\pi} \frac{\pi}{4}(\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\left(\frac{x^2}{2}\right)_0^{\pi/2} + \frac{\pi}{4}\left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{2}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right] = \frac{2\pi^2}{2.8} = \frac{\pi^2}{8}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\int_0^{\pi/2} x\cos nx\,dx + \frac{\pi}{4}\int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{2}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2}$$

$$+ \left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{2}\left[\cancel{\frac{\pi}{2n}\sin\frac{n\pi}{2}} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} - \cancel{\frac{\pi}{2n}\sin\frac{n\pi}{2}} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{1}{2}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$\therefore f(x) = \frac{1}{2}\left(\frac{\pi^2}{8}\right) + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]\cos nx.$$

$$= \frac{\pi^2}{16} + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]\cos nx.$$

3. Obtain Cosine and sine series for $f(x) = x$ in the interval $0 \le x \le \pi$. Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Half Range Cosine Series

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{ x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ \frac{(-1)^n}{n^2} - \frac{1}{n^2} \right\}$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{-4}{\pi n^2}, & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3}^{\infty} \frac{\cos nx}{n^2}$$

$x=0$ is a pt of Continuity

Putting $x=0$ in the Fourier Series,

$$\frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos 0}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = 0$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi}{4}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi^2}{8}.$$

<u>Half range sine Series</u>

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left\{ x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\frac{-\pi(-1)^n}{n}\right\} = \frac{-2(-1)^n}{n}.$$

$$\therefore \quad f(x) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

4. Obtain the half-range Cosine series of $f(x) = \pi^2 - x^2$ in $(0,\pi)$. ~~Deduce the sum of the series~~ $\cancel{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \dots \infty}$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi^2 - x^2)\, dx$$

$$= \frac{2}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^3 - \frac{\pi^3}{3}\right] = \frac{2}{\pi}\left[\frac{2\pi^3}{3}\right] = \frac{4\pi^2}{3}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi^2 - x^2) \cos nx \, dx$$

$$= \frac{2}{\pi}\left\{ \pi^2\left(\frac{\sin nx}{n}\right)_0^{\pi} - \left\{ x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right\}_0^{\pi}\right.$$

$$= -\frac{2}{\pi}\left\{ 2\pi \frac{(-1)^n}{n^2} \right\} = \frac{-4(-1)^n}{n^2}$$

$$f(x) = \frac{1}{2}\left(\frac{4\pi^2}{3}\right) - 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$$

$$= \frac{2\pi^2}{3} - 4\left\{ -\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots \right\}$$

$$= \frac{2\pi^2}{3} + 4\left\{ \frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots \right\}$$

$x = 0$ is an end pt.

$\pi^2 = 0 + \pi^2 - \pi^2$

The value of the Fourier series at $x=0$ $\Big\} = \dfrac{f(0) + f(\pi)}{2}$

$\dfrac{\pi^2}{12}$

$$= \frac{\pi^2 + 0}{2}$$

Putting $x=0$ in the Fourier series,

$$\frac{2\pi^2}{3} + 4\left\{ \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \right\} = \frac{\pi^2}{2}$$

$$4\left\{ \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \right\} =$$

$$\frac{\pi^2}{2} - \frac{2\pi^2}{3}$$

$$\frac{3\pi^2 - 4\pi^2}{6}$$

$$\frac{-\pi^2}{6 \cdot 4}$$

5. Find the Fourier sine series of $f(x) = e^{ax}$ in $(0, \pi)$

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi} \int_0^{\pi} e^{ax} \sin nx \, dx$$

$$= \frac{2}{\pi} \left[\frac{e^{ax}}{a^2+n^2} [a \sin nx - n \cos nx] \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{e^{a\pi}}{a^2+n^2} [-n(-1)^n] + \frac{n}{a^2+n^2} \right]$$

$$= \frac{2}{\pi(a^2+n^2)} \left[n - n e^{a\pi} (-1)^n \right]$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n}{a^2+n^2} \left[1 - (-1)^n e^{a\pi} \right] \sin nx$$

6. Find the half range cosine series of $f(x) = \cos ax$ in $(0, \pi)$, where a is neither zero nor an integer.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos ax \, dx$$

$$= \frac{2}{\pi} \left[\frac{\sin ax}{a} \right]_0^{\pi} = \frac{2}{a\pi} \sin a\pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos ax \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} \cos nx \cos$$

$$= \frac{2}{2\pi}\int_0^{\pi}\left[\cos(n+a)x + \cos(n-a)x\right] dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+a)x}{n+a} + \frac{\sin(n-a)x}{n-a}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+a)\pi}{n+a} + \frac{\sin(n-a)\pi}{n-a}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi\cos a\pi + \cos n\pi \sin a\pi}{n+a} + \frac{\sin n\pi \cos a\pi - \cos n\pi \sin a\pi}{n-a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n \sin a\pi}{n+a} - \frac{(-1)^n \sin a\pi}{n-a}\right]$$

$$= \frac{(-1)^n \sin a\pi}{\pi}\left[\frac{1}{n+a} - \frac{1}{n-a}\right]$$

$$= \frac{(-1)^n \sin a\pi}{\pi(n^2-a^2)}\left[n-a-n-a\right]$$

$$= \frac{-2a(-1)^n \sin a\pi}{\pi(n^2-a^2)} = \frac{2a(-1)^{n+1}\sin a\pi}{\pi(n^2-a^2)}$$

$$\therefore f(x) = \frac{\sin a\pi}{a\pi} + \frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos nx}{n^2-a^2}$$

odd & even

Find the Fourier series of $f(x) = |\sin x|$, $-\pi < x < \pi$. Hence deduce the sum of the series $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\cdots\infty$

$f(x) = |\sin x|$

$f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)$.

$\therefore f(x)$ is an even function

Hence $b_n = 0$.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\, dx$$

$$= \frac{2}{\pi}\left(-\cos x\right)_0^{\pi}$$

$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$$= \frac{2}{\pi}(1+1) = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{-\frac{\cancel{\cos}\,(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right\}$$

$$= \frac{1}{\pi}\left\{ \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} + \frac{n-1-n-1}{n^2-1} \right\}$$

$$= \frac{1}{\pi}\left\{ (-1)^n \left[\frac{n-1-n-1}{n^2-1}\right] - \frac{2}{n^2-1} \right\}$$

$$= \frac{-2}{\pi(n^2-1)}\left\{ (-1)^n + 1 \right\}, \quad n \neq 1$$

$$= \begin{cases} 0, & \text{when } n \text{ is odd}, n \neq 1 \\ \frac{-4}{\pi(n^2-1)}, & \text{when } n \text{ is even} \end{cases}$$

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x \, dx$$

$$= \frac{2}{\pi}\int_0^\pi \frac{\sin 2x}{2} \, dx$$

$$= \frac{1}{\pi}\int_0^\pi \sin 2x \, dx$$

$$= \frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^\pi$$

$$= -\frac{1}{2\pi}(1-1) = 0.$$

$$f(x) = \frac{1}{2}\left(\frac{4}{\pi}\right) - \frac{4}{\pi}\sum_{n=2,4}^{\infty} \frac{1}{n^2-1}\cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left\{ \cancel{\cos} \sum_{n=2,4}^{\infty} \frac{\cos nx}{(n-1)(n+1)} \right.$$

$x=0$ is a pt of Continuity.

Putting $x=0$ in the Fourier Series,

$$\frac{2}{\pi} - \frac{4}{\pi}\left\{\frac{1}{1.3} + \frac{1}{3.5} + \dots\dots\right\} = 0$$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \dots\dots\dots = -\frac{2}{\pi}\left(-\frac{\pi}{4}\right)$$

$$= \frac{1}{2}$$

Change of interval

Sometimes we require the expansion of a function defined in an interval of length $2l$ say $c \le x \le c+2l$.

In such cases, we transform the variable by a suitable substitution by changing the interval of length $2l$ into an interval of length 2π and then the Fourier series of $f(x)$ is obtained.

The Fourier series of $f(x)$ in $c \le x \le c+2l$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin\frac{n\pi x}{l}\,dx$$

Note: 1. When $c=0$, the interval becomes $0 \le x \le 2l$ and the expressions for Fourier Coefficients are given by

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx \quad \& \quad b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

2. When $C=-l$, the interval becomes $-l \le x \le l$ and the expressions for a_0, a_n & b_n are given by

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

3. If $f(x)$ is an even function of x in $(-l, l)$, the Fourier expansion of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{where}$$

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$$

$$\& \quad a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

4. If $f(x)$ is an odd function of x in $(-l, l)$, the Fourier expansion $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{2\pi x}{l} \cancel{dx}$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$

5. Any function $f(x)$ defined in the half-range 0 to l admits of

i) a Cosine expansion, where $a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} \, dx$

ii) a Sine expansion, where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx.$

1. Find the Fourier series expansion of period 2 for the function $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2. \end{cases}$

Deduce the sum of $\sum_{1,3,\ldots}^{\infty} \frac{1}{n^2}$

Here $2l = 2$
$\Rightarrow l = 1$

The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum a_n \cos n\pi x + \sum b_n \sin n\pi x$$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x) \, dx.$$

$$= \int_0^2 f(x)\,dx$$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left(\frac{x^2}{2}\right)_0^1 + \pi\left(2x-\frac{x^2}{2}\right)_1^2$$

$$= \frac{\pi}{2} + \pi\left(4-2-2+\frac{1}{2}\right)$$

$$= \pi$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos n\pi x\,dx$$

$$= \frac{1}{1}\int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \pi\left\{x\left(\frac{\sin n\pi x}{n\pi}\right) - 1\left(\frac{-\cos n x\pi}{n^2\pi^2}\right)\right\}_0^1$$

$$+ \pi\left\{(2-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(\frac{-\cos n x\pi}{n^2\pi^2}\right)\right\}_1^2$$

$$= \pi\left\{\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right\} + \pi\left\{-\frac{1}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2}\right\}$$

$$= \frac{2\pi}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ -\dfrac{4}{n^2\pi}, & \text{when } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \int_0^{2} f(x) \sin n\pi x\,dx$$

$$= \int_0^{1} \pi x \sin n\pi x\,dx + \int_1^{2} \pi(2-x) \sin n\pi x\,dx.$$

$$= \pi\left\{x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1.\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right\}_0^1$$

$$+ \pi\left\{(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right\}_1^2$$

$$= \pi\left\{-\frac{(-1)^n}{n\pi}\right\} + \pi(2-1)\frac{\cos n\pi}{\pi}$$

$$= -(-1)^n + (-1)^n = 0.$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3\ldots}^{\infty} \frac{\cos n\pi x}{n^2}$$

Deduction:

$x=1$ is a pt of Continuity.

Putting $x=1$ in the Fourier series,

$$\frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3}^{\infty}\frac{1}{n^2} = \pi$$

$$+\frac{4}{\pi}\sum\frac{1}{n^2} = \pi - \pi/2$$

$$= \frac{\pi}{2}$$

$$\sum\frac{1}{n^2} = \frac{\pi}{2}\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8}.$$

2. f(x) is defined in $(-2,2)$ as follows. Express f(x) in a F.S. of periodicity 4.

$$f(x) = \begin{cases} 0, & -2 < x < -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$

$a_0 = \frac{2}{l}\int_0^l f(x)dx$; $= \frac{2}{2}\int_0^2 f(x)dx$

$2l =$ Upper Limit − Lower Limit.

$\rightarrow \int_0^1 (1-x)dx$

$\left(x - \frac{x^2}{2}\right)_0^1$

$1 - \frac{1}{2} = \frac{1}{2}$

Solution: Here $2l = 4 \Rightarrow l = 2$

Let $f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{2} + \sum b_n \sin\frac{n\pi x}{2}$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)dx$$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0\,dx + \int_{-1}^{0}(1+x)\,dx + \int_0^1 (1-x)\,dx + \int_1^2 0\,dx\right]$$

$$= \frac{1}{2}\left\{\left[x + \frac{x^2}{2}\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_0^1\right\}$$

$$= \frac{1}{2}\left\{+1 - \frac{1}{2}\right\} + \frac{1}{2}\left(1 - \frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{4} = \frac{1}{2}$$

$$\boxed{a_0 = \frac{1}{2}}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left\{\int_{-1}^{0}(1+x)\cos\frac{n\pi x}{2}\,dx + \int_{0}^{1}(1-x)\cos\frac{n\pi x}{2}\,dx\right\}$$

$$= \frac{1}{2}\left\{(1+x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\cdot\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{-1}^{0}$$

$$+ \frac{1}{2}\left\{(1-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{0}^{1}$$

$$= \frac{1}{2}\left\{\frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right\} + \frac{1}{2}\left\{\frac{-4}{n^2\pi^2}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\right\}$$

$$= \frac{4}{n^2\pi^2}\left[1-\cos\frac{n\pi}{2}\right] = \frac{4}{n^2\pi^2}\,2\sin^2\frac{n\pi}{4}$$

$2x = \frac{n\pi}{2}$

$x = \frac{n\pi}{4}$

$$= \frac{8}{n^2\pi^2}\sin^2\frac{n\pi}{4}$$

$\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1-\cos 2x}{2}.$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\,\cancel{\cos}\,\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left\{\int_{-1}^{0}(1+x)\sin\frac{n\pi x}{2}\,dx + \int_{0}^{1}(1-x)\sin\frac{n\pi x}{2}\,dx\right\}$$

$$= \frac{1}{2}\left\{(1+x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\cdot\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{-1}^{0}$$

$$+ \frac{1}{2}\left\{(1-x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{0}^{1}$$

$= \frac{1}{2}\left\{\frac{-2}{n\pi} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}\right\}$

$= 0.$

$\therefore f(x) = \frac{1}{4} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\frac{n\pi}{4}\cos\frac{n\pi x}{2}$

3. Find the Fourier series for $f(x) = \begin{cases} x, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$.

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$
$$- \frac{2}{\pi}\left[\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \cdots\right]$$

4. Find a Fourier series for $f(x) = 2x - x^2$ with period 3 in the range $(0,3)$.

$2l = 3$ $\quad l = 3/2,$ $\quad a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$ $\quad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$

$$f(x) = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{3} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{3}$$

5. Find a Fourier series to represent $f(x) = \pi x$, in $-l \le x \le l$.

$f(x) = \pi x$

$f(-x) = -\pi x = -f(x).$

Hence $f(x)$ is an odd function.

The Fourier series of $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{2}{l}\int_0^l \pi x \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2\pi}{l}\left\{ x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1.\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^l$$

$$= \frac{2\pi}{l}\left\{-\frac{l^2}{n\pi}(-1)^n\right\} = \frac{2l}{n}(-1)^{n+1}.$$

$$\therefore f(x) = 2l\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}$$

6. Find the F.S. to represent $f(x) = x^2 - 2$ in $-2 < x < 2$.

$$f(x) = x^2 - 2$$

$$f(-x) = (-x)^2 - 2 = x^2 - 2 = f(x).$$

$\therefore$ $f(x)$ is even. $\Rightarrow$ $b_n = 0$.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$ [Here $2l = 4 \Rightarrow l = 2$]

$$a_0 = \frac{2}{l}\int_0^l f(x)\, dx$$

$$= \frac{2}{2}\int_0^2 (x^2 - 2)\, dx$$

$$= \left(\frac{x^3}{3} - 2x\right)_0^2 = \frac{8}{3} - 4 = -\frac{4}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\, dx.$$

$$= \frac{2}{2}\int_0^2 (x^2-2)\, \cos\frac{n\pi x}{2}\, dx$$

$$= \left\{ (x^2-2)\left(+\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (2x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right) - 1.\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{8}}\right)\right\}_0^2$$

$$= \frac{16}{n^2\pi^2}(-1)^n$$

The required Fourier series is

$$f(x) = \frac{1}{2}\left(\frac{4}{3}\right) + \frac{16}{\pi^2}\sum\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}$$

7. Find the Fourier Cosine series of $f(x) = \begin{cases} x^2, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \end{cases}$

Here $l = 2$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{2}\int_0^2 f(x)\,dx$$

$$= \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx.$$

$$= \frac{1}{3} + \left(2x - \frac{x^2}{2}\right)_1^2 = \frac{1}{3} + \left(4 - 2 - 2 + \frac{1}{2}\right) = \frac{1}{3} + \frac{1}{2}$$

$$= \frac{5}{6}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 x^2\cos\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\cos\frac{n\pi x}{2}\,dx$$

$$= \left\{x^2\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 2x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right) + 2\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right\}_0^1$$

$$+ \left\{(2-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_1^2$$

$$= \cancel{\frac{2}{n\pi}\sin\frac{n\pi}{2}} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi}{2}$$

$$\cancel{+} - \frac{4}{n^2\pi^2}(-1)^n - \cancel{\frac{2}{n\pi}\sin\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}$$

$$= \frac{12}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}(-1)^n$$

$$\therefore f(x) = \frac{5}{12} + \sum_{n=1}^{\infty}\left[\frac{12}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}(-1)^n\right]\cos\frac{n\pi x}{2}.$$

Expand $f(x) = (x-1)^2$, $0 < x < 1$ in a F.S. of sines only.

Here $l = 1$.

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{1}$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{1}\int_0^1 (x-1)^2 \sin n\pi x\,dx$$

$$= 2\left\{(x-1)^2\left(-\frac{\cos n\pi x}{n\pi}\right) - 2(x-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right\}_0^1$$

$$= 2\left\{\frac{2}{n^3\pi^3}(-1)^n + \frac{1}{n\pi} - \frac{2}{n^3\pi^3}\right]$$

$$= 2\left[\frac{2}{n^3\pi^3}\left[(-1)^n - 1\right] + \frac{1}{n\pi}\right]$$

$$= \frac{4}{n^3\pi^3}\left[(-1)^n - 1\right] + \frac{2}{n\pi}$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\left[\frac{4}{n^3\pi^3}\left[(-1)^n - 1\right] + \frac{2}{n\pi}\right]\sin n\pi x.$$

Find the half range Sine Series and Cosine Series of period 4 for $f(x)$ where $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 4-2x, & 1 < x < 2 \end{cases}$

Here $l = 2$

Half range Sine Series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= 2\int_0^1 x\sin\frac{n\pi x}{2}\,dx + \int_1^2 (4-2x)\sin\frac{n\pi x}{2}\,dx$$

$$= 2\left\{x\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1.\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_0^1$$

$$+\left\{(4-2x)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-2)\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_1^2$$

$$= 2\left\{\frac{-2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$+\frac{4}{n\pi}\cos\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{-4}{n\pi}\cos\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{4}{n\pi}\cos\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2}$$

<u>Cosine Series:</u>

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{2}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{2}\int_0^2 f(x)\,dx$$

$$= \int_0^1 2x\,dx + \int_1^2 (4-2x)\,dx$$

$$= \left(x^2\right)_0^1 + \left(4x - x^2\right)_1^2$$

$$= 1 + [8 - 4 - 4 + 1] = 2$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 2x\cos\frac{n\pi x}{2}\,dx + \int_1^2 (4-2x)\cos\frac{n\pi x}{2}\,dx$$

$$= 2\left\{x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_0^1$$

$$+ \left\{(4-2x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-2)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_1^2$$

$$= 2\left\{\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\right\}$$

$$+ \left\{-\frac{8}{n^2\pi^2}(-1)^n - 2\cdot\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}\right\}$$

$$= \frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2}$$

$$-\frac{8}{n^2\pi^2}(-1)^n - \frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}$$

$$= \frac{16}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2}\left[1+(-1)^n\right]$$

$$\therefore f(x) = 1 + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left\{\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2}\left[1+(-1)^n\right]\right\}\cos nx.$$

Find the half-range cosine series of $f(x) = (\pi - x)^2$ in $(0,\pi)$. Hence find the sum of the series $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\cdots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi-x)^2\,dx = \frac{2}{\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{\pi} = \frac{2}{3}\pi^2.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi-x)^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{(\pi-x)^2\left(\frac{\sin nx}{n}\right) - \left(2(\pi-x)(-1)\right)\left(-\frac{\cos nx}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{+\frac{2\pi}{n^2}\right\} = \frac{4}{n^2}.$$

$$\frac{1}{4}\left(\frac{2}{3}\pi^2\right)^2 + \frac{1}{2}\sum\left(\frac{4}{n^2}\right)^2 = \frac{1}{\pi}\int_0^{\pi}\left[(\pi-x)^2\right]^2 dx$$

$$\frac{\pi^4}{9} + \frac{1}{2}\sum\frac{16}{n^4} = \frac{1}{\pi}\int_0^{\pi}(\pi-x)^4\,dx$$

$$= \frac{1}{\pi}\left\{\frac{(\pi-x)^5}{-5}\right\}_0^{\pi}$$

$$\frac{\pi^4}{9} + \sum_{n=1}^{\infty}\frac{8}{n^4} = \frac{\pi^4}{5} \Rightarrow \sum\frac{1}{n^4} = \frac{1}{8}\left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) = \frac{4\pi^4}{45\times 8} = \frac{\pi^4}{90}$$

Root mean square value of a function

If a function $y = f(x)$ is defined in $(c, c+2l)$, then $\sqrt{\frac{1}{2l}\int_c^{c+2l} y^2\,dx}$ is called the root mean square (R.M.S.) or effective value of y in $(c, c+2l)$ and is denoted by $\bar{y}$.

(ie) $\bar{y} = \sqrt{\frac{1}{2l}\int_c^{c+2l} y^2\,dx}$

$$\sqrt{\frac{\int_c^{c+2l} y^2\,dx}{c+2l-c}} = \sqrt{\frac{1}{2l}\int_c^{c+2l} y^2\,dx}$$

$$\Rightarrow \bar{y}^2 = \frac{1}{2l}\int_c^{c+2l} y^2\,dx$$

If $y = f(x)$ can be expanded as a Fourier series in $(c, c+2l)$, then $\bar{y}^2$ can be expressed in terms of Fourier coefficients a_0, a_n & b_n. The formula that expresses $\bar{y}^2$ in terms of a_0, a_n & b_n is known as Parseval's formula which is stated as a theorem.

Parseval's theorem:

If $y = f(x)$ can be expanded as Fourier series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ in $(c, c+2l)$, then the root-mean square value $\bar{y}$ of $y = f(x)$ in $(c, c+2l)$ is given by

$$\bar{y}^2 = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2$$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$, $a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Note: 1 If the Fourier half range Cosine Series of $y = f(x)$ in $(0,l)$

$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$, then

$f(x)$ defined in (a,b)

Then $\sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$ $\bar{y}$

$$\bar{y}^2 = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2,$$

where $\bar{y}^2 = \frac{1}{l}\int_0^l y^2\, dx$

2. If the Fourier half-range sine series of $y = f(x)$ in $(0,l)$ is $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$, then

$$\bar{y}^2 = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2, \text{ where } \bar{y}^2 = \frac{1}{l}\int_0^l y^2\, dx$$

Proof:
for thm.

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ in $(c, c+2l)$

By Euler's formula for the Fourier Coefficients,

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\, dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos\frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l}\int_a^{c+2l} f(x) \sin\frac{n\pi x}{l}\, dx.$$

Proof By definition,

$$\bar{y}^2 = \frac{1}{2l}\int_c^{c+2l} y^2\, dx$$

$$= \frac{1}{2l}\int_c^{c+2l} [f(x)]^2\, dx.$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\right]dx$$

$$= \frac{a_0}{4}\left[\frac{1}{l}\int_c^{c+2l} f(x)\,dx\right] + \sum_{n=1}^{\infty}\frac{a_n}{2}\left[\frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$+ \sum_{n=1}^{\infty}\frac{b_n}{2}\left[\frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{a_0}{4}\cdot a_0 + \sum_{n=1}^{\infty}\frac{a_n}{2}\cdot a_n + \sum_{n=1}^{\infty}\frac{b_n}{2}\cdot b_n$$

$$= \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2$$

Find the F.S. of periodicity 2π for $f(x) = x^2$ in $-\pi < x$
Hence show that

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}.$$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2\pi^2}{3}$$

$\frac{1}{2l}$ $\frac{1}{2\pi}\int_{-\pi}^{\pi}$ [

$$a_n = \frac{4(-1)^n}{n^2}, \quad b_n = 0.$$

By Parseval's theorem,

$$\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2 = \bar{y}^2,$$ the square of R.M.S. value of

$$\frac{1}{4}\left(\frac{2\pi^2}{3}\right) + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[x^2\right]^2 dx$$

$$\frac{\pi^2}{6} + \sum\frac{8}{n^4} = \frac{2}{2\pi}\left(\frac{x^5}{5}\right)_0^{\pi}$$

$$8\sum\frac{1}{n^4} = \frac{1}{\pi}\left(\frac{\pi^5}{5}\right) - \cancel{\frac{\pi^4}{6}}\ \frac{\pi^2}{6}$$

$$= \frac{\pi^4}{5} - \frac{\pi^2}{6} = \frac{\pi^2}{30}$$

$$= \cancel{\frac{2\pi^2 - \pi^2}{6}}$$

$$\frac{1}{4}\left(\frac{2\pi^2}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\right]^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[x^2\right]^2 dx$$

$$\frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4} = \frac{1}{\pi}\int_{0}^{\pi} x^4\, dx$$

$$\frac{\pi^4}{9} + \sum\frac{8}{n^4} = \frac{1}{\pi}\left(\frac{x^5}{5}\right)_0^{\pi}$$

$$8\sum\frac{1}{n^4} = \frac{\pi^4}{5} - \frac{\pi^4}{9}$$

$$8\sum\frac{1}{n^4} = \frac{4\pi^4}{45}$$

$$\sum\frac{1}{n^4} = \frac{\pi^4}{}$$

Express $f(x) = x$ in half range Cosine series & sine Series of periodicity $2l$ in the range $0<x<l$ and deduce the value of $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$ & $\sum \frac{1}{n^2}$ using Passeval's thm.

<u>Cosine Series:</u>

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left(\frac{l^2}{2}\right) = l$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left\{x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\cdot\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^l$$

$$= \frac{2}{l}\left\{\frac{l^2}{n\pi}\sin n\pi + \frac{l^2}{n^2\pi^2}(-1)^n - \frac{l^2}{n^2\pi^2}\right\}$$

$$= \frac{2l^2}{l\,n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{-4l}{n^2\pi^2}, & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{l}{2} - \frac{4l}{\pi^2} \sum_{n=1,3\ldots}^{\infty} \frac{1}{n^2} \cos\frac{n\pi x}{l}.$$

By Parseval's thm,

$$\frac{a_0^2}{4} + \frac{1}{2}\sum a_n^2 = \frac{1}{l}\int_0^l [f(x)]^2\,dx$$

$$\frac{l^2}{4} + \frac{1}{2}\sum_{n=1,3\ldots}^{\infty} \frac{16l^2}{n^4\pi^4} = \frac{1}{l}\int_0^l x^2\,dx$$

$$\frac{l^2}{4} + \frac{8l^2}{\pi^4}\sum_{n=1,3..}^{\infty}\frac{1}{n^4} = \frac{1}{l}\left(\frac{l^3}{3}\right)$$

$$l^2\left[\frac{1}{4} + \frac{8}{\pi^4}\sum_{n=1,3..}^{\infty}\frac{1}{n^4}\right] = \frac{l^2}{3}$$

$$\frac{1}{4} + \frac{8}{\pi^4}\sum\frac{1}{n^4} = \frac{1}{3}$$

$$\sum_{n=1,3\ldots}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{8}\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{8}\left[\frac{4-3}{12}\right]$$

$$= \frac{\pi^4}{8.12} = \frac{\pi^4}{96}$$

(ie) $\dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \cdots\cdots = \dfrac{\pi^4}{96}.$

Sine Series:

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \left\{ x \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - 1 \cdot \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right.$$

$$= \frac{2}{l} \left\{ \frac{-l^2}{n\pi} \cos n\pi + 0 \right\}$$

$$= \frac{-2l}{n\pi} (-1)^n = \frac{2l}{n\pi} (-1)^{n+1}$$

By Parseval's thm,

$$\frac{1}{2} \sum b_n^2 = \frac{1}{l} \int_0^l [f(x)]^2 \, dx$$

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} = \frac{1}{l} \frac{l^3}{3}$$

$$\sum_{n=1}^{\infty} \frac{2l^2}{n^2\pi^2} = \frac{l^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} = \frac{1}{6}$$

Expand $f(x) = x - x^2$ as a F.S. in $-1 < x < 1$ & using this series find the R.M.S. value of $f(x)$ in the interval.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{1}\int_{-1}^{1} f(x)\,dx$$

$$= \int_{-1}^{1} (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{2} - \frac{1}{3} = -\frac{2}{3}.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\,dx$$

$$= \int_{-1}^{1} (x - x^2) \cos n\pi x\,dx$$

$$= \left\{(x - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1 - 2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{-\sin n\pi x}{n^3\pi^2}\right)\right\}_{-1}^{1}$$

$$= -\frac{1}{n^2\pi^2} - \frac{3}{n^2\pi^2}(-1)^n$$

$$= \frac{-4}{n^2\pi^2}(-1)^n = \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \int_{-1}^{1} (x - x^2) \sin n\pi x \, dx$$

$$= \int_{-1}^{1} x \sin n\pi x \, dx - \int_{-1}^{1} x^2 \sin n\pi x \, dx$$

$$= 2\int_{0}^{1} x \sin n\pi x \, dx + 0 \quad (\text{odd})$$

$$= 2\left\{ x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1\cdot\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right\}_0^1$$

$$= 2\left\{-\frac{(-1)^n}{n\pi}\right\} = \frac{2(-1)^{n+1}}{n\pi}.$$

$$\therefore f(x) = -\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos n\pi x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x$$

R.M.S. value of $f(x) = \sqrt{\frac{1}{2l}\int_{-l}^{l}[f(x)]^2\,dx}$

$$= \sqrt{\frac{1}{2}\int_{-1}^{1}(x-x^2)^2\,dx} \qquad \begin{matrix}(x-x^2)^2 \\ x^2 - 2x^3 + x^4\end{matrix}$$

$$= \sqrt{\frac{1}{2}\cdot 2\int_0^1 (x^2+x^4)\,dx}$$

$$= \sqrt{\left(\frac{x^3}{3}+\frac{x^5}{5}\right)_0^1}$$

$$= \sqrt{\frac{1}{3}+\frac{1}{5}} = \sqrt{\frac{8}{15}}$$

Find the Fourier series of period l for the function

$$f(x) = \begin{cases} x, & (0, l/2) \\ l-x, & (l/2, l) \end{cases}$$

Hence deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$

$2L \rightarrow$ period

Here $2L = l$
$\rightarrow L = l/2.$

The F.S. of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l/2} + \sum b_n \sin \frac{n\pi x}{l/2}.$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi x}{l}$$

$$a_0 = \frac{1}{l/2} \int_0^{l} f(x)\, dx \qquad a_0 = \frac{1}{L} \int_0^{2L} f(x)\, dx.$$

$$a_0 = \frac{1}{L} \int_0^{2L} f(x)\, dx$$

$$= \frac{2}{l} \int_0^{l} f(x)\, dx$$

$$= \frac{2}{l} \left[\int_0^{l/2} x\, dx + \int_{l/2}^{l} (l-x)\, dx \right]$$

$$= \frac{2}{l} \left[\left(\frac{x^2}{2} \right)_0^{l/2} + \left(lx - \frac{x^2}{2} \right)_{l/2}^{l} \right]$$

$$= \frac{2}{l} \left[\frac{l^2}{8} + l^2 - \frac{l^2}{2} - \frac{l^2}{2} + \frac{l^2}{8} \right]$$

$$= \frac{2}{l} \left(\frac{2l^2}{8} \right) = \frac{l}{2}.$$

$$\frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx$$

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↓

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Ex. 16

$$a_n = \frac{1}{l/2}\int_0^{l} f(x)\cos\frac{n\pi x}{l/2}\,dx$$

$$= \frac{2}{l}\int_0^{l} f(x)\cos\frac{2n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left\{\int_0^{l/2} x\cos\frac{2n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\cos\frac{2n\pi x}{l}\,dx\right\}$$

$$= \frac{2}{l}\left[\left\{x\left(\frac{\sin\frac{2n\pi x}{l}}{\frac{2\pi n}{l}}\right) - 1\cdot\left(\frac{-\cos\frac{2n\pi x}{l}}{\frac{4\pi^2 n^2}{l^2}}\right)\right\}_0^{l/2}\right.$$

$$\left. + \left\{(l-x)\left(\frac{\sin\frac{2n\pi x}{l}}{\frac{2\pi n}{l}}\right) - (-1)\left(\frac{-\cos\frac{2n\pi x}{l}}{\frac{4\pi^2 n^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\cancel{\frac{l^2}{4n\pi}(0) + \frac{l^2}{4\pi^2 n^2}\cos n\pi - \frac{l^2}{4\pi n^2} - \frac{l^2}{4\pi^2 n^2} + 0 + \frac{l^2}{4\pi^2 n^2}}\right]$$

$$= \frac{2}{l}\left[0 + \frac{l^2}{4n^2\pi^2}\cos n\pi + 0 - \frac{l^2}{4n^2\pi^2} + 0 - \frac{l^2}{4n^2\pi^2}(-1)^{2n} + 0 + \frac{l^2}{4n^2\pi^2}(-1)^n\right]$$

$$= \frac{2}{l}\left[\frac{l^2}{4n^2\pi^2}(-1)^n - \frac{l^2}{4n^2\pi^2} - \frac{l^2}{4n^2\pi^2} + \frac{l^2}{4n^2\pi^2}(-1)^n\right]$$

$$= \frac{2}{l}\left[\frac{2l^2}{4n^2\pi^2}(-1)^n - \frac{2l^2}{4n^2\pi^2}\right]$$

$$= \frac{l}{n^2\pi^2}\left\{(-1)^n - 1\right\}.$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{-2l}{n^2\pi^2}, & \text{when } n \text{ is odd.} \end{cases}$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{l/2}\int_0^{l} f(x) \sin\frac{n\pi x}{l/2}\, dx$$

$$= \frac{2}{l}\left\{\int_0^{l/2} x \sin\frac{2n\pi x}{l}\, dx + \int_{l/2}^{l} (l-x)\sin\frac{2n\pi x}{l}\, dx\right\}$$

$$= \frac{2}{l}\left[\left\{x\left(\frac{-\cos\frac{2n\pi x}{l}}{\frac{2n\pi}{l}}\right) - 1\cdot\left(\frac{-\sin\frac{2n\pi x}{l}}{\frac{4n^2\pi^2}{l^2}}\right)\right\}_0^{l/2} + \left\{(l-x)\left(\frac{-\cos\frac{2n\pi x}{l}}{\frac{2n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{2n\pi x}{l}}{\frac{4n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[-\frac{l}{2}\cdot\frac{l}{2n\pi}(-1)^n + 0 + \frac{l}{2}\cdot\frac{l}{2n\pi}(-1)^n\right]$$

$$= 0$$

FP.

$$\therefore \quad f(x) = \frac{l}{4} - \frac{2l}{\pi^2}\sum_{n=1,3\ldots}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{l}.$$

$$\frac{a_0^2}{4} + \frac{1}{2}\sum a_n^2 + \frac{1}{2}\sum b_n^2 = \frac{1}{2L}\int_0^{2L}[f(x)]^2\, dx.$$

$$\frac{1}{2l}\int_0^{2l}[f(x)]^2\,dx$$

$$\downarrow$$

$$\frac{1}{4}\left(\frac{l^2}{4}\right)+\frac{4l^2}{2\pi^4}\sum_{1,3\ldots}^{\infty}\frac{1}{n^2}=\frac{1}{l}\int_0^{l}[f(x)]^2\,dx$$

$$\frac{l^2}{16}+\frac{2l^2}{\pi^4}\sum_{n=1,3\ldots}^{\infty}\frac{1}{n^4}=\frac{1}{l}\left[\int_0^{l/2}x^2\,dx+\int_{l/2}^{l}(l-x)^2\,dx\right] \quad l-l/2$$

$$=\frac{1}{l}\left\{\left[\frac{x^3}{3}\right]_0^{l/2}+\left[\cancel{lx-\frac{x^2}{2}}\ \frac{(l-x)^3}{-3}\right]_{l/2}^{l}\right\}$$

$$=\frac{1}{l}\left\{\frac{l^3}{8\cdot 3}+\cancel{l^2-\frac{l^2}{2}-\frac{l^2}{2}-\frac{l^2}{8}}\right\}$$

$$+\frac{l^3}{8\cdot 3}$$

$$=\frac{1}{l}\left\{\frac{l^3}{24}+\frac{l^3}{24}\right\}$$

$$=\frac{2l^3}{24l}=\frac{l^2}{12}$$

$$\frac{2l^2}{\pi^4}\sum_{n=1,3}^{\infty}\frac{1}{n^4}=\frac{l^2}{12}-\frac{l^2}{16}$$

2 | 12, 16
2 | 6, 8
3, 4

16×3

$\boxed{48}$

$$=\frac{4-3}{48}$$

$$\frac{2}{\pi^4}\sum_{n=1,3}^{\infty}\frac{1}{n^4}=\frac{1}{48}$$

$$\sum_{n=1,3}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{96}$$

$$\Longrightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{96}.$$

The representation of periodic phenomena using complex number leads to Complex form of the Fourier Series

Complex form of Fourier Series:

The F.S. of $f(x)$ in $(c, c+2l)$ can also be put in the exponential form with Complex coefficients as explained below.

The trignometric form of the F.S. of $f(x)$ defined in $(c, c+2l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Using the exponential values of $\cos\frac{n\pi x}{l}$ & $\sin\frac{n\pi x}{l}$, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{\frac{in\pi x}{l}} + e^{-\frac{in\pi x}{l}}}{2}\right) + b_n\left(\frac{e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}}}{2i}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{\frac{in\pi x}{l}} + e^{-\frac{in\pi x}{l}}}{2}\right) + b_n\left(\frac{e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}}}{2i}\right)\frac{(-i)}{-i}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-\frac{in\pi x}{l}} \quad \text{——(1)}$$

Let $\frac{a_0}{2} = c_0$, $\frac{a_n - ib_n}{2} = c_n$ & $\frac{a_n + ib_n}{2} = c_{-n}$.

$$(1) \Rightarrow f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} c_{-n} e^{-\frac{in\pi x}{l}}$$

$$= c_0 + \sum_{1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=-\infty}^{-1} c_n e^{\frac{in\pi x}{l}}$$

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

This is called the Complex form or exponential form of the F.S of $f(x)$ in $(c, c+2l)$. The Coefficient c_n is given by

$$c_n = \frac{1}{2l}\int_{c}^{c+2l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx.$$

$(-l, l)$

Note: When $l = \pi$, the Complex form of F.S. of $f(x)$ in $(c, c+2\pi)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\, e^{-inx}\, dx.$$

Find the Complex form of the Fourier Series of $f(x) = e^x$ in $(0, 2)$

Here $2l = 2 \Rightarrow l = 1$.

The Complex form of the Fourier Series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx$$

$$= \frac{1}{2}\int_0^2 e^x e^{-in\pi x}\, dx$$

$e^{+ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$$= \frac{1}{2}\int_0^2 \left[e^{(1-in\pi)x} \right] dx$$

$$= \frac{1}{2}\left\{ \frac{e^{(1-in\pi)x}}{1-in\pi} \right\}_0^2$$

$(1-in\pi)(1+in\pi)$

$1^2 - i^2 n^2 \pi^2$

$$= \frac{1}{2(1-in\pi)}\left\{ e^{2(1-in\pi)} - 1 \right\}$$

$$= \frac{(1+in\pi)}{2(1-in\pi)(1+in\pi)}\left\{ e^2 . e^{-2in\pi} - 1 \right\}$$

$$= \frac{1+in\pi}{2(1+n^2\pi^2)}\left\{ e^2(\cos 2n\pi - i\sin 2n\pi) - 1 \right\}$$

$$= \frac{1+in\pi}{2(1+n^2\pi^2)}\left\{e^2\left[(-1)^{2n} - 0\right] - 1\right\}$$

$$= \frac{(e^2-1)(1+in\pi)}{2(1+n^2\pi^2)}$$

$$\therefore f(x) = \left(\frac{e^2-1}{2}\right)\sum_{n=-\infty}^{\infty} \frac{(1+in\pi)}{(1+n^2\pi^2)} e^{in\pi x}$$

Find the Complex form of the F.S. of $f(x) = \cos x$ in $(0, \pi)$.

Here $2l = \pi \Rightarrow l = \frac{\pi}{2}$.

The Complex form of F.S. is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2nx}$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x) e^{-\frac{in\pi x}{l}} dx.$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos x \, e^{-\frac{in\pi x}{\pi/2}} dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos x \, e^{-i2nx} dx \qquad 1^2 + (-i2n)^2 \quad 1^2 + i^2 4n^2$$

$$= \frac{1}{\pi}\left[\frac{e^{-i2nx}}{1^2 - 4n^2}\left[-i2n\cos x + 1 \cdot \sin x\right]\right]_0^{\pi}$$

$$= \frac{1}{\pi\cancel{(1+4n^2)}}\left\{\frac{e^{-i2n\pi}}{(-4n^2+1)}\left[2ni + 0\right] - \frac{1}{-4n^2+1}\left[2ni\right]\right\}$$

$$= \frac{1}{\pi\cancel{(-4n^2+1)}}\left[\frac{+2ni}{-4n^2+1} + \frac{2ni}{-4n^2+1}\right]$$

$$= \frac{+4in}{\pi(-4n^2+1)} \qquad \therefore f(x) = \frac{4i}{\pi}\sum_{n=-\infty}^{\infty} \frac{n}{1-4n^2} e^{i2nx}$$

Find the Complex form of the F.S. of $f(x) = \cos ax$ in $(-\pi, \pi)$, where a is neither zero nor an integer.

Here $2l = 2\pi \Rightarrow l = \pi$.

The Complex form of the F.S. of is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\cancel{c_n = \frac{1}{2l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\,dx}$$

$$\cancel{= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax \cos\frac{n\pi x}{\pi}\,dx}$$

$$\cancel{= \frac{1}{2\pi}\int_{-\pi}^{\pi}}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax\, e^{-\frac{in\pi x}{\pi}}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax\, e^{-inx}\,dx$$

$$\cancel{c_n = \frac{1}{2l}\int_{-l}^{2l} f(x)\, e^{-\frac{in\pi x}{l}}\,dx}$$

$$\cancel{= \frac{1}{2\pi}\int_{-l}^{2l} \cos ax\, e^{-\frac{in\pi x}{l}}\,dx}$$

$$= \frac{1}{2\pi}\left\{ \frac{e^{-inx}}{a^2-n^2}\left[-in\cos ax + a\sin ax\right] \right\}_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a^2-n^2)}\left\{ e^{-in\pi}\left[-in\cos a\pi + a\sin a\pi\right] - e^{in\pi}\left[-in\cos a\pi - a\sin a\pi\right] \right\}$$

$$= \frac{1}{2\pi(a^2-n^2)} \left\{ (\cos n\pi - i\sin n\pi)\left[-in\cos a\pi + a\sin a\pi\right] \right.$$

$$\left. - (\cos n\pi + i\sin n\pi)\left[-in\cos a\pi - a\sin a\pi\right] \right\}$$

$$= \frac{1}{2\pi(a^2-n^2)} \left\{ (-1)^n\left[-in\cos a\pi\right] + (-1)^n a\sin a\pi \right.$$

$$\left. + in(-1)^n\cos a\pi + a(-1)^n\sin a\pi \right\}$$

$$= \frac{2a(-1)^n\sin a\pi}{2\pi(a^2-n^2)} = \frac{a(-1)^n\sin a\pi}{\pi(a^2-n^2)}$$

$$\therefore f(x) = \frac{a\sin a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2-n^2} e^{inx}$$

Find the Complex form of the F.S. of $f(x) = e^{ax}$ in $(0, 2l)$.

The Complex form of F.S. is given by

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

$$C_n = \frac{1}{2l}\int_0^{2l} f(x) e^{-\frac{in\pi x}{l}}\, dx$$

$$= \frac{1}{2l}\int_0^{2l} e^{ax} e^{-\frac{in\pi x}{l}}\, dx$$

$$= \frac{1}{2l}\int_0^{2l} e^{\left(a - \frac{in\pi}{l}\right)x}\, dx.$$

$$= \frac{1}{2l}\left[\frac{e^{\left(a-\frac{in\pi}{l}\right)x}}{a-\frac{in\pi}{l}}\right]_0^{2l}$$

$$= \frac{1}{2l}\left[\frac{e^{\left(a-\frac{in\pi}{l}\right)2l}}{a-\frac{in\pi}{l}} - \frac{1}{a-\frac{in\pi}{l}}\right]$$

$$= \frac{1}{2l\left[\frac{la-in\pi}{l}\right]}\left\{e^{2al}\cdot e^{-i2n\pi} - 1\right\}$$

$$= \frac{la+in\pi}{2(l^2a^2+n^2\pi^2)}\left\{e^{2al}-1\right\}$$

$$f(x) = \left(\frac{e^{2al}-1}{2}\right)\sum_{n=-\infty}^{\infty}\left(\frac{al+in\pi}{l^2a^2+n^2\pi^2}\right)\cdot e^{\frac{in\pi x}{l}}$$

Find the Complex form of the F.S. of $f(x) = \sin 2x$ in $(0,1)$.

The Complex form of the F.S. is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

$2l = 1$

$l = 1/2$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx$$

$$= \frac{1}{1}\int_0^{1} \sin 2x\, e^{-\frac{in\pi x}{1/2}} dx$$

$$= \frac{1}{1}\left\{\frac{e^{-\frac{in2\pi x}{1}}}{-4n^2\pi^2+4}\left[-in\pi 2\sin 2x - 2\cos 2x\right]\right\}_0^{1}$$

$$= \frac{1}{2}\left\{ \frac{e}{4-n^2\pi^2 4}\left[-2in\pi\sin 2 - 2\cos 2\right] - \frac{1}{4-n^2\pi^2}\left[-2\right] \right\}$$

$$= \frac{1}{\cancel{2}(4-n^2 4\pi^2)}\left\{-(-1)^{2n} 2in\pi\sin 2 - 2(-1)^{2n}\cos 2 + 2\right\}$$

$$= \frac{2}{4(1-n^2\pi^2)}\left\{-in\pi\sin 2 - \cancel{2}\cos 2 + 1\right\}$$

$$= \frac{1}{2(n^2\pi^2-1)}\left[\cos 2 - 1 + in\pi\sin 2\right].$$

The representation of periodic signals as a linear combination of complex exponentials leads to Fourier transform.

Harmonic Analysis

Sometimes the function is not given by a formula, but by a graph or by a table of corresponding values. The process of finding the F.S. for a function given by such values of the function and independent variable is known as Harmonic analysis.

The process of finding the harmonics in the Fourier expansion of a function numerically is known as harmonic analysis.

Let $f(x)$ be defined in $(0, 2l)$ in a tabular form as below.

x	x_0	x_1	x_2	$\cdots$	x_{k-1}
$y=f(x)$	y_0	y_1	y_2	$\cdots$	y_{k-1}

Here $x_1 - x_0 = x_2 - x_1 \cdots\cdots x_k - x_{k-1} = \frac{2l}{k}$ & $x_0 = 0,\ x_k = 2l$.

When $y = f(x)$ is defined in a tabular form as above, a_0 & a_n cannot be evaluated exactly by mathematical integration, but are evaluated approximately by numerical integration as explained below.

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx$$

$$= \frac{2}{\cancel{2}l}\cdot\frac{1}{(2l-0)}\int_0^{2l} f(x)\, dx$$

$$= 2\left[\frac{1}{2l-0}\int_0^{2l} f(x)\, dx\right]$$

$$= 2\,\frac{\Sigma f(x)}{k} = 2\,\frac{\Sigma y}{k}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= 2\times\left[\frac{1}{2l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= 2\left[\frac{\sum f(x)\cos\frac{n\pi x}{l}}{k}\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$= 2\times\left[\frac{1}{2l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{k}\sum f(x)\sin\frac{n\pi x}{l}$$

Note: 1. When the interval $(0,2l)$ is divided into k equal sub-intervals, each of length $\frac{2l}{k}$, only k values of $y=f(x)$ are taken into consideration for numerical computation of a_n & b_n.

2. In most situations, the amplitudes (enlargement) of the successive harmonics will decrease very rapidly. Hence in most harmonic analysis problems, we may have to find the first few harmonics only.

Fundamental or first harmonic: The term $a_1\cos x + b_1\sin x$ in Fourier series is called the fundamental or first harmonic.

Second harmonic: The term $a_2\cos 2x + b_2\sin 2x$ in F.S. is called the second harmonic and so on.

Type I : Given datas are in π form

Type II : Given datas are in degree form $2c=2\pi$ $l=\pi$

Type III : Given datas are in T form $\cos\frac{n\pi x}{\pi/l}$

Type IV : Given datas are in l form.

1. Find the F.S. upto the third harmonic for $y=f(x)$ in $(0,2\pi)$ defined by the table of values given below.

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1	1.4	1.9	1.7	1.5	1.2	1.0

$2l=6$, $l=3$.

no. of datas

Solution: Since the last value of y is a repetition of the first, only the first six values will be used. Here $k=6$

We know that the Fourier series for first three harmonics is given by

$$y=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+a_3\cos 3x + b_1\sin x+b_2\sin 2x+b_3\sin 3x. \quad \text{—(1)}$$

To evaluate the coefficients, we form the following table.

$$2l = 2\pi$$
$$l = \pi$$

Radians

$y\cos 3x$	$y\sin 3x$	x	y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$\cos 3x$	$\sin 3x$	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
1	0	0	1	1	0	1	0	+1	0	1	0	1	0
-1.4	0	$\frac{\pi}{3}$	1.4	0.5	0.866	-0.5	0.866	-1	0	0.7	1.212	-0.7	1.212
1.9	0	$\frac{2\pi}{3}$	1.9	-0.5	0.866	-0.5	-0.866	1	0	-0.95	1.65	-0.95	-1.645
-1.7	0	π	1.7	-1	0	1	0	-1	0	-1.7	0	1.7	0
1.5	0	$\frac{4\pi}{3}$	1.5	-0.5	-0.866	-0.5	0.866	1	0	-0.75	-1.299	-0.75	1.299
-1.2	0	$\frac{5\pi}{3}$	1.2	0.5	-0.866	-0.5	-0.866	-1	0	0.6	-1.039	-0.6	-1.039
0.1													

$\Sigma y = 8.7$

$\Sigma y\cos 3x = 0.1$ $\quad \Sigma y\cos x = -1.1$ $\quad \Sigma y\cos 2x = -0.3$

$\Sigma y\sin 3x = 0$ $\quad \Sigma y\sin x = 0.5196$ $\quad \Sigma y\sin 2x = -0.1732$

$$a_0 = 2\left[\frac{\Sigma y}{k}\right] = 2\left[\frac{8.7}{6}\right] = 2.9$$

$$a_1 = 2\left[\frac{\Sigma y\cos x}{k}\right] = 2\left[\frac{-1.1}{6}\right] = -0.37$$

$$a_2 = 2\left[\frac{\Sigma y\cos 2x}{k}\right] = 2\left[\frac{-0.3}{6}\right] = -0.1$$

$$a_3 = 2\left[\frac{\Sigma y\cos 3x}{k}\right] = 2\left[\frac{+0.1}{6}\right] = 0.03$$

$$b_1 = 2\left[\frac{\Sigma y\sin x}{k}\right] = 2\left[\frac{0.5196}{6}\right] = 0.17$$

$$b_2 = 2\left[\frac{\Sigma y\sin 2x}{k}\right] = 2\left[\frac{-0.1732}{6}\right] = -0.06$$

$b_3 = 2\left[\frac{\Sigma y \sin 3x}{k}\right] = 2\left[\frac{0}{6}\right] = 0.$

Sub these values in ①,

$y = \frac{2.9}{2} + (-0.37 \cos x + 0.17 \sin x)$

$+ (-0.1 \cos 2x - 0.06 \sin 2x)$

$+ (0.03 \cos 3x)$

$= 1.45 + (-0.37 \cos x + 0.17 \sin x) - (0.1 \cos 2x + 0.06 \sin 2x)$

$+ 0.03 \cos 3x.$

2. Determine the first two harmonic of the F.S. for the following values.

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
y	1.98	1.30	1.05	1.30	-0.88	-0.25

Radians

$2l = 6.$

Here $k = 6$

The F.S. for 1^{st} 2 harmonics is given by

$y = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x.$

x	y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$y \cos x$	$y \sin x$	$y \cos 2x$	$y \sin 2x$
0	1.98	1.00	0	1	0	1.98	0	1.98	0
$\frac{\pi}{3}$	1.30	0.5	0.866	-0.5	0.866	0.65	1.126	-0.65	1.126
$\frac{2\pi}{3}$	1.05	-0.5	0.866	-0.5	-0.866	-0.525	0.909	-0.525	-0.909
π	1.30	-1.00	0	1.0	0	-1.30	0	-1.30	0
$\frac{4\pi}{3}$	-0.88	-0.5	-0.866	-0.5	0.866	0.44	0.762	0.44	-0.762
$\frac{5\pi}{3}$	-0.25	0.5	-0.866	-0.5	-0.866	-0.125	0.217	0.125	0.2167
						1.12	3.014	0.07	-0.328

$\Sigma y = 4.5$

$$a_0 = 2\left[\frac{\Sigma y}{k}\right] = 2\left[\frac{4.5}{6}\right] = 1.5$$

$$a_1 = 2\left[\frac{\Sigma y \cos x}{k}\right] = 2\left[\frac{1.12}{6}\right] = 0.373$$

$$a_2 = 2\left[\frac{\Sigma y \cos 2x}{k}\right] = 2\left[\frac{0.07}{6}\right] = 0.023$$

$$b_1 = 2\left[\frac{\Sigma y \sin x}{k}\right] = 2\left[\frac{3.014}{6}\right] = 1.005$$

$$b_2 = 2\left[\frac{\Sigma y \sin 2x}{k}\right] = 2\left[\frac{-0.328}{6}\right] = -0.109$$

$$\therefore y = \frac{1.5}{2} + (0.373\cos x + 1.005\sin x) + (0.023\cos 2x - 0.109\sin 2x)$$

3. Find an emprical formula of the form

$f(x) = a_0 + a_1\cos x + b_1 \sin x$ for the following data given that $f(x)$ is periodic with period 2π.

x in degrees	0	60	120	180	240	300	360
$f(x)$	40	31	-13.7	20	3.7	-21	40

Sol: Since the last value of $f(x)$ is a repetition of the first, only the first 6 values will be used.

$k = 6$

x	$f(x)$	$\cos x$	$\sin x$	$f(x)\cos x$	$f(x)\sin x$
0	40	1	0	40	0
60	31	0.5	0.866	15.50	26.846
120	-13.7	-0.5	0.866	6.85	-11.864
180	20	-1.0	0	-20.0	0
240	3.7	-0.5	-0.866	-1.85	-3.204
300	-21	0.5	-0.866	-10.50	18.186
~~360~~	~~40~~				

$\Sigma f(x) = 60 \qquad \Sigma y \cos = 30 \qquad \Sigma y \sin x = 29.964$

$$a_0 = 2\left[\frac{\Sigma \overset{f(x)}{y}}{k}\right]$$

$$= 2\left[\frac{60}{6}\right] = 20$$

$$a_1 = 2\left[\frac{\Sigma \overset{f(x)}{y}\cos x}{k}\right] = 2\left[\frac{30}{6}\right] = 10$$

$$b_1 = 2\left[\frac{\Sigma \overset{f(x)}{y}\sin x}{k}\right] = 2\left[\frac{29.964}{6}\right] = 9.988$$

$\therefore f(x) = 10 + 10\cos x + 9.988 \sin x.$

Datas are in T form. $\theta = \frac{2\pi x}{T}$.

The values of x and the corresponding values of $f(x)$ over a period T are given below. S.t.

$f(x) = 0.75 + 0.37\cos\theta + 1.004 \sin\theta$ where $\cancel{\theta = \frac{2\pi x}{T}}$ $\theta = \frac{2\pi x}{T}$

x	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5T}{6}$	T
y	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Since the last value of y is a repetition of the first, Only first 6 values will be used. Here $k=6$.

Let $f(x) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta$

when $x \to 0$ to T, $\theta \to 0$ to 2π with an increase of $\frac{2\pi}{6}$

Radians

x	$\theta = \frac{2\pi x}{T}$	y	$\cos\theta$	$\sin\theta$	$y\cos\theta$	$y\sin\theta$
0	0	1.98	1.0	0	1.98	0
$\frac{T}{6}$	$\frac{\pi}{3}$	1.30	0.5	0.866	0.65	1.1258
$\frac{T}{3}$	$\frac{2\pi}{3}$	1.05	-0.5	0.866	-0.525	0.9093
$\frac{T}{2}$	π	1.30	-1	0	-1.3	0
$\frac{2T}{3}$	$\frac{4\pi}{3}$	-0.88	-0.5	-0.866	0.44	0.762
$\frac{5T}{6}$	$\frac{5\pi}{3}$	-0.25	0.5	-0.866	-0.125	0.2165
		4.6			1.12	3.013

$$a_0 = 2\frac{\sum y}{k} = 2\left(\frac{4.6}{6}\right) = 1.5$$

$$a_1 = 2\frac{\sum y\cos\theta}{k} = 2\left(\frac{1.12}{6}\right) = 0.37$$

$$b_1 = 2\frac{\sum y\sin\theta}{k} = 2\left(\frac{3.013}{6}\right) = 1.0\cancel{08}4$$

$$\therefore f(x) = 0.75 + 0.37\cos\theta + 1.004\sin\theta.$$

$f(x) = 1.45 - 0.367\cos\frac{\pi x}{3} + 0.173\sin\frac{\pi x}{3} - 0.1\cos\frac{2\pi x}{3} - 0.05\sin\frac{2\pi x}{3} + 0.033\cos\pi x$

Given datas are ~~kl~~ form

Find the first 3 harmonics in the F.S. of $y = f(x)$, which is defined in the following table in $(0,6)$.

x	0	1	2	3	4	5	6
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Since the last value of y is a repetition of the first, Only first 6 values will be used. $k = 6$.

Length of the ~~to~~ interval is $2l = 6 \Rightarrow l = 3$.

The F.S. is given by

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l} + a_2\cos\frac{2\pi x}{l} + b_2\sin\frac{2\pi x}{l} + a_3\cos\frac{3\pi x}{l} + b_3\sin\frac{3\pi x}{l}$$

$$= \frac{a_0}{2} + a_1\cos\frac{\pi x}{3} + b_1\sin\frac{\pi x}{3} + a_2\cos\frac{2\pi x}{3} + b_2\sin\frac{2\pi x}{3} + a_3\cos\pi x + b_3\sin\pi x.$$

8.7

x	y	$\cos\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$	$\cos\frac{2\pi x}{3}$	$\sin\frac{2\pi x}{3}$	$\cos\pi x$	$\sin\pi x$	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$
0	1	1	0	1	0	1	0	1	0
1	1.4	0.5	0.866	-0.5	0.866	-1	0	0.7	1.212
2	1.9	-0.5	0.866	-0.5	-0.866	1	0	-0.95	1.645
3	1.7	-1	0	1	0	-1	0	-1.7	0
4	1.5	-0.5	-0.866	-0.5	0.866	1	0	-0.75	-1.299
5	1.2	0.5	-0.866	-0.5	-0.866	-1	0	0.6	-1.0392
	8.7							-1.1	0.5188

$y\cos\frac{2\pi x}{3}$	$y\sin\frac{2\pi x}{3}$	$y\cos\pi x$	$y\sin\pi x$
1	0	1	0
-0.7	1.212	-1.4	0
-0.95	-1.645	1.9	0
1.7	0	-1.7	0
-0.75	1.299	1.5	0
-0.6	-1.0392	-1.2	0
-0.3	-0.1732	0.1	0

$$a_0 = 2\left(\frac{\Sigma y}{k}\right) = 2\left(\frac{8.7}{6}\right) = 2.9$$

$$a_1 = 2\left(\frac{\Sigma y\cos\frac{\pi x}{3}}{k}\right) = 2\left(\frac{-1.1}{6}\right) = -0.367$$

$$a_2 = 2\left(\frac{\Sigma y\cos\frac{2\pi x}{3}}{k}\right) = 2\left(\frac{-0.3}{6}\right) = -0.1$$

$$a_3 = 2\left(\frac{\Sigma y\cos\pi x}{k}\right) = 2\left(\frac{0.1}{6}\right) = 0.033$$

$$b_1 = 2\left(\frac{\Sigma y\sin\frac{\pi x}{3}}{k}\right) = 2\left(\frac{0.5188}{6}\right) = 0.1729 = 0.173$$

$$b_2 = 2\left(\frac{\Sigma y\sin\frac{2\pi x}{3}}{k}\right) = 2\left(\frac{-0.1732}{6}\right) = -0.058$$

$$b_3 = 2\left(\frac{\Sigma y\sin\pi x}{k}\right) = 0$$

$$\therefore f(x) = \frac{2.9}{2} + \{ -0.367 \cos\frac{\pi x}{3} + 0.173 \sin\frac{\pi x}{3}$$
$$- 0.1 \cos\frac{2\pi x}{3} - 0.058 \sin\frac{2\pi x}{3} + 0.033 \cos \pi x .$$

Turning angle

The turning moment T is given for a Series of values of the Crank angle $\theta^\circ = 75^\circ$.
L-Shaped to & fro to circular motion

θ° :	0	30	60	90	120	150	180
T :	0	5224	8097	7850	5499	2626	0

Obtain the 1st 4 terms in a Series of sines to represent T and Calculate T for $\theta = 75^\circ$

Sol: Let the Fourier sine series to represent T in (0,180) be
$$T = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta + \dots\dots$$

θ°	T	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$
0	0	0	0	0	0
30	5224	0.500	0.866	1	0.866
60	8097	0.866	0.866	0	-0.866
90	9850	1.000	0	-1	0
120	5499	0.866	-0.866	0	0.866
150	2626	0.500	-0.866	1	-0.866

$$b_1 = \frac{2}{6} \sum y \sin\theta = \frac{1}{3}\left[(5224 + 2626)\,0.5 + (8097 + 5499)\,0.866 + 9850 \right]$$

$$= 785$$

$$b_2 = \frac{2}{6} \Sigma y \sin 2\theta$$

$$= \frac{1}{3}\left[(5224+8097)\,0.866 + (5499+2626)(-0.866)\right] = 150$$

$$b_3 = \frac{2}{6} \Sigma y \sin 3\theta$$

$$= \frac{1}{3}\left[5224 - 7850 + 2626\right] = 0.$$

$$b_4 = \frac{2}{6} \Sigma y \sin 4\theta$$

$$= \frac{1}{3}\left[(5224+5499)(0.866) + (8097+2626)(-0.866)\right] = 0.$$

Hence $T = 785 \sin\theta + 150 \sin 2\theta$

For $\theta = 75^\circ$, $T = 785 \sin 75^\circ + 150 \sin 150^\circ$

$$= 785(0.9659) + 150(0.5) = 833.2.$$

UNIT-III Boundary value problems.

Partial differential equations arise in several physical and engineering problems in which the functions involved depend on two or more independent variables such as time and coordinates in space.

Ex: Wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

One dimensional heat flow: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

<u>Initial and boundary value problems</u>

In ODEs, first we get the general solution which contains the arbitrary constants, and then we determine these constants from the given initial values. This type of problem is called initial value problems.

In many physical problems, we always seek a solution of the differential eqns, whether it is ordinary or partial, which satisfies some specified conditions called boundary conditions. Any differential eqns together with these boundary conditions is called boundary value problems.

Classification of partial differential equations of the Second order

Let a second order p.d.e. in the function u of the two independent variables x, y be of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad \text{---} (1)$$

This equation is linear in the second order terms but the term $f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ may be linear or nonlinear. In the former case, the eqn (1) is said to be linear, in the latter case to be quasi-linear.

~~Classify the following equations~~:

The above equation of second order (linear) (1) is said to be

1) elliptic if $B^2 - 4AC < 0$
2) parabolic if $B^2 - 4AC = 0$
3) hyperbolic if $B^2 - 4AC > 0$

Classify the following equations:

1. $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$

Here $A = 1, B = 2, C = 1.$

$B^2 - 4AC = 4 - 4 = 0$, for all x, y.

Hence, the equation is parabolic at all points.

2. $x^2 f_{xx} + (1-y^2) f_{yy} = 0.$

Here $A = x^2$, $B = 0$, $C = 1-y^2$.

$B^2 - 4AC = -4x^2(1-y^2)$

$= 4x^2(y^2-1)$

If $-1 \leq y < 1$, y^2-1 is -ve.

$\therefore$ B^2-4AC is -ve ~~hence~~ if $-1 < y < 1$, $x \neq 0$.

For $-\infty < x < \infty$, $(x \neq 0)$, $-1 < y < 1$, the equation is ellipt

For $-\infty < x < \infty$ $(x \neq 0)$, $y < -1$ or $y > 1$, the equation i

For $x = 0$ for all y or for all x, $y = \pm 1$, the equation

parabolic.

$u_{xx} + 4u_{xy} + (x^2+4y^2) u_{yy} = \sin(x+y)$.

$A = 1$, $B = 4$, $C = x^2+4y^2$.

$B^2 - 4AC = 16 - 4(x^2+4y^2)$

$= 4[4 - x^2 - 4y^2]$

The equation is elliptic if $B^2 - 4AC < 0$

(ie) $4 - x^2 - 4y^2 < 0$

$4 < x^2 + 4y^2$

$\frac{x^2}{4} + \frac{y^2}{1} > 1.$

$\therefore$ it (equ) is elliptic outside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} =$

The equation is hyperbolic if $B^2 - 4AC > 0$

$\frac{x^2}{4} + \frac{y^2}{1} < 1.$

$\therefore$ it is hyperbolic inside the ellipse $\frac{x^2}{4} +$

It is parabolic on the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$.

$$\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}-12\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+7u=x^2+y^2$$ $=0$ parabolic

$(1+x^2)f_{xx}+(5+2x^2)f_{xy}+(4+x^2)f_{yy}=2\sin(x+y)$ 9 hy.

Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y}=0 \rightarrow -4$ e

Poisson equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$ -4 e

One dimensional heat equation $\alpha^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$ p

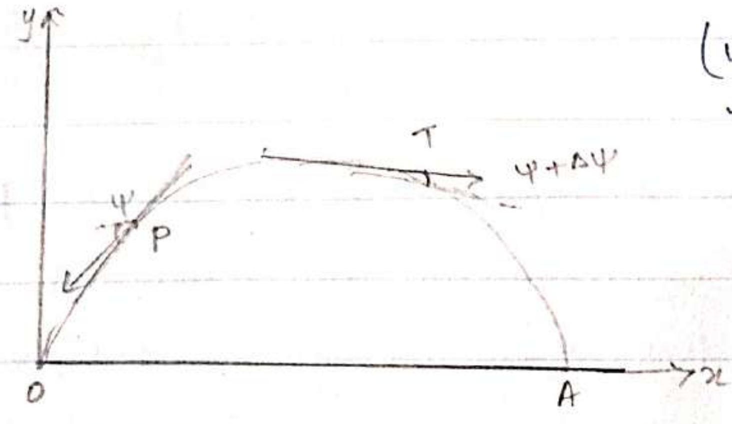
One dimensional wave equation $\alpha^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$ $4\alpha^2>0$ hy.

One of the most fundamental and common phenomena that is found in nature is the phenomenon of wave motion. When a stone is dropped into a pond, the surface of water is disturbed and waves of displacement travel radially outward. When a bell or tuning fork is struck, sound waves are propagated from the source of sound. The electrical oscillations of a radio antenna generate electromagnetic waves that are propagated thro' space. Whatever be the nature of wave phenomenon, whether it be the displacement of a tightly stretched string, the deflection of a stretched membrane, the propagation of currents and potentials along an electrical transmission line or the propagation of currents and potentials along an electrical transmission line or the propagation of ~~etc~~ electromagnetic waves in free space, these entities are governed by a certain partial differential equation, known as the wave equation.

Transverse vibrations of a stretched string

– One dimensional wave equation

Let us derive the p.d.e. governing small transverse vibrations of an elastic string which is stretched to a length l, and then fixed at its 2 ends O & A.



(we use partial derivative because y is a fun of x & t)

The end O of the string is taken as the origin, the position of the string at equilibrium as the x-axis and the line thro' perpendicular to the x-axis and lying in the plane of motion the string as the y-axis.

By disturbing the equilibrium of the string at a certain instant, say $t=0$, it is allowed to vibrate transversely, (ie) at right angles to the equilibrium position of the string, in the xy-plane. Our aim is to study the vibrations of the string, i.e., to find the deflection (displacement) of the string $y(x,t)$ at any pt x and at any time $t>0$.

In order to derive the p.d.e. satisfied by (x,t) in the simplest form, we make the following assumptions.

The motion takes place entirely in one plane. This plane is chosen as the xy plane

1. The tension T caused by stretching the string before fixing it at the end pts is constant at all pts of the deflected string and at all times.
2. T is so large that other external forces such as weight of the string and friction may be considered negligible.
3. The string is homogeneous (i.e. the mass of the string per unit length is constant) and perfectly elastic and so ~~on~~ does not offer resistance to bending.
4. Deflection y and the slope $\frac{\partial y}{\partial x}$ at every pt of the string are small, so that their higher powers may be neglected.

Let us consider the motion of an element PQ of the string, where $P(x,y)$ and $Q(x+\Delta x, y+\Delta y)$ are two neighbouring pts. Let ψ and $\psi+\Delta\psi$ be the angles made by the tgts at P & Q resp. with the x-axis and $\overset{\frown}{PQ} = \Delta s$.

Acceleration of PQ in the positive y direction $\Big\} = \dfrac{\partial^2 y}{\partial t^2}$.

The force acting on PQ in the positive direction $\Big\} = m\,\Delta s \dfrac{\partial^2 y}{\partial t^2}$ —— (1)

by Newton's second law, where m is the mass per unit length of the string.

The actual external force acting on PQ in the positive y direction

$$= T\sin(\psi+\Delta\psi) - T\sin\psi$$

$$= T\left[(\psi+\Delta\psi)-\psi\right]$$

$$= T\Delta\psi. \quad \text{—— (2)}$$

Equating (1) & (2), we get the equation of motion of the element PQ as
$$m\frac{\partial^2 y}{\partial t^2} = T\frac{\Delta\psi}{\Delta s}.$$

Taking limits on both sides as ~~P → Q (ie)~~ $Q \to P$ (ie) $\Delta s \to 0$,

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\lim_{\Delta s\to 0}\frac{\Delta\psi}{\Delta s}$$

$$= \frac{T}{m}\frac{d\psi}{ds}. \quad \text{—— (3)}$$

$\dfrac{d\psi}{ds}$ = Curvature at P of the deflection curve.

$$= \frac{\dfrac{\partial^2 y}{\partial x^2}}{\left[1+\left(\dfrac{\partial y}{\partial x}\right)^2\right]^{3/2}}$$

$$= \frac{\partial^2 y}{\partial x^2} \quad \text{—— (4)} \quad (\text{by assumption (4)})$$

$$a^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass per unit length of the string}}$$

Since T and m are both positive, $\frac{T}{m}$ is positive and hence $\frac{T}{m}$ Can be taken as a^2 — (5)

Sub (4) & (5) in (3), we get the p.d.e. of the vibrating string as

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

This is known as the one dimensional wave equation.

<u>Solution of the wave equation:</u>

By the method of separation of variables.

Consider the equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$. — (1)

Let $y(x,t) = X(x) . T(t)$ be a solution of (1), where $X(x)$ is a function of x only and $T(t)$ is a function of t only.

Then $\frac{\partial y}{\partial t} = XT'$ $\quad \frac{\partial y}{\partial x} = X'T$

$\frac{\partial^2 y}{\partial t^2} = XT''$ $\quad \frac{\partial^2 y}{\partial x^2} = X''T$. where ~~das~~ dashes denote ordinary derivatives with respect to the concerned variables

(1) becomes,

$$XT'' = a^2 X'' T$$

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} \quad \text{— (2)}$$

The L.H.S. of (2) is a function of x only whereas the R.H.S. is a function of time t only. But x & t are independent ~~tin~~ variables, Hence ② is true only if each is equal to a Constant.

$\therefore \quad \frac{X''}{X} = \frac{T''}{a^2T} = k$ (say) where k is any Constant.

$\Rightarrow X'' - kX = 0$ & $T'' - ka^2T = 0.$ } — ③

Case 1: Let $k = \lambda^2$, a positive value

③ $\Rightarrow \quad X'' - \lambda^2 X = 0 \quad$ & $\quad T'' - a^2\lambda^2 T = 0.$

$m^2 - \lambda^2 = 0 \qquad m^2 - a^2\lambda^2 = 0$

$m^2 = \lambda^2 \qquad m^2 = a^2\lambda^2$

$m = \pm\lambda \qquad m = \pm a\lambda.$

$X = A_1 e^{\lambda x} + B_1 e^{-\lambda x}$ & $T = C_1 e^{\lambda a t} + D_1 e^{-\lambda a t}$

Case 2: Let $k = -\lambda^2$, a negative number

③ $\Rightarrow \quad X'' + \lambda^2 X = 0 \quad$ & $\quad T'' + a^2\lambda^2 T = 0$

$m^2 + \lambda^2 = 0 \qquad m^2 = -a^2\lambda^2$

$m^2 = -\lambda^2 \qquad m = \pm i a\lambda$

$m = \pm i\lambda$

$\therefore \quad X = A_2 \cos\lambda x + B_2 \sin\lambda x$

$T = C_2 \cos a\lambda t + D_2 \sin a\lambda t$

Case 3: Let $k = 0.$

③ $\Rightarrow \quad X'' = 0 \quad$ & $\quad T'' = 0$

$m^2 = 0 \qquad m^2 = 0$

$X = A_3 x + B_3 \qquad T = C_3 t + D_3$

Thus the various possible solutions of the wave equation are

$y = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 e^{\lambda a t} + D_1 e^{-\lambda a t})$ —— I

$y = (A_2 \cos\lambda x + B_2 \sin\lambda x)(C_2 \cos\lambda a t + D_2 \sin\lambda a t)$ —— II

$$y = (A_3 x + B_3)(C_3 t + D_3) \quad \text{——— (III)}$$

Out of these solutions, we have to select that particular solution which suits the physical nature of the problem and the given boundary conditions. In the case of vibration of string, it is evident that y must be a periodic function of x and t. Hence we select the solution (II) as the probable solution of the wave equation. The constants are determined by using the boundary conditions in the problem. In doing problems, we shall select the solution (II) directly.

Next probl.

Problems on vibrating string with zero initial velocity: (DT.0)

Type III, Non-zero, IY, $y(x,0)$ & $\left(\frac{\partial y}{\partial t}\right)$ both non-zero.

A tightly stretched string with fixed end pts $x=0$ & $x=l$ is initially in the position $y=f(x)$. It is set vibrating by giving to each of its pts a velocity $\frac{\partial y}{\partial t} = g(x)$ at $t=0$. Find $y(x,t)$ in the form of Fourier Series.

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$.

The boundary conditions are

i) $y(0,t)=0$ } There is no displacement at the end pts
ii) $y(l,t)=0$ }

iii) $y(x,0) = f(x), \quad 0<x<l$

iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x), \quad 0<x<l.$

The solution is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \quad \text{—— (1)}$$

Applying i)

$$0 = A(C\cos\lambda at + D\sin\lambda at) \Rightarrow A = 0.$$

$$(1) \Rightarrow y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \quad \text{—— (2)}$$

Applying ii)

$$0 = B\sin\lambda l(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow \sin\lambda l = 0 = \sin n\pi$$

$$\Rightarrow \lambda l = n\pi$$

$$\Rightarrow \lambda = \frac{n\pi}{l}.$$

$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left[C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right]$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[B_n\cos\frac{n\pi at}{l} + C_n\sin\frac{n\pi at}{l}\right] \quad \text{—— (3)}$$

Applying iii)

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

This is a Fourier Sine Series.

$$\therefore B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[B_n\left(-\sin\frac{n\pi at}{l}\right)\cdot\frac{n\pi a}{l} + C_n\cos\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)\right]$$

Applying iv)

$$g(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\cdot\left[C_n \frac{n\pi a}{l}\right]$$

This is a Fourier Sine Series.

Note:1 The B.C. with non zero value on the R.H.S. should be as the last B.C.

after getting the most g.s., we should use the non zero boundary Condition.

2. $\therefore C_n \frac{n\pi a}{l} = \frac{2}{l}\int_0^l g(x) \sin\frac{n\pi x}{l}\,dx$

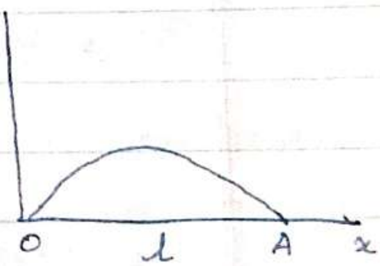
$$C_n = \frac{2}{n\pi a}\int_0^l g(x) \sin\frac{n\pi x}{l}\,dx$$

Sub. the values of B_n and C_n in (5), we get the solution of the wave equation satisfying the given boundary Conditions.

2. A string is stretched and fastenend to two pts $x=0$ & $x=l$. apart. Motion is started by displacing the string into the form $y = k(lx-x^2)$ from which it is released at time $t=0$. Find the displacement of any pt on the string at a distance of x from one end at time t.

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$.

The boundary conditions are



i) $y(0,t)=0$ for $t\geq 0$

ii) $y(l,t)=0$ for $t\geq 0$

iii) $\frac{\partial y}{\partial t}(x,0) = 0$ ($\because$ I.V is zero)

iv) $y(x,0) = k(lx-x^2)$.

The solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \quad \text{——} (1)$$

Applying i), we get

$$0 = A(C\cos\lambda at + D\sin\lambda at)$$

$\Rightarrow A = 0$

① $\Rightarrow y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at)$ —— ②

Applying ii), $0 = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at)$

$\Rightarrow \quad \sin\lambda l = 0 = \sin n\pi \quad \Rightarrow \lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}$

② $\Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right)$ —— ③

$$\frac{\partial y}{\partial t} = B\sin\frac{n\pi x}{l}\left\{C\left(-\sin\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right) + D\cos\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)\right\}$$

Applying iii) $\quad 0 = B\sin\frac{n\pi x}{l}\left\{D.\frac{n\pi a}{l}\right\}$

$B \neq 0$, $\frac{n\pi a}{l} \neq 0$ ($\because$ all Consts) $\sin\frac{n\pi x}{l} \neq 0$ ($\because$ It is defined for all x.

$\therefore D = 0.$

③ $\Rightarrow y(x,t) = BC\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.$

$$\cancel{= B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}} \quad \text{—— ④}$$

The most g.s. is

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

Applying iv)

$$k(lx - x^2) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}$$

This is a Fourier Sine Series.

$$B_n = \frac{2}{l}\int_0^l k(lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left\{(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right\}_0^l$$

$$= \frac{2k}{l}\left\{\frac{-2\cdot l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right\}$$

$$= \frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{8kl^2}{n^3\pi^3}, & \text{when } n \text{ is odd} \end{cases}$$

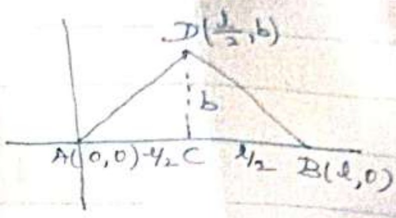
Sub in (4),

$$y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1,3\ldots}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.$$

A string is tightly stretched and its ends are fastened at pts $x=0$ & $x=l$. The midpt of the string is displaced transversely thro' a small distance 'b' and the string is released from rest in that position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.

To find the equation of the string in its initial position.

The equation of the string (line) AD is



$$\frac{x-0}{0-\frac{l}{2}} = \frac{y-0}{0-b}$$

$$bx = \frac{yl}{2} \Rightarrow y = \frac{2bx}{l}, \quad 0 < x < \frac{l}{2}.$$

The equation of DB is

$$\frac{x-\frac{l}{2}}{\frac{l}{2}-l} = \frac{y-b}{b-0}$$

$$b\left(x-\frac{l}{2}\right) = (y-b)\left(\frac{l}{2}-l\right)$$

$$y-b = \frac{bx-\frac{lb}{2}}{\frac{l}{2}-l}$$

$$= \frac{2bx-lb}{l-2l} = \frac{2bx-lb}{-l}$$

$$\Rightarrow \quad y = \frac{bl-2bx}{l} + b$$

$$= \frac{2bl-2bx}{l} = \frac{2b}{l}(l-x).$$

Hence ~~the~~ initially the displacement of the string is in the form

$$y(x,0) = \begin{cases} \frac{2bx}{l}, & 0 < x < l/2 \\ \frac{2b}{l}(l-x), & \frac{l}{2} < x < l \end{cases}$$

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ —— (1)

The boundary conditions are

i) $y(0,t)=0$

ii) $y(l,t)=0$

iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0}=0$

iv) $y(x,0) = \begin{cases} \frac{2bx}{l}, & 0<x<\frac{l}{2} \\ \frac{2b}{l}(l-x), & \frac{l}{2}<x<l. \end{cases}$

Applying first (3) B.C.'s the most g.s. is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l} \quad \text{——— (2)}$$

Applying iv),

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

This is a Fourier sine series.

$$\therefore B_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2bx}{l} \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2b}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2b\,2}{l\,l}\left\{\left[\cancel{\frac{2b}{l}}x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \cancel{\frac{2b}{l}}\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/2} + \left[\cancel{\frac{2b}{l}}(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \cancel{\frac{2b}{l}}(-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{4b}{l^2}\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

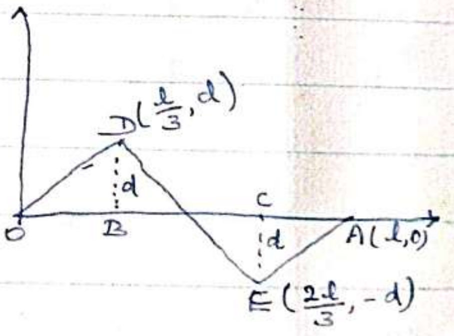
$$= \frac{4b}{l^2}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{8bl}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Sub in (2), the most g.s. is

$$y(x,t) = \frac{8bl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}.$$

The pts of trisection of a tightly stretched string of length l with fixed ends are pulled aside thro' a distance d on opposite sides of the position of equilibrium, and the string is released from rest. Obtain an expression for the displacement of the string at any subsequent time and show that the midpt of the string always remains at rest.

Let B and C be the pts of trisection of the string OA. The initial position of the string is shown by the lines ODEA, where $BD = CE = d$



The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}.$$

The boundary Conditions are

i) $y(0,t)=0$

ii) $y(l,t)=0$

iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.$

To find the initial position of the string, we require the eqn of ODEA.

The equation of OD is $\dfrac{x-0}{0-\frac{l}{3}} = \dfrac{y-0}{0-d}$

$$xd = \frac{yl}{3} \Rightarrow y = \frac{3dx}{l}$$

The equation of DE is

$$\frac{x-\frac{l}{3}}{\frac{2l}{3}-\frac{l}{3}} = \frac{y-d}{-d-d}$$

$$\frac{x-\frac{l}{3}}{\frac{l}{3}} = \frac{y-d}{-2d}$$

$$-2dx + \frac{2dl}{3} = \frac{yl}{3} - \frac{dl}{3}$$

$$= (y-d)\frac{l}{3}$$

$$(-6dx + 2dl)\frac{1}{l} = y-d$$

$$y = \frac{-6dx + 2dl}{l} + d$$

$$= \frac{-6dx + 3dl}{l}$$

$$= \frac{3d}{l}(l-2x)$$

Equation of EA is $\dfrac{x-\frac{2l}{3}}{l-\frac{2l}{3}} = \dfrac{y+d}{0+d}$

$$\left(\frac{3x-2l}{3}\right)d = (y+d)\left(\frac{l}{3}\right)$$

$$y + d = \frac{3dx - 2dl}{l}$$

$$y = \frac{3dx - 3dl}{l}$$

$$= \frac{3d}{l}(x - l)$$

The 4th I.C. is

$$y(x,0) = \begin{cases} \frac{3dx}{l}, & 0 < x < \frac{l}{3} \\ \frac{3d}{l}(l - 2x), & \frac{l}{3} < x < \frac{2l}{3} \\ \frac{3d}{l}(x - l), & \frac{2l}{3} < x < l \end{cases}$$

Applying 1st 3 B.C's, we get the most g.s.

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}$$

Applying iv)

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = y(x,0)$$

This is F. Sine Series.

$$\therefore \quad B_n = \frac{2}{l}\int_0^l y(x,0)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left\{\int_0^{l/3} \frac{3d}{l} x \sin\frac{n\pi x}{l}\,dx + \frac{3d}{l}\int_{l/3}^{\frac{2l}{3}} (l - 2x)\sin\frac{n\pi x}{l}\,dx \right.$$

$$\left. + \frac{3d}{l}\int_{\frac{2l}{3}}^{l} (x - l)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{6d}{l^2}\left\{\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-1.\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{\frac{l}{3}}\right.$$

$$+\left[(l-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{\frac{l}{3}}^{\frac{2l}{3}}$$

$$\left.+\left[(x-l)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-1.\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{\frac{2l}{3}}^{l}\right\}$$

$$= \frac{6d}{l^2}\left[\frac{l}{3}\left(-\cos\frac{n\pi}{3}\right)\frac{l}{n\pi}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right.$$

$$+\left(l-\frac{4l}{3}\right)\left(-\cos\frac{2n\pi}{3}\right)\frac{l}{n\pi}-\frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}$$

$$+\left(l-\frac{2l}{3}\right)\cos\frac{n\pi}{3}\left(\frac{l}{n\pi}\right)+\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$$\left.0+\left(\frac{2l}{3}-l\right)\left(\cos\frac{2n\pi}{3}\right)\frac{l}{n\pi}-\frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right]$$

$$= \frac{6d}{l^2}\left[\cancel{-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}}+\frac{3l^2}{n^2\pi^2}\sin\frac{n\pi}{3}+\cancel{\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}}\right.$$

$$\left.-\frac{3l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}+\cancel{\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}}-\cancel{\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}}\right]$$

$$= \frac{6d}{l^2}\left[\frac{3l^2}{n^2\pi^2}\sin\frac{n\pi}{3}-\frac{3l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right]$$

We allow the string to vibrate by taking it to some position and then released from rest. ∴ in that case the initial vel. $\frac{\partial y}{\partial t}$ at $t=0$ is zero. We may allow the string to vibrate by giving

$$= \frac{6d}{l^2} \cdot \frac{3l^2}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right]$$

$$= \frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \sin\left(n\pi - \frac{n\pi}{3}\right)\right]$$

$\sin(n\pi)\cos\frac{n\pi}{3} - \cos n\pi \sin\frac{n\pi}{3}$

$$= \frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3} + \cos n\pi \sin\frac{n\pi}{3}\right]$$

$$= \frac{18d}{n^2\pi^2}\sin\frac{n\pi}{3}\left[1 + (-1)^n\right]$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{36d}{n^2\pi^2}\sin\frac{n\pi}{3}, & \text{if } n \text{ is even.} \end{cases}$$

$$y(x,t) = \frac{36d}{\pi^2}\sum_{n=2,4,\ldots}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$= \frac{36d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\sin\frac{2n\pi}{3}\sin\frac{2n\pi x}{l}\cos\frac{2n\pi at}{l}$$

$$= \frac{36d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{4n^2}\sin\frac{2n\pi}{3}\sin\frac{2n\pi x}{l}\cos\frac{2n\pi at}{l}$$

$$= \frac{9d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{2n\pi}{3}\sin\frac{2n\pi x}{l}\cos\frac{2n\pi at}{l}.$$

By putting $x = \frac{l}{2}$, we get the displacement of the midpt.

$y\left(\frac{l}{2}, t\right) = 0$ since $\sin\frac{2n\pi x}{l}$ becomes $\sin n\pi = 0$ when $x = \frac{l}{2}$

∴ The midpt of the string is at rest.

velocity to the string in its equilibrium position. ∴ the velocity may be ~~the~~ a function of x and hence there displacement at time $t=0$. (ie) $y(x,0) = 0 \; \forall x$.

Problems on vibrating string with nonzero initial velocity.

A tightly stretched string with fixed end pts $x=0$ & $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each pt a velocity $3x(l-x)$, find its displacement.

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ — (1)

The boundary conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $y(x,0) = 0$

iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 3x(l-x)$

Let

$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$ be the solution of (1). — (2)

Applying i), $0 = A(C\cos\lambda at + D\sin\lambda at)$

$\Rightarrow A = 0$

(2) $\Rightarrow y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at)$ — (3)

Applying ii) $0 = B\sin\lambda l(C\cos\lambda at + D\sin\lambda at)$

$\Rightarrow \sin\lambda l = 0 = \sin n\pi$

$\lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}$

(3) $\Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left[C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right]$ — (4)

Applying iii), $0 = BC \sin\frac{n\pi x}{l}$

$\Rightarrow C = 0.$

(4) (5) $\Rightarrow y(x,t) = BD \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \qquad \text{—— (5)}$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l}$$

$$3x(l-x) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l}$$

This is Fourier Sine Series of $3x(l-x)$ in $(0,l)$

$$\therefore \quad B_n \frac{n\pi a}{l} = \frac{2}{l}\int_0^l 3x(l-x) \sin\frac{n\pi x}{l}\,dx$$

$$B_n = \frac{2}{n\pi a}\int_0^l 3x(l-x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{6}{n\pi a}\int_0^l (lx-x^2) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{6}{n\pi a}\left\{ (lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right\}_0^l$$

$$= \frac{6}{n\pi a}\left\{\frac{-2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right\}$$

$$= \frac{12l^3}{n^4\pi^4 a}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{24l^3}{n^4\pi^4 a}, & \text{if } n \text{ is odd.} \end{cases}$$

Sub. in (5),

$$y(x,t) = \frac{24l^3}{a\pi^4}\sum_{n=1,3..}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= \frac{24l^3}{a\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{l}\sin\frac{(2n-1)\pi at}{l}$$

A string is stretched between 2 fixed pts at a distance $2l$ apart and the pts of the string are given I.V.'s v where

$$v = \begin{cases} \frac{cx}{l}, & 0<x<l \\ \frac{c}{l}(2l-x), & l<x<2l \end{cases}$$

x being the distance from an end pt. Find the displacement of the string at any subsequent time.

Solution: The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$.

The boundary conditions are

i) $y(0,t)=0$

ii) $y(2l,t)=0$

iii) $y(x,0)=0$

iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \begin{cases} \frac{cx}{l}, & 0<x<l \\ \frac{c}{l}(2l-x), & l<x<2l \end{cases}$

~~Applying i)~~

The solution is given by

$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$ —— ①

Applying i)

$0 = A(C\cos\lambda at + D\sin\lambda at)$

$\Rightarrow A = 0.$

① $\Rightarrow y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at)$

Applying ii)

$0 = B\sin 2\lambda l(C\cos\lambda at + D\sin\lambda at)$

$\Rightarrow \sin 2\lambda l = 0 = \sin n\pi$

$2\lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{2l}.$

$\therefore y(x,t) = B\sin\frac{n\pi x}{2l}\left[C\cos\frac{n\pi at}{2l} + D\sin\frac{n\pi at}{2l}\right]$ —— ②

Applying iii)

$0 = BC\sin\frac{n\pi x}{2l}$

$\Rightarrow C = 0$

② $\Rightarrow y(x,t) = BD\sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l}$

The most g.s. is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l} \sin\frac{n\pi at}{2l} \quad \text{--- (3)}$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l} \cos\frac{n\pi at}{2l}\left(\frac{n\pi a}{2l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{2l} \sin\frac{n\pi x}{2l}$$

This is a F. Sine Series.

$$\frac{n\pi a}{2l} B_n = \frac{2}{2l}\int_0^{2l} f(x) \sin\frac{n\pi x}{2l}\, dx$$

$$= \frac{1}{l}\left\{\frac{c}{l}\int_0^{l} x \sin\frac{n\pi x}{2l}\, dx + \frac{c}{l}\int_l^{2l} (2l-x) \sin\frac{n\pi x}{2l}\, dx\right\}$$

$$= \frac{c}{l^2}\left\{\left[x\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right]_0^l\right.$$

$$\left. + \left[(2l-x)\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right]_l^{2l}\right\}$$

$$= \frac{c}{l^2}\left\{-\frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right.$$

$$\left. + \frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{c^2}{l^2} \cdot \frac{8l^2}{n^2\pi^2} \sin\frac{n\pi}{2}$$

$$B_n = \frac{8c^2}{n^2\pi^2} \left(\frac{2l}{n\pi a}\right) \sin\frac{n\pi}{2}$$

$$= \frac{16cl}{an^3\pi^3} \sin\frac{n\pi}{2}.$$

Sub in (3),

$$y(x,t) = \frac{16cl}{a\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{2l} \sin\frac{n\pi at}{2l}$$

Solve the problem of the vibrating string for the following boundary conditions.

Type(ii)

i) $y(0,t) = 0$ ii) $y(l,t) = 0$

iii) $\frac{\partial y}{\partial t}(x,0) = x(x-l)$, $0 < x < l$

iv) $y(x,0) = \begin{cases} x, & 0 < x < \frac{l}{2} \\ l-x, & \frac{l}{2} < x < l. \end{cases}$

$y(x,0)$ & $\left(\frac{\partial y}{\partial t}\right) \neq 0$

The solution is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \quad (1)$$

Applying i),

$$0 = A(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow A = 0.$$

$$(1) \Rightarrow y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at)$$

Applying ii)

The most g.s. is
$$y(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[C_n \cos\frac{n\pi a t}{l} + D_n \sin\frac{n\pi a t}{l}\right] \quad (2)$$

$$0 = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at)$$
$$\Rightarrow \sin\lambda l = 0 = \sin n\pi \Rightarrow \lambda = \frac{n\pi}{l}$$
$$\therefore y(x,t) = B\sin\frac{n\pi x}{l}\left[C\cos\frac{n\pi a t}{l} + D\sin\frac{n\pi a t}{l}\right]$$
$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[C_n\left(-\sin\frac{n\pi a t}{l}\right)\cdot\frac{n\pi a}{l} + D_n\cos\frac{n\pi a t}{l}\left(\frac{n\pi a}{l}\right)\right]$$

Applying iii)
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\, D_n \frac{n\pi a}{l}$$
$$x(x-l) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l}\left(\frac{n\pi a}{l}\right)$$

This is a F. Sine Series.

$$\frac{n\pi a}{l} D_n = \frac{2}{l}\int_0^l x(x-l)\sin\frac{n\pi x}{l}\,dx.$$

$$D_n = \frac{2}{n\pi a}\int_0^l (x^2 - lx)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{n\pi a}\left\{(x^2-lx)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2x-l)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right\}_0^l$$

$$= \frac{2}{n\pi a}\left\{\frac{2l^3}{n^3\pi^3}(-1)^n - \frac{2l^3}{n^3\pi^3}\right\}$$

$$= \frac{4l^3}{n^4\pi^4 a}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{-8l^3}{n^4\pi^4 a}, & \text{if } n \text{ is odd} \end{cases}$$

Applying iv), in ②

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}$$

This is a Fourier Sine Series.

$$C_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{2}{l}\left\{\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{2}{l}\left\{\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1.\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/2} + \left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{2}{l}\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{2}{l}\left\{\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4l}{n^2\pi^2} \sin\frac{n\pi}{2}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore\ y(x,t) = \sum_{n=1,3,\ldots}^{\infty} \frac{4l}{n^2\pi^2} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}$$

$$- \frac{8l^3}{\pi^4 a} \sum_{n=1,3..}^{\infty} \frac{1}{n^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}$$

If a String of length l is initially at rest in equillibrium position and each pt of it is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3\frac{\pi x}{l}$, $0 < x < l$, determine the transverse displacement of $y(x,t)$ $\quad \sin 3A = 3\sin A - 4\sin^3 A$. p-19

$y(0,t) = 0 \quad y(l,t) = 0 \quad y(x,0) = 0, \quad \left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3\frac{\pi x}{l}$

After applying 1st 3 B.C's the most general Solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l} \quad \text{---} (I)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l} \left(\frac{n\pi a}{l}\right)$$

Applying Condition (iv) $\quad \sin 3A = 3\sin A - 4\sin^3 A$

$$\sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} = v_0 \sin^3\frac{\pi x}{l}$$

$$\sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} = \frac{v_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

Comparing like terms we get

$$B_1 \cdot \frac{\pi a}{l} = \frac{3v_0}{4}, \quad B_3 \frac{3\pi a}{l} = -\frac{v_0}{4}$$

$$B_1 = \frac{3v_0 l}{4\pi a} \qquad B_3 = \frac{-v_0 l}{12\pi a}$$

$B_n = 0$, $n \neq 1, n \neq 3$.

Sub in (I),

$$y(x,t) = \frac{3v_0 l}{4\pi a} \sin\frac{\pi x}{l} \sin\frac{\pi a t}{l} - \frac{v_0 l}{12\pi a} \sin\frac{3\pi x}{l} \sin\frac{3\pi a t}{l}$$

Zero IV A tightly stretched string with end pts $x=0$ & $x=l$ is initially in a position given by $y(x,0) = y_0 \sin\frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x,t)$ at any pt of the string.

The B.C's are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$ iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

iv) $y(x,0) = y_0 \sin\frac{\pi x}{l}$.

Applying 1st 3 B.C's we have

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}$$

Applying iv $y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$

$$\sum B_n \sin\frac{n\pi x}{l} = y_0 \sin\frac{\pi x}{l}$$

Comparing like terms

$B_1 = y_0 \qquad B_n = 0, \; n \neq 1.$

$\therefore \; y(x,t) = y_0 \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l}.$

1. A string is stretched between 2 fixed pts at a distance of l cm and the pts of the string are given initial velocity $v = \lambda(lx - x^2)$, for $0 < x < l$. Find the displacement function $y(x,t)$.

$$y(x,t) = \frac{8\lambda l^3}{\pi^4 a} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi x}{l} \sin \frac{(2n-1)\pi a t}{l}$$

2. The ends of a uniform string of length $2l$ are fixed. The initial displacement is $y(x,0) = kx(2l - x)$, $0 < x < 2l$, while the initial velocity is zero. Find the displacement at any distance x from the end $x = 0$ at any time t.

$$y(x,t) = \frac{32kl^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{2l} \cos \frac{(2n-1)\pi a t}{2l}.$$

One dimensional heat equation

We consider the flow of heat and the accompanying variation of temperature with position and with time in conducting solids.

Heat flows from a higher to lower temperature. The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and to the temperature change. This constant of proportionality is known as the specific heat (c) of the conducting material.

The rate at which heat flows thro' an area is proportional to the area and to the temperature gradient normal to the area. This ~~φ~~ constant of proportionality is known as the thermal conductivity (k) of the material. This is [The rate of change of temperature with respect to distance is called temperature gradient and is denoted by $\frac{\partial u}{\partial x}$] known as Fourier's law of heat conduction.

$\frac{k}{\rho c}$ is a positive value and hence we consider α^2 instead of α.

The one dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } \alpha^2 = \frac{k}{\rho c} \quad k\text{- thermal conductivity}$$

$\frac{k}{\rho c}$ is called diffusivity of the substance ρ - density, c - specific heat

<u>Solution of one dimensional heat equation</u>

<u>Mtd of Separation of variables.</u>

We know that one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

Let $u(x,t) = X(x)\,T(t)$ --- (2) be the solution of (1), where X is a function of x only and T is a function of t only.

From (2) $\frac{\partial u}{\partial x} = X'T \qquad \frac{\partial^2 u}{\partial x^2} = X''T$

$\frac{\partial u}{\partial t} = XT' \qquad \cancel{\frac{\partial^2 u}{\partial t^2} = XT''}$

(1) $\Rightarrow$ ~~u(x,t)~~ $XT' = \alpha^2 X''T$

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = k$$

$X'' - kX = 0$, $T' - \alpha^2 kT = 0$

case i) $k = \lambda^2$, a positive no.

$X'' - \lambda^2 X = 0 \qquad \frac{dT}{dt} = \alpha^2\lambda^2 T$

$m^2 = \lambda^2 \qquad \frac{dT}{T} = \alpha^2\lambda^2\,dt$

$X = A_1 e^{\lambda x} + B_1 e^{-\lambda x}$ Integrating, $\log T = \alpha^2\lambda^2 t + \log c_1$

$\log\left(\frac{T}{c_1}\right) = \alpha^2\lambda^2 t$

$\frac{T}{c_1} = e^{\alpha^2\lambda^2 t}$

$\Rightarrow T = c_1 e^{\alpha^2\lambda^2 t}$

Case ii) $k = -\lambda^2$, a -ve no.

$X'' + \lambda^2 X = 0 \quad T' + \alpha^2\lambda^2 T = 0$

$m^2 = -\lambda^2$

$X = A_2 \cos\lambda x + B_2 \sin\lambda x \qquad T = C_2 e^{-\alpha^2\lambda^2 t}$

Case iii) $k = 0$

$X'' = 0, \quad T' = 0.$

$X = A_3 x + B_3, \quad T = C_3.$

The possible solutions of (1) are $(A e^{\lambda x} + B e^{-\lambda x}) e^{\alpha^2\lambda^2 t}$

$u(x,t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x}) C_1 e^{\alpha^2\lambda^2 t}$ ——— I

$u(x,t) = (A_2 \cos\lambda x + B_2 \sin\lambda x) C_2 e^{-\alpha^2\lambda^2 t}$ ——— II

$u(x,t) = (A_3 x + B_3) C_3$ ——— III

Note: Consider I. If t increases, then $u(x,t)$ also increases i.e. If $t \to \infty$, $u(x,t) \to \infty$. $\therefore$ It is not a correct solution.

$u(x,t)$ must decrease with increase of time. $\therefore$ II is the correct solution.

In the steady state conditions, when the temperature no longer varies with time, the solution of (1) is III.

In unsteady state, the temperature at any pt of the body depends on the position of the pt and also the time t. In steady state, the temperature at any pt depends only on the position of the pt and is independent of the time t.

1 Dimensional wave equ.	One dimensional heat equ.
1. It is a hyperbolic p.d.e	It is a parabolic p.d.e.
2. Wave motion is a periodic motion with respect to 't' and hence in the solution of ① there will be trignometric terms in t.	The solution $u(x,t)$ of the heat equation ② is a transient (passes away quickly) Solution i.e. u decrease with increase of time.
3. $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$	$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$
4. 4 B.C's	3 B.C's.

Find the solution to the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ that satisfies the Conditions

i) $u(0,t) = 0$

ii) $u(l,t) = 0$

iii) $u(x,0) = \begin{cases} x, & 0 < x < \frac{l}{2} \\ l-x, & \frac{l}{2} < x < l \end{cases}$

The heat equation is given by $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ —— (1)

The B.C's are i) $u(0,t) = 0$

ii) $u(l,t) = 0$

iii) $u(x,0) = \begin{cases} x, & 0 < x < \frac{l}{2} \\ l-x, & \frac{l}{2} < x < l \end{cases}$

Let the solution of (1) be $u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t}$ —— (2)

Applying i), $0 = Ae^{-\alpha^2\lambda^2 t} \Rightarrow \boxed{A=0}$

(2) $\Rightarrow$ $u(x,t) = B\sin\lambda x \; e^{-\alpha^2\lambda^2 t}$ —— (3)

Applying ii), $0 = B\sin\lambda l \; e^{-\alpha^2\lambda^2 t}$

$\Rightarrow \sin\lambda l = 0 = \sin n\pi$

$\Rightarrow \lambda l = n\pi \Rightarrow \boxed{\lambda = \frac{n\pi}{l}}$

(3) $\Rightarrow$ $u(x,t) = B\sin\frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}}$ —— (4)

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2}{l^2}t}$$

Applying iv),

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x) \text{ where } f(x) = \begin{cases} x, & 0 < x < \frac{l}{2} \\ l-x, & \frac{l}{2} < x < l \end{cases}$$

half range
This is a Fourier sine series of f(x) in $(0, l)$.

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left\{\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\cdot\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/2}\right.$$

$$\left. + \left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{2}{l}\left\{\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{2}{l}\left\{\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$\therefore\ u(x,t) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\,e^{-\frac{\alpha^2 n^2\pi^2}{l^2}t}$$

Solve $\frac{\partial \theta}{\partial t} = \alpha^2 \frac{\partial^2 \theta}{\partial x^2}$

i) θ is finite when $t \to \infty$

ii) $\theta = 0$ when $x = 0$ & $x = \pi$ for all values of t.

iii) $\theta = x$ from $x = 0$ to $x = \pi$ when $t = 0$.

The heat equation is $\frac{\partial \theta}{\partial t} = \alpha^2 \frac{\partial^2 \theta}{\partial x^2}$

The B.C's are i) $\theta(0,t) = 0$

ii) $\theta(\pi, t) = 0$

iii) $\theta(x, 0) = x$.

Let the solution be $\theta(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t}$ —(1)

Applying i), $0 = Ae^{-\alpha^2\lambda^2 t} \Rightarrow \boxed{A = 0}$

(1) $\Rightarrow$ $\theta(x,t) = B\sin\lambda x\, e^{-\alpha^2\lambda^2 t}$ —(2)

Applying ii), $0 = B\sin\lambda\pi\, e^{-\alpha^2\lambda^2 t}$

$\Rightarrow \sin\lambda\pi = 0 = \sin n\pi$

$\Rightarrow \lambda\pi = n\pi \Rightarrow \boxed{\lambda = n}$

(2) $\Rightarrow$ $\theta(x,t) = B\sin nx\, e^{-\alpha^2 n^2 t}$

The general ~~of~~ solution is $\theta(x,t) = \sum_{n=1}^{\infty} B_n \sin nx\, e^{-\alpha^2 n^2 t}$

This is a half range Fourier Sine Series

~~App I~~ Applying iii), $\theta(x,0) = \sum_{n=1}^{\infty} B_n \sin nx$

$$B_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$$

$$= \frac{2}{\pi}\left\{ x\left(\frac{-\cos nx}{n}\right) - 1.\left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\frac{-\pi(-1)^n}{n}\right\} = \frac{2}{n}(-1)^{n+1}$$

$$\therefore\ \theta(x,t) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx\, e^{-\alpha^2 n^2 t}$$

A rod l cm with insulated lateral surface is initially at temperature $f(x)$ at an inner pt of distance x cm from one end. If both the ends are kept at zero temperature, find the temperature at any pt of the rod at any subsequent time.

The heat equation is given by $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

The B.C's are i) $u(0,t) = 0$

ii) $u(l,t) = 0$

iii) $u(x,0) = f(x),\ 0 < x < l.$

Let the solution of ① be $u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t}$ — ②

Applying i), $0 = Ae^{-\alpha^2\lambda^2 t} \Rightarrow \boxed{A=0}$

② $\Rightarrow u(x,t) = B\sin\lambda x\, e^{-\alpha^2\lambda^2 t}$ — ③

Applying ii), $0 = B\sin\lambda l\, e^{-\alpha^2\lambda^2 t}$

$\Rightarrow \sin\lambda l = 0 = \sin n\pi \Rightarrow \boxed{\lambda = \frac{n\pi}{l}}$

③ $\Rightarrow u(x,t) = B\sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2\pi^2}{l^2}t}$

The most general solution is $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2}{l^2}t}$

Applying iii), $u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$

$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$

This is a half range F.S.S. $\quad B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx\right)\sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2\pi^2}{l^2}t}$$

Refer back 1 problem below defn of Temp gradient in k/cc.

Steady state Conditions and Zero boundary Conditions.

Suppose a rod is heated at both ends by a Const. temperature c_1 & c_2. After sometime the temperature in the rod remains Constant. Hence there is no change in temperature in the rod if time t varies. ∴ the temperature function $u(x,t)$ is a function of x alone. Or it is independent of time. This state in which the temperature does not vary with respect to time 't' is called steady state. When steady state exists, $u(x,t)$ becomes $u(x)$.

A rod of length l has its ends A and B kept at 0°C & 100°C until steady state condition prevail. If the temperature at B is reduced suddenly to 0°C & kept so while that of A is maintained, find the temperature $u(x,t)$ at a distance x from A and at time 't'.

The heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ —— (1)

When Steady state condition prevail, $u(x,t)$ is a function of x alone.

$$\therefore \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \left(\because \frac{\partial u}{\partial t} = 0, \ u \text{ is free from 't'}\right)$$

Since 'u' is a function of x alone, $\frac{d^2 u}{dx^2} = 0$ —— (2)

The B.C's are i) $u(0) = 0$

ii) $u(l) = 100$

Solution of (2) is $u(x) = ax + b$ —— (3)

Applying i), $0 = b$

Sub $b = 0$ in (3), $u(x) = ax$ —— (4)

Applying ii), $100 = al \Rightarrow a = \frac{100}{l}$

(4) $\Rightarrow$ $u(x) = \frac{100x}{l}$

This is the temperature function in the steady state. The end B is reduced to zero. For the unsteady state, the initial temperature distribution is $u(x) = \frac{100x}{l}$

The B.C's for unsteady state are

i) $u(0,t) = 0$

ii) $u(l,t) = 0$

iii) $u(x,0) = \frac{100x}{l}$

Let $u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t}$ be the solution of (1) —— (5)

Applying i), $0 = Ae^{-\alpha^2\lambda^2 t} \Rightarrow A = 0$

(5) $\Rightarrow$ $u(x,t) = B\sin\lambda x\, e^{-\alpha^2\lambda^2 t}$ —— (6)

Applying ii), $0 = B\sin\lambda l\, e^{-\alpha^2\lambda^2 t}$

$\Rightarrow \sin\lambda l = 0 = \sin n\pi \Rightarrow \lambda = \frac{n\pi}{l}$

(6) $\Rightarrow$ $u(x,t) = B\sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2\pi^2}{l^2}t}$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2\pi^2}{l^2}t}$$

Applying iii), $u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$

$$\frac{100x}{l} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

This is a half range F.S.S.

$$B_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l}\,dx$$

$$= \frac{200}{l^2}\int_0^l x \sin \frac{n\pi x}{l}\,dx$$

$$= \frac{200}{l^2}\left\{ x\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1.\left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^l$$

$$= \frac{200}{l^2}\left\{ -\frac{l^2}{n\pi}(-1)^n \right\} = \frac{200(-1)^{n+1}}{n\pi}$$

$$\therefore u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2}{l^2}t}.$$

A rod of length l has its ends A and B kept at $0^\circ C$ & $120^\circ C$ respectively until steady state conditions prevail. If the temp at B is reduced at $0^\circ C$ and kept so while that of A is maintained, find the temp. dis. in the rod.

The heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ —— (1)

When steady state condition prevail, u is a fun of x alone.

$$\frac{d^2u}{dx^2} = 0 \quad \text{—— (2)}$$

The B.C's are i) $u(0)=0$ ii) $u(l)=120$

Sol of (2) is $u(x) = ax+b$ —— (3)

Applying i), $b=0$. (3) $\Rightarrow u = ax$.

Applying ii), $120 = al \Rightarrow a = \frac{120}{l}$. $\therefore u(x) = \frac{120x}{l}$

This is the temp distribution in the steady state. The end B is reduced to zero.

For the unsteady state, the initial temp dis is $u(x) = \frac{120x}{l}$

The B.C's are

i) $u(0,t)=0$ ii) $u(l,t)=0$ iii) $u(x,0) = \frac{120x}{l}$

Let $u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t}$

Applying i) $0 = Ae^{-\alpha^2\lambda^2 t} \Rightarrow A = 0$.

Applying ii) $0 = B\sin\lambda l \, e^{-\alpha^2\lambda^2 t} \Rightarrow \lambda = \frac{n\pi}{l}$.

$$\therefore u(x,t) = B\sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2\pi^2}{l^2} t}$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2}{l^2} t}$$

Applying iii), $\frac{120x}{l} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$

$$B_n = \frac{2}{l}\int_0^l \frac{120x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{240}{l^2}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{240}{l^2}\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\cdot\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^l$$

$$= \frac{240}{l^2}\left\{-\frac{l^2}{n\pi}(-1)^n\right\} = \frac{240(-1)^{n+1}}{n\pi}$$

$$u(x,t) = \frac{240}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2\pi^2}{l^2} t}$$

A rod 30cm long has its ends A and B kept at 20°C & 80°C resp, until steady state conditions prevail. The temp. at each end is then suddenly reduced to 0°C & kept so. Find the resulting temp. function $u(x,t)$ taking $x=0$ at A

The heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ —— ①

When steady conditions prevail, u is a function of x alone.

$$\frac{d^2u}{dx^2} = 0 \text{ —— } ②$$

The B.C's are i) $u(0) = 20$ ii) $u(30) = 80$

Solution of ② is $u = ax+b$

Applying ①, $20 = b$

$\therefore u = ax + 20$

Applying ii), $80 = 30a + 20 \Rightarrow a = \frac{60}{30} = 2$

$\therefore u(x) = 2x+20.$

This is the temp. distribution in the steady state. The ends A & B are reduced to 0.

For the unsteady state, the B.C's are

i) $u(0,t)=0$ ii) $u(30,t)=0$ iii) $u(x,0) = 2x+20$

Let $u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t}$ be the solution of ①

Applying i) $A=0$

Applying ii) $0 = B\sin\lambda 30\, e^{-\alpha^2\lambda^2 t}$

$\Rightarrow \sin 30\lambda = 0 = \sin n\pi \Rightarrow \lambda = \frac{n\pi}{30}$

$\therefore u(x,t) = B \sin\frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2\pi^2}{900}t}$

The general solution is $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2\pi^2}{300}t}$

Using iii), $2x+20 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30}$

$$B_n = \frac{2}{30}\int_0^{30} (2x+20)\sin\frac{n\pi x}{30}\,dx$$

$$= \frac{1}{15}\left\{(2x+20)\left(\frac{-\cos\frac{n\pi x}{30}}{\frac{n\pi}{30}}\right) - (2)\left(\frac{-\sin\frac{n\pi x}{30}}{\frac{n^2\pi^2}{900}}\right)\right\}_0^{30}$$

$$= \frac{1}{15}\left\{-\frac{80.30}{n\pi}(-1)^n + \frac{20.30}{n\pi}\right\}$$

$$= \frac{1}{15n\pi}\left\{-2400(-1)^n + 600\right\}$$

$$= \frac{600}{15n\pi}\left\{-4(-1)^n + 1\right\} = \frac{40}{n\pi}\left[1-4(-1)^n\right]$$

$$\therefore\ u(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty}\left[1-4(-1)^n\right]\sin\frac{n\pi x}{30}\, e^{\frac{-\alpha^2 n^2\pi^2}{900}t}.$$

Steady State Conditions and non zero boundary Conditions

A bar 10 cm long with insulated sides, has its ends A and B kept at 20°C and 40°C resp, until steady state conditions prevail. The temperature at A is then suddenly raised to 50°C and at the same instant that at B is lowered to 10°C. Find the subsequent temp. function $u(x,t)$ at any time.

The heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ ——— ①

In steady state, $\frac{d^2 u}{dx^2} = 0$

$\Rightarrow \quad u = ax + b$ ——— ②

$u = \left(\frac{b-a}{l}\right)x + a$

$\frac{20}{10}x + 20$

$2x + 20$

The B.C's are i) $u(0) = 20$ ii) $u(10) = 40$

Applying i) $20 = b \Rightarrow u = ax + 20$

Applying ii) $40 = a\overset{10}{x} + 20 \Rightarrow 10a = 20 \Rightarrow a = \frac{20}{10} = 2$

The temperature function in steady state,

$u(x) = 2x + 20$

When the temperatures at A and B are changed, the state is no longer steady. Then the temp function $u(x,t)$ satisfies ①.

The B.C's in the unsteady state are

i) $u(0,t) = 50$

ii) $u(10,t) = 10$

iii) $u(x,0) = 2x + 20$

We break up the required function $u(x,t)$ into 2 parts & write.

$$u(x,t) = u_s(x) + u_t(x,t) \quad \text{——— (2)}$$

where $u_s(x)$ is the Sol of heat equ involving x ~~and~~ only & Satisfying i) & ii), $u_t(x,t)$ is a transient Solution satisfying (2) which decreases as t increases. → lasting only for short time.

As $u_s(x)$ satisfies the heat equ involving x only, we have

$$\frac{d^2u_s}{dx^2} = 0 \quad \text{where } u_s(0) = 50 \text{ — (I)}, \quad u_s(10) = 10 \text{ — (II)}$$

$$\Rightarrow u_s = Ax + B \quad \text{——— (3)}$$

~~$u_s(0) = 50$~~ Applying (I), $50 = 10(0) + B \Rightarrow \boxed{B = 50}$

$$u_s = Ax + 50$$

Applying (II), $10 = 10A + 50 \Rightarrow 10A = -40 \Rightarrow \boxed{A = -4}$

$$\therefore u_s(x) = -4x + 50$$

(2) $\Rightarrow u_t(x,t) = u(x,t) - u_s(x)$

$$u_t(0,t) = u(0,t) - u_s(0) = 50 - 50 = 0 \quad \text{——— iv)}$$

$$u_t(10,t) = u(10,t) - u_s(10) = 10 - 10 = 0 \quad \text{——— v)}$$

$$u_t(x,0) = u(x,0) - u_s(x)$$

$$= 2x + 20 - 50 + 4x = 6x - 30 \quad \text{——— vi)}$$

we know that $u_t(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t}$ ——— (I)

Applying iv), $0 = A e^{-\alpha^2\lambda^2 t} \Rightarrow \boxed{A = 0}$

(I) $\Rightarrow u_t(x,t) = B\sin\lambda x \, e^{-\alpha^2\lambda^2 t}$ ——— (II)

Applying v), $0 = B\sin 10\lambda \, e^{-\alpha^2\lambda^2 t}$

$$\Rightarrow \sin 10\lambda = 0 = \sin n\pi$$

$$\lambda = \frac{n\pi}{10}$$

$(ii) \Rightarrow u_t(x,t) = B \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$

The most general solution is

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$

Applying vi),

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}$$

$$6x - 30 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}$$

This is a F.S. Series.

$$B_n = \frac{2}{10}\int_0^{10} (6x-30) \sin\frac{n\pi x}{10}\, dx$$

$$= \frac{1}{5}\left\{ (6x-30)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - 6\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right\}_0^{10}$$

$$= \frac{1}{5}\left\{ -\frac{30.10}{n\pi}(-1)^n - \frac{30.10}{n\pi} \right\}$$

$$= -\frac{300}{5n\pi}\left\{(-1)^n + 1\right\} = -\frac{60}{n\pi}\left[1 + (-1)^n\right]$$

$$u_t(x,t) = \frac{-60}{\pi}\sum_{n=1}^{\infty} \frac{[1+(-1)^n]}{n} \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$

$$\therefore u(x,t) = u_s(x) + u_t(x,t)$$

$$= -4x + 50 - \frac{60}{\pi}\sum_{n=1}^{\infty}\left[\frac{1+(-1)^n}{n}\right] \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$

2. A rod of length l has its ends A and B kept at 0°C & 100°C resp, until steady state conditions prevail. If the temp at of A is suddenly raised to 50°C and that of B to 150°C, find

the temp dis. at any pt of the rod and at any time.

$$50+\frac{100x}{l}+\frac{100}{\pi}\sum_{n=1}^{\infty}\left[\frac{-1+(-1)^n}{n}\right]\sin\frac{n\pi x}{l}\; e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$$

Temperature gradient Consider a bar of uniform cross section of length 'x' cm. Let the 2 ends of the rod be maintained at temperatures u_1 and u_2 where $u_1>u_2$. The quantity $\frac{u_1-u_2}{x}$ represents the rate of fall of temp. with respect to distance. This rate of change of temp with respect to distance is called temperature gradient & is denoted by $\frac{\partial u}{\partial x}$.

* ~~Solve the problem of heat conduction in a rod given~~ that

I Type 4th problem

Find the temperature $u(x,t)$ in a silver bar (of length 10cm constant cross section of $1\,cm^2$ area, density $10.6\ gm/cm^3$, thermal conductivity 1.04 cal/cm deg. sec; specific heat 0.056 cal/gm. deg) which is perfectly insulated laterally, if the ends are kept at 0°c & if initially the temperature is 5°c, at the centre of the bar and falls uniformly to zero at its ends.

Here $l=10cm$, $\rho=10.6\ gm/cm^3$, $k=1.04$ cal/cm deg sec, $c=0.056$ cal/gm

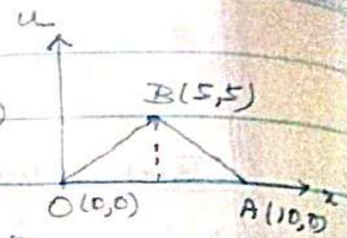
$$\alpha^2=\frac{k}{\rho c}$$

$$=\frac{1.04}{(10.6)(0.056)}=1.75$$

The B.C's are

i) $u(0,t)=0$ ii) $u(10,t)=0$

The heat equ is $\frac{\partial u}{\partial t} = 1.75 \frac{\partial^2 u}{\partial x^2}$ —— ①



The equation of OB is

$\frac{x-0}{5} = \frac{y-0}{5} \Rightarrow y = x$ (ie) $u = x$, $0 < x < 5$
where $y = u$.

The equation of BA is

$\frac{x-5}{10-5} = \frac{y-5}{-5}$

$-\cancel{5}(x-5) = \cancel{5}(y-\cancel{10}\,5)$

~~$-5x+25 = 5y-50$~~

$-x+5 = y-\cancel{10}\,5$

$-x+\cancel{5}\,10 = y \Rightarrow u = 10-x$, $5 < x < 10$.

Let $u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t}$ be the Solution of ①
—— ②

Applying i) $0 = Ae^{-\alpha^2\lambda^2 t} \Rightarrow \boxed{A=0}$

② $\Rightarrow$ $u(x,t) = B\sin\lambda x\, e^{-\alpha^2\lambda^2 t}$

Applying ii) $0 = B\sin 10\lambda\, e^{-\alpha^2\lambda^2 t}$
$\Rightarrow \lambda = \frac{n\pi}{10}$

$\therefore u(x,t) = B\sin\frac{n\pi x}{10}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{100}}$

The g.s. is $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{100}}$ —— ③

Applying iii),

$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}$

$B_n = \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}\,dx$

$$= \frac{2}{10}\left\{\int_0^5 x \sin\frac{n\pi x}{10}\,dx + \int_5^{10}(10-x)\sin\frac{n\pi x}{10}\,dx\right\}$$

$$= \frac{1}{5}\left\{\left[x\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - 1\cdot\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_0^5\right.$$

$$\left. + \left[(10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_5^{10}\right\}$$

$$= \frac{1}{5}\left\{-\frac{5.10}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{5.10}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right.$$

$$= \frac{1}{5}\left\{\frac{200}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} = \frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$\therefore u(x,t) = \frac{40}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}\,e^{-\frac{\alpha^2 n^2\pi^2 t}{100}}$$

$$= \frac{40}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}\,e^{-\frac{n^2\pi^2(1.75)^2 t}{100}}$$

1. Solve the problem of heat Conduction in a rod given that

i) $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ ii) u is finite at $t \to \infty$

iii) $\frac{\partial u}{\partial x} = 0$ for $x=0$ & $x=l$, $t>0$

iv) $u = lx - x^2$ for $t=0$, $0 \le x \le l$.

The One dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ — (1)

Let $u(x,t) = (A\cos\lambda x + B\sin\lambda x) e^{-\alpha^2\lambda^2 t}$ — (2) be the Sol of (1)

$$\frac{\partial u}{\partial x} = (-A\lambda\sin\lambda x + B\lambda\cos\lambda x) e^{-\alpha^2\lambda^2 t}$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow B\lambda e^{-\alpha^2\lambda^2 t} = 0 \Rightarrow B = 0. \Rightarrow \frac{\partial u}{\partial x} = -A\lambda\sin\lambda x e^{-\alpha^2\lambda^2 t}$$

$$\frac{\partial u}{\partial x}(l,t) = 0 \Rightarrow -A\lambda\sin\lambda l\, e^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow \sin\lambda l = 0 = \sin n\pi$$

$$\Rightarrow \lambda = \frac{n\pi}{l}$$

$$\therefore u(x,t) = A\cos\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$$

The most g.s. is $u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}} \quad \text{— (3)}$$

Applying iv),

$$lx - x^2 = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} \quad \text{— (4)}$$

To find A_0 & A_n expand $lx - x^2$ in half range cosine series

($\because$ L.H.S. Contains Cosine terms)

$$lx - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{— (5)}$$

Comparing (4) & (5),

$$A_0 = \frac{a_0}{2}, \quad A_n = a_n.$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l (lx - x^2)\,dx = \frac{2}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^3}{2} - \frac{l^3}{3}\right] = \frac{2l^2}{6} = \frac{l^2}{3}$$

$$A_0 = \frac{a_0}{2} = \frac{l^2}{6}.$$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left\{(lx - x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right\}_0^l$$

$$= \frac{2}{l}\left\{-\frac{l^3}{n^2\pi^2}(-1)^n - \frac{l^3}{n^2\pi^2}\right\} = -\frac{2l^2}{n^2\pi^2}\left[1 + (-1)^n\right]$$

$$A_n = \frac{-2l^2}{n^2\pi^2}\left[1 + (-1)^n\right]$$

Sub A_0 & A_n in ③

$$\therefore\ u(x,t) = \frac{l^2}{6} + -\frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1 + (-1)^n}{n^2}\right]\cos\frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}}.$$

2. Solve the following boundary value problem

i) $\frac{\partial u}{\partial t} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial x^2}$, $0 < x < 5$ ii) $\frac{\partial u(0,t)}{\partial x} = 0$ iii) $\frac{\partial u(5,t)}{\partial x} = 0$

iv) $u(x,0) = x$.

The given eq is $\frac{\partial u}{\partial t} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial x^2}$

$$= c^2\frac{\partial^2 u}{\partial x^2} \text{ where } \frac{1}{\alpha^2} = c^2 \quad \text{——①}$$

Let $u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t}$ the Sol of ①.

——②

$$\frac{\partial u}{\partial x} = (-A\lambda \sin\lambda x + B\lambda\cos\lambda x)\, e^{-\alpha^2\lambda^2 t}$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow \quad B\lambda e^{-\alpha^2\lambda^2 t} = 0 \Rightarrow \quad B = 0.$$

$$\frac{\partial u}{\partial x} = -\lambda A \sin\lambda x \; e^{-\alpha^2\lambda^2 t}$$

$$\frac{\partial u}{\partial x}(5,t) = 0 \Rightarrow -\lambda A \sin 5\lambda\, e^{-\alpha^2\lambda^2 t} = 0 \Rightarrow \lambda = \frac{n\pi}{5}$$

Sub the values of B & λ in (2)

$$u(x,t) = A\cos\frac{n\pi x}{5}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{25}}$$

The general Sol is $u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{5}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{25}}$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{5}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{25}} \quad \text{—— (3)}$$

Applying iv) $\quad x = A_0 + \sum_{n=1} A_n \cos\frac{n\pi x}{5}$ —— (4)

To find A_0 & A_n, expand x in half range Cosine Series.

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{5} = x \quad \text{—— (5)}$$

Comparing (1) & (2) $\quad A_0 = \frac{a_0}{2}, \quad A_n = a_n.$

$$a_0 = \frac{2}{5}\int_0^5 x\, dx = \frac{2}{5}\left(\frac{x^2}{2}\right)_0^5 = 5$$

$$\therefore A_0 = 5/2.$$

$$a_n = \frac{2}{5}\int_0^5 x \cos\frac{n\pi x}{5}\, dx$$

$$= \frac{2}{5}\left\{ x\left(\frac{\sin\frac{n\pi x}{5}}{\frac{n\pi}{5}}\right) - 1.\left(\frac{-\cos\frac{n\pi x}{5}}{\frac{n^2\pi^2}{25}}\right)\right\}_0^5$$

$$= \frac{2}{5}\left\{\frac{25}{n^2\pi^2}(-1)^n - \frac{25}{n^2\pi^2}\right\}$$

$$= \frac{10}{n^2\pi^2}\left\{(-1)^n - 1\right\} \quad \text{(ie)}\; A_n = \frac{10}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$\therefore u(x,t) = \frac{5}{2} + \frac{10}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2}\right] \cos\frac{n\pi x}{5}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{25}}$

3. The temperature at one end of a bar 50cm long with insulated sides is kept at 0°c and that at the other end is kept at 100°c until steady state conditions prevail. The 2 ends are then suddenly insulated, So that the temp gradient is zero at each end thereafter. Find the temp. distribution.

The heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ —— ①

In Steady State Condition, $u(x,t)$ is a function of x alone.

$\frac{d^2u}{dx^2} = 0 \Rightarrow u = ax + b$.

The boundary conditions are $u(0) = 0$ —i), $u(50) = 100$ —ii)

Applying i) $0 = b$

$\therefore u = ax$

Applying ii), $100 = 50a \Rightarrow = 2$.

$\therefore$ the initial temperature function is $u(x,0) = 2x$

The boundary Conditions are

$\frac{\partial u}{\partial x}(0,t) = 0$ —— iii)

$\frac{\partial u}{\partial x}(50,t) = 0$ —— iv) $u(x,0) = 2x$ —— v)

Let $u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t}$ —② be the Sol of ①.

$\frac{\partial u}{\partial x} = [-A\lambda\sin\lambda x + B\lambda\cos\lambda x]\, e^{-\alpha^2\lambda^2 t}$

Applying iii), $B\lambda e^{-\alpha^2\lambda^2 t} = 0 \Rightarrow \boxed{B=0}$

$\therefore \frac{\partial u}{\partial x} = -A\lambda \sin\lambda x \, e^{-\alpha^2\lambda^2 t}$

Applying iv), $0 = -A\lambda \sin 50\lambda \, e^{-\alpha^2\lambda^2 t}$

$\Rightarrow \sin 50\lambda = 0 \Rightarrow \lambda = \frac{n\pi}{50}$

Sub the values of B & λ in (2),

$u(x,t) = A\cos\frac{n\pi x}{50} \, e^{\frac{-\alpha^2 n^2\pi^2}{50^2}t}$

The most general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{50} \, e^{-\alpha^2\left(\frac{n\pi}{50}\right)^2 t}$$

Applying v),

$$2x = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{50}$$

(ie) $A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{50} = 2x$ —— (3)

To find A_0 and A_n, expand $2x$ in half range Fourier cosine series

$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{50} = 2x$ —— (4)

From (3) & (4), $A_0 = \frac{a_0}{2}$, $A_n = a_n$.

$$a_0 = \frac{2}{50}\int_0^{50} 2x\,dx = \frac{4}{50}\left(\frac{x^2}{2}\right)_0^{50} = 2.50 = 100$$

$A_0 = 50$

$$a_n = \frac{2}{50}\int_0^{50} 2x \cos\frac{n\pi x}{50}\,dx$$

$$= \frac{4}{50}\left\{ x\left(\frac{\sin\frac{n\pi x}{50}}{\frac{n\pi}{50}}\right) - 1.\left(\frac{-\cos\frac{n\pi x}{50}}{\frac{n^2\pi^2}{50^2}}\right)\right\}_0^{50}$$

$$= \frac{4}{50}\left\{\frac{50^2}{n^2\pi^2}(-1)^n - \frac{50^2}{n^2\pi^2}\right\}$$

$$= \frac{4}{50 \cdot n^2\pi^2} \cdot 50^2 \left[(-1)^n - 1 \right]$$

$$= \frac{4 \cdot 50}{n^2\pi^2} \left[(-1)^n - 1 \right] = \frac{200}{n^2\pi^2} \left[(-1)^n - 1 \right]$$

$$\therefore u(x,t) = 50 + \frac{200}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2} \right] \cos\frac{n\pi x}{50}\, e^{-\alpha^2 \left(\frac{n\pi}{50}\right)^2 t}$$

4. Solve the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ subject to the following Conditions

i) u is finite when $t \to \infty$ (5)

ii) $\frac{\partial u}{\partial x} = 0$ when $x = 0$ for all $t > 0$ ii) $u(0,t) = 0$

iii) $u = 0$ when $x = l$ for all $t > 0$ iii) $\frac{\partial u}{\partial x}(l,t) = 0$.

iv) $u = u_0$ when $t = 0$ for all values of x between 0 & l. iv) $u = u_0$ when $t = 0$ for all

Sol: The heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ —— (1)

The B.C's are i) $\frac{\partial u}{\partial x}(0,t) = 0$ ii) $u(l,t) = 0$ iii) $u(x,0) = u_0$.

Let $u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t}$ —— (2) be the Sol of (1).

$$\frac{\partial u}{\partial x} = (-A\lambda \sin\lambda x + B\lambda\cos\lambda x)\, e^{-\alpha^2\lambda^2 t}$$

Applying i), $0 = B\lambda\, e^{-\alpha^2\lambda^2 t} \Rightarrow \boxed{B=0}$

(2) $\Rightarrow u(x,t) = A\cos\lambda x\, e^{-\alpha^2\lambda^2 t}$ —— (3)

Applying ii), $0 = A\cos\lambda l\, e^{-\alpha^2\lambda^2 t}$

$\Rightarrow \cos\lambda l = 0$

λl = an odd multiple of $\frac{\pi}{2}$.

$= (2n-1)\frac{\pi}{2}$

$$\lambda = \frac{(2n-1)\pi}{2l}$$

$$\cos\frac{(2n-1)\pi x}{2l}\cos\frac{\pi x}{2l} = \frac{1}{2}\left[\cos\frac{2n\pi x}{2l} + \cos\frac{2(n-1)\pi x}{2l}\right]$$

$$= \frac{1}{2}\left[\cos\frac{n\pi x}{l} + \cos\frac{(n-1)\pi x}{l}\right]$$

Sub the value of λ in (3),

$$u(x,t) = A\cos\frac{(2n-1)\pi x}{2l}\, e^{-\frac{\alpha^2(2n-1)^2\pi^2 t}{4l^2}}$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{(2n-1)\pi x}{2l}\, e^{-\frac{\alpha^2(2n-1)^2\pi^2 t}{4l^2}} \quad \text{—— (5)}$$

Applying iii),

$$u_0 = \sum_{n=1}^{\infty} A_n \cos\frac{(2n-1)\pi x}{2l}$$

There is no sine or cosine series.

$$u_0 = A_1\cos\frac{\pi x}{2l} + A_2\cos\frac{3\pi x}{2l} + \cdots\cdots + A_n\cos\frac{(2n-1)\pi x}{2l} + \cdots$$

Multiplying both sides by $\cos\frac{(2n-1)\pi x}{2l}$ and integrating with respect to x between 0 & l, we get

$$u_0\int_0^l \cos\frac{(2n-1)\pi x}{2l}\,dx = A_n\int_0^l \cos^2\frac{(2n-1)\pi x}{2l}\,dx \quad \text{—— (6)}$$

$$u_0\int_0^l \cos\frac{(2n-1)\pi x}{2l}\,dx = u_0\left[\frac{\sin\frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi}{2l}}\right]_0^l$$

$$= \frac{2lu_0}{(2n-1)\pi}\left[\sin\frac{(2n-1)\pi x}{2l}\right]_0^l$$

$$= \frac{2lu_0}{(2n-1)\pi}\left[\sin(2n-1)\frac{\pi}{2}\right] = \frac{2lu_0}{(2n-1)\pi}\sin\left(n\pi - \frac{\pi}{2}\right)$$

$$= \frac{2lu_0}{(2n-1)\pi}\left[\sin n\pi\cos\frac{\pi}{2} - \cos n\pi\sin\frac{\pi}{2}\right]$$

$$= \frac{2lu_0}{(2n-1)\pi}\left[-(-1)^n\right] = \frac{2lu_0}{(2n-1)\pi}(-1)^{n+1}$$

$$A_n \int_0^l \cos^2 \frac{(2n-1)\pi x}{2l}\, dx = A_n \int_0^l \frac{1}{2}\left[1 + \cos 2\frac{(2n-1)\pi x}{2l}\right] dx$$

$$= \frac{A_n}{2}\int_0^l \left[1 + \cos\frac{(2n-1)\pi x}{l}\right] dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{A_n}{2}\left\{ x + \frac{\sin\frac{(2n-1)\pi x}{l}}{\frac{(2n-1)\pi}{l}} \right\}_0^l$$

$$= \frac{A_n}{2}[l + 0]$$

$$④ \Rightarrow \frac{2l u_0 (-1)^{n+1}}{(2n-1)\pi} = \cancel{\frac{A_n l}{2}} \frac{A_n l}{2}$$

$$\Rightarrow \quad A_n = \frac{4u_0 (-1)^{n+1}}{(2n-1)\pi}$$

Sub the value of A_n in (I),

$$u(x,t) = 4u_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)\pi} \cos\frac{(2n-1)\pi x}{2l}\, e^{-\frac{\alpha^2 (2n-1)^2 \pi^2 t}{4l^2}}$$

i) $\Rightarrow A = 0$. $u(x,t) = B \sin \lambda x\, e^{-\alpha^2\lambda^2 t}$

$$\frac{\partial u}{\partial x} = B\lambda \cos\lambda x\, e^{-\alpha^2\lambda^2 t}$$

ii) $\Rightarrow$ $0 = B\lambda \cos\lambda l\, e^{-\alpha^2\lambda^2 t} \Rightarrow \lambda = \frac{(2n-1)\pi}{2l}$

g.s. is $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi x}{2l}\, e^{-\alpha^2\left(\frac{(2n-1)\pi}{2l}\right)^2 t}$

Applying iv) $u_0 = \sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi x}{2l}$

$$B_n = \frac{2}{l}\int_0^l u_0 \sin\frac{(2n-1)\pi x}{2l}\, dx = \frac{2u_0}{l}\left[\frac{-\cos\frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi}{2l}}\right]_0^l$$

$$= \frac{4u_0}{(2n-1)\pi}\left[-\cos\frac{(2n-1)\pi}{2} + 1\right] = \frac{4u_0}{(2n-1)\pi}\left[1 - \cos\left(n\pi - \pi/2\right)\right]$$

$$= \frac{4u_0}{(2n-1)\pi} \quad \therefore u(x,t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\frac{(2n-1)\pi x}{2l}\, e^{-\alpha^2\left(\frac{(2n-1)\pi}{2l}\right)^2 t}$$

When the heat flow is along plane curves, lying in the same parallel planes, instead of along straight lines, then the heat flow is said to be 2 dimensional.

Two dimensional heat flow equation

$$\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \text{ — (1)} \quad \text{where } \alpha^2 = \frac{k}{\rho c}$$

When steady state exists, the temperature func $u(x,t)$ is independent of t. $\therefore \frac{\partial u}{\partial t} = 0$. (1) $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

This is called Laplace equation in 2 dimensions.

<u>Solution of 2 dimensional heat equation</u> (By mtd of Separation variable)

The 2 dimensional heat flow equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ —

where u is a function of x & y.

Let $u = X(x)Y(y)$ be the solution of (1).

$$\frac{\partial u}{\partial x} = X'Y \Rightarrow \frac{\partial^2 u}{\partial x^2} = X''Y$$

$$\frac{\partial u}{\partial y} = XY' \Rightarrow \frac{\partial^2 u}{\partial y^2} = XY''$$

(1) $\Rightarrow$ $X''Y + XY'' = 0$

$$X''Y = -XY''$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = k. \quad \text{(ie) } X'' - kX = 0 \quad Y'' + kY = 0.$$

<u>Case i)</u> $k = \lambda^2$

$X'' - \lambda^2 X = 0 \quad Y'' + \lambda^2 Y = 0$

$m^2 = \lambda^2$

$X = A_1 e^{\lambda x} + B_1 e^{-\lambda x} \qquad Y = C_1 \cos\lambda y + D_1 \sin\lambda y$

<u>Case ii)</u> $k = -\lambda^2$

$X'' + \lambda^2 X = 0 \qquad Y'' - \lambda^2 Y = 0$

$X = A_2 \cos\lambda x + B_2 \sin\lambda x \quad , Y = C_2 e^{\lambda y} + D_2 e^{-\lambda y}$

<u>Case iii)</u> $k = 0$

$X'' = 0, \quad Y'' = 0$

$X = A_3 x + B_3 \qquad Y = C_3 y + D_3$

The possible Solutions of ① are

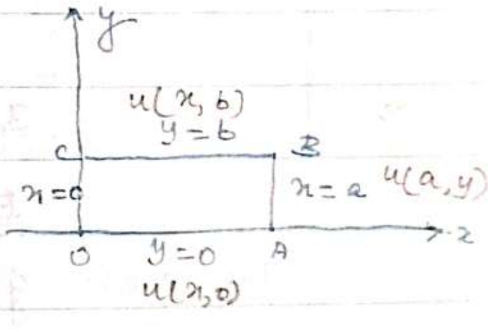
$u(x,y) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 \cos\lambda y + D_1 \sin\lambda y)$

$u(x,y) = (A_2 \cos\lambda x + B_2 \sin\lambda x)(C_2 e^{\lambda y} + D_2 e^{-\lambda y})$

$u(x,y) = (A_3 x + B_3)(C_3 y + D_3)$

In the problems, where the B.c's are given, we have to select a suitable solution to satisfy ① and B.c's.

1. A rectangular plate is bounded by lines $x=0$, $x=a$, $y=0$ & $y=b$ and the edge temperatures are $u(0,y)=0$, $u(x,b)=0$, $u(a,y)=0$ and $u(x,0) = 5\sin\frac{5\pi x}{a} + 3\sin\frac{3\pi x}{a}$. Find the Steady State temp distribution at any pt of the plate.



The two dimensional heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——} ①$$

The B.c's are

i) $u(0,y) = 0, \quad 0 < y < b$

ii) $u(a,y) = 0, \quad 0 < y < b$

iii) $u(x,b) = 0, \quad 0 < x < a$

iv) $u(x,0) = 5\sin\frac{5\pi x}{a} + 3\sin\frac{3\pi x}{a}, \quad 0 < x < a$

The possible Sol is $u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$ —②

Using i), $0 = A(Ce^{\lambda y} + De^{-\lambda y}) \Rightarrow A = 0.$

② $\Rightarrow$ $u(x,y) = B\sin\lambda x(Ce^{\lambda y} + De^{-\lambda y})$ —③

Using ii) $0 = B\sin\lambda a\,(Ce^{\lambda y} + De^{-\lambda y})$

$\Rightarrow \sin\lambda a = 0 \Rightarrow \lambda = \frac{n\pi}{a}.$

$(3) \Rightarrow u(x,y) = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi y}{a}} + De^{-\frac{n\pi y}{a}}\right)$ ——— (4)

Using iii)

$$0 = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi b}{a}} + De^{-\frac{n\pi b}{a}}\right)$$

$\Rightarrow Ce^{\frac{n\pi b}{a}} + De^{-\frac{n\pi b}{a}} = 0$

$\Rightarrow Ce^{\frac{n\pi b}{a}} = -De^{-\frac{n\pi b}{a}}$

$\Rightarrow D = -\frac{Ce^{\frac{n\pi b}{a}}}{e^{-\frac{n\pi b}{a}}} = -Ce^{\frac{2n\pi b}{a}}$

$(4) \Rightarrow u(x,y) = B\sin\frac{n\pi x}{a}\left[Ce^{\frac{n\pi y}{a}} - Ce^{\frac{2n\pi b}{a}}e^{-\frac{n\pi y}{a}}\right]$

$$= BC\sin\frac{n\pi x}{a}\left[e^{\frac{n\pi y}{a}} - e^{\frac{n\pi b}{a}}e^{\frac{n\pi b}{a}}e^{-\frac{n\pi y}{a}}\right]$$

$$= BC\sin\frac{n\pi x}{a}\cdot e^{\frac{n\pi b}{a}}\left[\frac{e^{\frac{n\pi y}{a}}}{e^{\frac{n\pi b}{a}}} - e^{\frac{n\pi b}{a}}e^{-\frac{n\pi y}{a}}\right]$$

$$= BC\sin\frac{n\pi x}{a}\,e^{\frac{n\pi b}{a}}\left[e^{\frac{n\pi}{a}(y-b)} - e^{-\frac{n\pi}{a}(y-b)}\right]$$

$$= BC\sin\frac{n\pi x}{a}\,e^{\frac{n\pi b}{a}}\,2\sinh\frac{n\pi}{a}(y-b)$$

The most general Solution is $= \underbrace{2BCe^{\frac{n\pi b}{a}}}_{B_n}\sin\frac{n\pi x}{a}\sinh\frac{n\pi}{a}(y$

$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi}{a}(y-b)$ ——— (5)

Using iv), $u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi}{a}(-b)$

$$= -\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

(ie) $-\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a} = 5\sin\frac{5\pi x}{a} + 3\sin\frac{3\pi x}{a}$

$B_1 e^{\frac{n\pi b}{a}} = 5$

$5e^{-\frac{n\pi b}{a}}$

$$-B_1 \sin\frac{\pi x}{a}\sinh\frac{\pi b}{a} - B_2 \sin\frac{2\pi x}{a}\sinh\frac{2\pi b}{a} - B_3 \sin\frac{3\pi x}{a}\sinh\frac{3\pi b}{a}$$

$$-B_4 \sin\frac{4\pi x}{a}\sinh\frac{4\pi b}{a} - B_5 \sin\frac{5\pi x}{a}\sinh\frac{5\pi b}{a} - \cdots\cdots$$

$$= 5\sin\frac{5\pi x}{a} + 3\sin\frac{3\pi x}{a}$$

Equating like coefficients,

$$-B_3 \sinh\frac{3\pi b}{a} = 3 \Rightarrow B_3 = \frac{-3}{\sinh\frac{3\pi b}{a}}$$

$$-B_5 \sinh\frac{5\pi b}{a} = 5 \Rightarrow B_5 = \frac{-5}{\sinh\frac{5\pi b}{a}}, \quad B_n = 0, \ n \neq 3, 5$$

Sub in (5),

$$u(x,y) = \frac{-3}{\sinh\frac{3\pi b}{a}} \sin\frac{3\pi x}{a} \sinh\frac{3\pi}{a}(y-b)$$

$$- \frac{5}{\sinh\frac{5\pi b}{a}} \sin\frac{5\pi x}{a} \sinh\frac{5\pi}{a}(y-b)$$

$$= \frac{3}{\sinh\frac{3\pi b}{a}} \sin\frac{3\pi x}{a} \sinh\frac{3\pi}{a}(b-y) + \frac{5}{\sinh\frac{5\pi b}{a}} \sin\frac{5\pi x}{a} \sinh\frac{5\pi}{a}(b-y).$$

2\. A square plate has its faces and the edge $y=0$ insulated. Its edges $x=0$ & $x=\pi$ are kept at zero temp and its 4th edge $y=\pi$ is kept at temp $f(x)$. Find the steady state temp at any pt of the plate.

Out of syllabus

The 2 dimensional heat equation is $\nabla^2 u = 0$ —— (1)

The B.C's are

i) $u(0,y) = 0, \quad 0 < y < \pi$

ii) $u(\pi,y) = 0, \quad 0 < y < \pi$

iii) $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0, \quad 0 < x < \pi$ ($\because$ $y=0$ is insulated)

iv) $u(x,\pi) = f(x), \quad 0 < x < \pi$

Let $u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$ be Sol. of ①.

Using i), $0 = A(Ce^{\lambda y} + De^{-\lambda y}) \Rightarrow A = 0.$

$\therefore u(x,y) = B\sin\lambda x(Ce^{\lambda y} + De^{-\lambda y})$

Using ii) $0 = B\sin\lambda\pi\,(Ce^{\lambda y} + De^{-\lambda y}) \Rightarrow \boxed{\lambda = n.}$

$\therefore u(x,y) = B\sin nx(Ce^{ny} + De^{-ny})$

$\dfrac{\partial u}{\partial y} = B\sin nx\,(nCe^{ny} - nDe^{-ny})$

$\left(\dfrac{\partial u}{\partial y}\right)_{y=0} \Rightarrow nB\sin nx\,(C - D) \Rightarrow C - D = 0 \Rightarrow C = D.$

$u(x,y) = B\sin nx\,(Ce^{ny} + Ce^{-ny})$

$= BC\sin nx(e^{ny} + e^{-ny}) = BC\,2\sin nx\cosh ny$

The most g.s. is $u(x,y) = \sum_{n=1}^{\infty} B_n \sin nx \cosh ny$

$u(x,\pi) = f(x) \Rightarrow u(x,\pi) = \sum_{n=1}^{\infty} B_n \sin nx \cosh n\pi$

$B_n\cosh n\pi = \dfrac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$\Rightarrow B_n = \dfrac{2}{\pi\cosh n\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$\therefore u(x,y) = \dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{\sin nx\cosh ny}{\cosh n\pi}\left(\int_0^{\pi} f(x)\sin nx\,dx\right).$

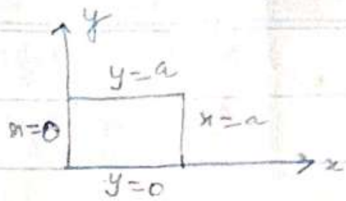
3. The three sides $x=0, x=a, y=0$ of a square plate bounded by the lines $x=0, x=a, y=0$ and $y=a$ are kept at $0°c$. The side $y=a$ is kept at steady temp given by $u(x,a) = bx(x-a), 0 \le x \le a$ where b is a constant. Find the steady state temp $u(x,y)$ in the plate.

The equation is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ —— ①

The B.C's are

i) $u(0,y)=0$, $0<y<a$ ii) $u(a,y)=0$, $0<y<a$

iii) $u(x,0)=0$, $0<x<a$ iv) $u(x,a)=bx(x-a)$



The possible solutions of ① are

a) $u(x,y) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 \cos\lambda y + D_1 \sin\lambda y)$

b) $u(x,y) = (A_2 \cos\lambda x + B_2 \sin\lambda x)(C_2 e^{\lambda y} + D_2 e^{-\lambda y})$

c) $u(x,y) = (A_3 x + B_3)(C_3 y + D_3)$

Of these solutions, we have to select a solution which suits the boundary conditions.

Consider the sol a) $u(x,y) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 \cos\lambda y + D_1 \sin\lambda y)$

Using i), $0 = (A_1 + B_1)(C_1 \cos\lambda y + D_1 \sin\lambda y) \Rightarrow A_1 = -B_1$.

$u(x,y) = A_1(e^{\lambda x} - e^{-\lambda x})(C_1 \cos\lambda y + D_1 \sin\lambda y)$

not necessary

Using ii), $0 = A_1(e^{\lambda a} - e^{-\lambda a})(C_1 \cos\lambda y + D_1 \sin\lambda y) \Rightarrow A_1 = 0$.

$\therefore u(x,y) = 0$ is a trivial sol. Hence a) is not a correct solu

Consider c): $u(x,y) = (A_3 x + B_3)(C_3 y + D_3)$

Using i), $0 = B_3(C_3 y + D_3) \Rightarrow B_3 = 0$.

$\therefore u(x,y) = A_3 x(C_3 y + D_3)$

Using ii), $0 = A_3 a(C_3 y + D_3) \Rightarrow A_3 = 0$.

$\therefore u(x,y) = 0$ is a trivial sol. Hence c) is not a correct sol

So the correct sol is b).

Let $u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$ —— ② be sol of ①

Using i), $A = 0$

$u(x,y) = B\sin\lambda x\,(Ce^{\lambda y} + De^{-\lambda y})$ —— ③

Using ii), $0 = B\sin\lambda a\,(Ce^{\lambda y} + De^{-\lambda y}) \Rightarrow \lambda = \frac{n\pi}{a}$

$$③ \Rightarrow u(x,y) = B\sin\frac{n\pi x}{a}\left[Ce^{\frac{n\pi y}{a}} + De^{-\frac{n\pi y}{a}}\right] \quad \text{———} ④$$

Applying iii), $0 = B\sin\frac{n\pi x}{a}\,[C+D]$

$$\Rightarrow C+D = 0 \Rightarrow -C = +D$$

$$④ \Rightarrow u(x,y) = B\sin\frac{n\pi x}{a}\left[Ce^{\frac{n\pi y}{a}} - Ce^{-\frac{n\pi y}{a}}\right]$$

$$= BC\sin\frac{n\pi x}{a}\left[e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right]$$

$$= BC\sin\frac{n\pi x}{a}\; 2\sinh\frac{n\pi y}{a}$$

The most g.s. is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

Using iv),

$$bx(x-a) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh n\pi \quad \text{———} ⑤$$

To find B_n, expand $bx(x-a)$ in a half range F.S.S.

(ie) $$bx(x-a) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \quad \text{———} ⑥$$

From ⑤ & ⑥,

$$B_n \sinh n\pi = b_n$$

$$= \frac{2}{a}\int_0^a bx(x-a)\sin\frac{n\pi x}{a}\,dx$$

$$= \frac{2b}{a}\int_0^a (x^2-ax)\sin\frac{n\pi x}{a}\,dx$$

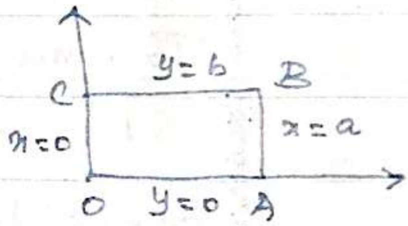
$$= \frac{2b}{a}\left\{(x^2-ax)\left(\frac{-\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right) - (2x-a)\left(\frac{-\sin\frac{n\pi x}{a}}{\frac{n^2\pi^2}{a^2}}\right) + 2\left(\frac{\cos\frac{n\pi x}{a}}{\frac{n^3\pi^3}{a^3}}\right)\right\}_0^a$$

$$= \frac{2b}{a}\left\{ \frac{2a^3}{n^3\pi^3}(-1)^n - \frac{2a^3}{n^3\pi^3} \right\} = \frac{4ba^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$B_n = \frac{4ba^2}{n^3\pi^3 \sinh n\pi}\left[(-1)^n - 1\right]$$

$$\therefore\ u(x,y) = \frac{4ba^2}{\pi^3}\sum_{n=1}^{\infty} \frac{\left[(-1)^n - 1\right]}{n^3 \sinh n\pi} \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

4. A rectangular plate is bounded by the lines $x=0, y=0, x=a, y=b$. Its surfaces are insulated. The temperature along $x=0, y=0$ are kept at 0°C and the others at 100°C. Find the steady state temp. at any pt of the plate.



The equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ——— ①

The B.C's are

i) $u(0,y) = 0, \ 0 < y < b$

ii) $u(a,y) = 100, \ 0 < y < b$

iii) $u(x,0) = 0, \ 0 < x < a$

iv) $u(x,b) = 100, \ 0 < x < a$.

Now we split the solution into 2 solutions

$u(x,y) = u_1(x,y) + u_2(x,y)$ ——— ②

where $u_1(x,y)$ and $u_2(x,y)$ are solutions of ①.

$u_1(x,y)$ is the temp. at any pt with the edge BC maintai[ned] at 100°C and the other 3 edges at 0°C while $u_2(x,y)$ is the temp. at any pt with the edge AB maintained at 10[0°C] and the other 3 edges at 0°C.

The B.C's for the functions $u_1(x,y)$ & $u_2(x,y)$ are

$a_1)\ u_1(0,y)=0$ $\qquad a_2)\ u_2(x,0)=0$

$b_1)\ u_1(a,y)=0$ $\qquad b_2)\ u_2(x,b)=0$

$c_1)\ u_1(x,0)=0$ $\qquad c_2)\ u_2(0,y)=0$

$d_1)\ u_1(x,b)=100$ $\qquad d_2)\ u_2(a,y)=100$

Let $u_1(x,y)=(A\cos\lambda x+B\sin\lambda x)(Ce^{\lambda y}+De^{-\lambda y})$ be sol of ① for u_1

Applying a_1, b_1, c_1, we get the most g.s.

$$u_1(x,y)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

$$u_1(x,b)=100 \Rightarrow \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}=100 \quad \text{—— ③}$$

To find B_n, expand 100 in H.R.F.S.S.

$$100=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \quad \text{—— ④}$$

From ③ & ④ $\quad B_n \sinh\frac{n\pi b}{a}=b_n$

$$b_n=\frac{2}{a}\int_0^a 100 \sin\frac{n\pi x}{a}\,dx$$

$$=\frac{200}{a}\left\{\frac{-\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right\}_0^a$$

$$=\frac{200}{n\pi}\left\{-(-1)^n+1\right\}=\begin{cases}0, & n \text{ is even}\\ \frac{400}{n\pi}, & \text{odd}\end{cases}$$

$$B_n=\frac{200}{n\pi\sinh\frac{n\pi b}{a}}\left[1-(-1)^n\right] \qquad B_n=\frac{400}{n\pi\sinh\frac{n\pi b}{a}}$$

$$u_1(x,y)=\frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{\sinh\frac{n\pi b}{a}}\frac{\left[1-(-1)^n\right]}{n}\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

$$u(x,y)=\frac{400}{\pi}\sum_{n=1,3,\dots}\frac{1}{n\sinh\frac{n\pi b}{a}}\sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

Let $u_2(x,y) = (Ae^{\lambda x} + Be^{-\lambda x})(C\cos\lambda y + D\sin\lambda y)$ —

Using a_2), $0 = C(Ae^{\lambda x} + Be^{-\lambda x}) \Rightarrow C=0$.

$u_2(x,y) = D\sin\lambda y\,(Ae^{\lambda x} + Be^{-\lambda x})$

Using b_2), $0 = D\sin\lambda b\,(Ae^{\lambda x} + Be^{-\lambda x})$

$\Rightarrow \sin\lambda b = 0 \Rightarrow \lambda = \frac{n\pi}{b}$.

$\therefore u_2(x,y) = D\sin\frac{n\pi y}{b}\left(Ae^{\frac{n\pi x}{b}} + Be^{-\frac{n\pi x}{b}}\right)$

Using c_2), $0 = D\sin\frac{n\pi y}{b}(A+B)$

$\Rightarrow A+B = 0 \Rightarrow A = -B$

$u_2(x,y) = D\sin\frac{n\pi y}{b}\left[Ae^{\frac{n\pi x}{b}} - Ae^{-\frac{n\pi x}{b}}\right]$

$= AD\sin\frac{n\pi y}{b}\left[e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right]$

$= 2AD\sin\frac{n\pi y}{b}\sinh\frac{n\pi x}{b}$

The most g.s. is $u_2(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{b}\sinh\frac{n\pi x}{b}$

Using d_2), $100 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{b}\sinh\frac{n\pi a}{b}$ — (4)

To find B_n, expand 100 in H.R.F.S.S.

$100 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{b}$ — (5)

From (4) & (5), $B_n \sinh\frac{n\pi a}{b} = b_n$.

$b_n = \frac{2}{b}\int_0^b 100\sin\frac{n\pi y}{b}\,dy$

$= \frac{200}{b}\left\{\frac{-\cos\frac{n\pi y}{b}}{\frac{n\pi}{b}}\right\}_0^b = \frac{200}{n\pi}\left[-(-1)^n + 1\right]$

$= \begin{cases} 0, & n \text{ is even} \\ \frac{400}{n\pi}, & n \text{ is odd} \end{cases}$

$$B_n = \frac{400}{n\pi} \frac{1}{\sinh \frac{n\pi a}{b}}$$

$$u_2(x,y) = \frac{400}{\pi} \sum_{n=1,3\ldots}^{\infty} \frac{1}{n} \frac{1}{\sinh \frac{n\pi a}{b}} \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b}$$

$$\therefore u(x,y) = u_1(x,y) + u_2(x,y).$$

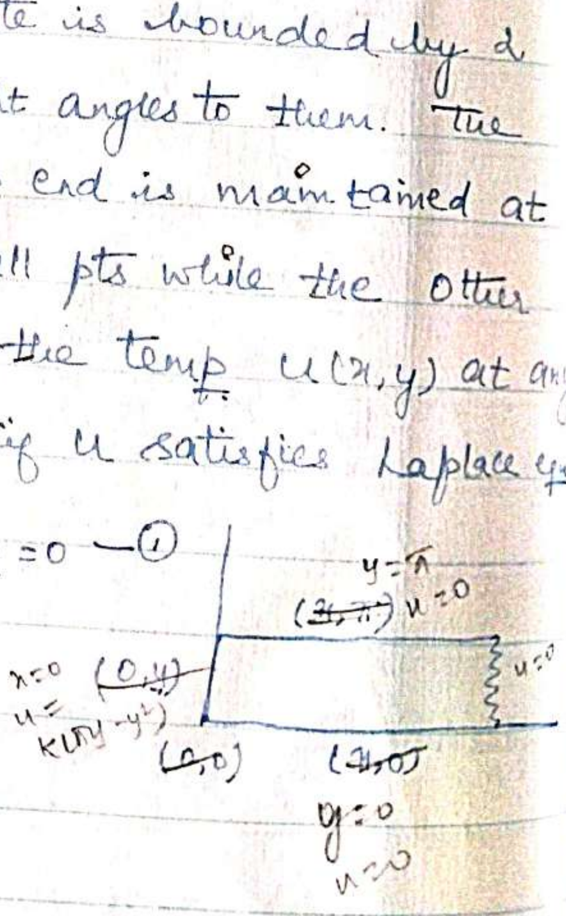
A Square plate is bounded by the lines $x=0, y=0, x=a, y=a$. Its surfaces are insulated and their temperatures along the edges $x=a$ & $y=a$ are each $100°c$ while the other two edges are kept at $0°c$. Find the steady state temp distribution at any pt on the plate.

Infinite plates

1. An infinitely long plane uniform plate is bounded by 2 parallel edges and an edge at right angles to them. The breadth of the edge $x=0$ is π. This end is maintained at temperature as $u = k(\pi y - y^2)$ at all pts while the other edges are at zero temp. Determine the temp $u(x,y)$ at any pt of the plate in the steady state if u satisfies Laplace eq.

The heat equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ —①

The B.C's are



i) $u(x,0) = 0$

ii) $u(x,\pi) = 0$

iii) $u(\infty,y) = 0$

iv) $u(0,y) = k(\pi y - y^2)$

Let $u(x,y) = (Ae^{\lambda x} + Be^{-\lambda x})(C\cos\lambda y + D\sin\lambda y)$ be Sol of ① —②

Using i), $0 = C(Ae^{\lambda x} + Be^{-\lambda x}) \Rightarrow C = 0$

② $\Rightarrow$ $u(x,y) = D\sin\lambda y\,(Ae^{\lambda x} + Be^{-\lambda x})$

Using ii), $0 = D\sin\lambda\pi\,(Ae^{\lambda x} + Be^{-\lambda x})$

$\Rightarrow \sin\lambda\pi = 0 = \sin n\pi \Rightarrow \boxed{\lambda = n}$

$\therefore u(x,y) = D\sin ny\,(Ae^{\lambda x} + Be^{-\lambda x})$ —③

Using iii), $0 = D\sin ny\,(Ae^{\infty} + Be^{-\infty})$

$\Rightarrow A = 0$

③ $\Rightarrow u(x,y) = BD\sin ny\; e^{-nx}$

The most general solution is $u(x,y) = \sum_{n=1}^{\infty} B_n \sin ny\, e^{-nx}$ —④

Using iv), $k(\pi y - y^2) = \sum_{n=1}^{\infty} B_n \sin ny$ —⑤

To find B_n, expand $k(\pi y - y^2)$ in H.R.F.S.S.

$$k(\pi y - y^2) = \sum_{n=1}^{\infty} \cancel{B_n}\, b_n \sin ny \quad \text{—⑥}$$

From ⑤ & ⑥, $B_n = b_n$

$$= \frac{2k}{\pi}\int_0^{\pi} (\pi y - y^2)\sin ny\, dy$$

$$= \frac{2k}{\pi}\left\{(\pi y - y^2)\left(\frac{-\cos ny}{n}\right) - (\pi - 2y)\left(-\frac{\sin ny}{n^2}\right) + (-2)\left(\frac{\cos ny}{n^3}\right)\right\}_0^{\pi}$$

$$= \frac{2k}{\pi}\left\{\frac{-2}{n^3}(-1)^n + \frac{2}{n^3}\right\}$$

$$= \frac{4k}{\pi n^3}\left[-(-1)^n + 1\right] = \begin{cases} \dfrac{8k}{\pi n^3}, & n \text{ is odd} \\ 0, & \text{odd} \end{cases}$$

$$\therefore u(x,y) = \frac{4k}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^3}\right]\sin ny\, e^{-nx}$$

$$\frac{8k}{\pi}\sum_{n=1,3}^{\infty} \frac{1}{n^3}\sin ny\, e^{-nx}$$

A rectangular plate with insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in the length without introducing an appreciable error. If the temperature along one short edge $y=0$ is given by $u(x,0) = 100 \sin \frac{\pi x}{8}, 0<x<8$ while the two longe edges $x=0$ & $x=8$ as well as the other short edges are kept at 0°c, find the steady state temperature function $u(x,y)$.

The heat flow equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ — ①

The B.c's are

i) $u(0,y) = 0$

ii) $u(8,y) = 0$

iii) $u(x,\infty) = 0$

iv) $u(x,0) = 100 \sin \frac{\pi x}{8}, \ 0<x<8$

Let $u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$

Using i), $0 = A(Ce^{\lambda y} + De^{-\lambda y}) \Rightarrow A = 0.$

$u(x,y) = B\sin\lambda x\,(Ce^{\lambda y} + De^{-\lambda y})$

Using ii), $0 = B\sin\lambda 8\,(Ce^{\lambda y} + De^{-\lambda y})$

$\sin 8\lambda = 0 = \sin n\pi \Rightarrow \lambda = \frac{n\pi}{8}$

$u(x,y) = B\sin\frac{n\pi x}{8}\left(Ce^{\frac{n\pi y}{8}} + De^{-\frac{n\pi y}{8}}\right)$

Using iii), $0 = B\sin\frac{n\pi x}{8}\left(Ce^{\infty} + De^{-\infty}\right)$

$\Rightarrow C = 0$

$u(x,y) = BDe^{-\frac{n\pi y}{8}} \sin\frac{n\pi x}{8}$

The most g.s. is $u(x,y) = \sum_{n=1}^{\infty} B_n e^{-\frac{n\pi y}{8}} \sin\frac{n\pi x}{8}$

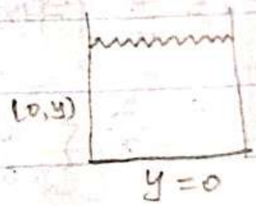
Using iv), $100 \sin\frac{\pi x}{8} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{8}$

Equating like coefficients, $B_1 = 100$, $B_n = 0$, $n \neq 1$.

$u(x,y) = 100 e^{-\frac{\pi y}{8}} \sin\frac{\pi x}{8}$.

3. A long rectangular plate has its surfaces insulated and the 2 long sides as well as one of the short sides are maintained at 0°C. Find an expression for the steady state temp. $u(x,y)$ if the short side $y=0$ is π cm long and is kept at u_0°C

The B.C.'s are



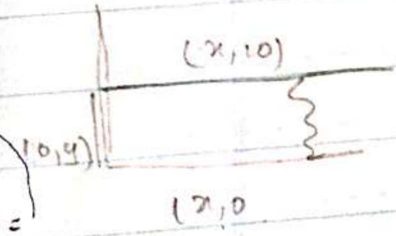
i) $u(0,y)=0, \forall y$ ii) $u(\pi,y)=0 \ \forall y$

iii) $u(x,\infty)=0$ iv) $u(x,0)=u_0$, $0<x<\pi$

$$u(x,y) = \sum_{n=1,3..}^{\infty} \frac{4u_0}{n\pi} \sin nx \, e^{-ny}$$

4. An infinitely long rectangular plate with insulated surfaces is 10cm wide. The 2 long edges and one short edge are kept at 0 temp, while the other short edge $x=0$ is kept at temp given by

$$u = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10-y), & 5 \le y \le 10 \end{cases}$$



Find the steady state temp dis in the plate.

BC's i) ~~$u(0,y)=0$~~ $u(x,0)=0$ ii) ~~$u(10,y)=0$~~ $u(x,10)=0$ iii) $u(\infty,y)=0$ iv) $u(0,y)=$

$$u(x,y) = \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{2} \sin\frac{n\pi y}{10} e^{-\frac{n\pi x}{10}}$$

An integral transform when applied to a p.d.e. reduces the no of its independent variables by one.

UNIT - IV Fourier Transforms.

Laplace transforms are used to find solution of o.d.e.s. Fourier transforms are used to find solution of p.d.es. The effect of applying an integral transform to a p.d.e. is to reduce the no. of independent variables by one.

The effect of mathematical representation of periodic phenomena using complex nos leads to complex form of the Fourier series representation of periodic function. The representation of periodic signals as a linear combination of harmonically related complex exponentials can be extended to develop a representation of a periodic signals as linear combination of complex exponentials. This leads to Fourier transforms.

Fourier integral (theorem) formula: If $f(x)$ is piecewise continuously differentiable and absolutely integrable in $(-\infty, \infty)$ then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i(x-t)s}\, dt\, ds.$$

$\int_{-\infty}^{\infty} |f(x)|\, dx$ is cgt.

(OR)

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\{(x-t)s\}\, dt\, ds$$

Complex Fourier Transform: (Infinite)

Let $f(x)$ be a function defined in $(-\infty, \infty)$ and be piecewise continuous in each finite partial interval and absolutely integrable in $(-\infty, \infty)$. Then the complex Fourier transform of $f(x)$ is defined by

$$F[f(x)] = \bar{f}(s) = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \{\cos(x-t)s + i \sin(x-t)s\} \, dt \, ds$$

Equating real & imaginary parts, $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos(x-t)s \, dt \, ds = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} \, dt \, ds$

$f(x) = 0$, (imaginary part) $\because f(x)$ is real

<u>Inversion theorem for Complex Fourier Transform:</u>

If $f(x)$ satisfies the Dirichlet's conditions in every finite interval $(-l, l)$ and if its is absolutely integrable in the range and if $F(s)$ denotes the CFT of $f(x)$, then at every pt of continuity of $f(x)$, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds$$

<u>Properties:</u>

1. Fourier transform is linear.

(ie) $F[af(x) + bg(x)] = aF[f(x)] + bF[g(x)]$ where F stands for Fourier Transform.

<u>proof:</u> $F[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{isx} \, dx$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{isx} \, dx$$

$$= aF[f(x)] + bF[g(x)].$$

2. <u>Shifting property:</u>

i) If $F[f(x)] = F(s)$, then $F[f(x-a)] = e^{isa} F(s)$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} \, dx$$

Put $x-a=t \Rightarrow x=a+t$, $dx=dt$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{is(a+t)}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{isa}\, e^{ist}\, dt$$

$$= e^{isa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= e^{isa} F(s).$$ next page property ii

3. Change of scale property *

If $F[f(x)] = F(s)$, then $F[f(ax)] = \frac{1}{a}F\left(\frac{s}{a}\right)$, $a>0$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

put $ax=t$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{is\left(\frac{t}{a}\right)}\, \frac{dt}{a}$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}\, dt$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right),\quad a>0.$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(t)\, e^{i(s/a)t}\, \frac{dt}{a} = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i(s/a)t}\, \frac{dt}{a} = -\frac{1}{a}F(s/a)$$

* 4. Modulation theorem.

If $F[f(x)] = F(s)$, then $F[f(x)\cos ax] = \frac{1}{2}\left[F(s+a) + F(s-a)\right]$

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax \, e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2} \right) e^{isx} \, dx$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[e^{i(s+a)x} + e^{+i(s-a)x} \right] dx$$

$$= \frac{1}{2} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} \, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s-a)x} \, dx \right\}$$

$$= \frac{1}{2} \left[F(s+a) + F(s-a) \right]$$

Shifting property.

* 5. $F[e^{iax} f(x)] = F(s+a)$

ii) $F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} \, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} \, dx = F(s+a)$$

$\cancel{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{is}}$ $F(s+a)$

a ≠ 0.

6. If $F[f(x)] = F(s)$, then $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$.

7. $F[f^n(x)] = (-is)^n F(s)$ if $f, f', f'', \ldots, f^{(n-1)} \to 0$ as $x \pm \infty$.

8. $F\left[\int_a^x f(x) \, dx \right] = \frac{F(s)}{(-is)}$

9. $F[\overline{f(x)}] = \overline{F(-s)}$

We know that $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} dx.$$

Taking Complex Conjugate on both Sides,

$$\overline{F(-s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)} e^{isx} dx = F[\overline{f(x)}]$$

Similarly we can prove $F[\overline{f(-x)}] = \overline{F(s)}$.

We know that

4. $$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$$

$$F[f(-x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x) e^{isx} dx$$

$-x = t \Rightarrow -dx = dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t) e^{-ist} (-dt)$$

$x \to -\infty \Rightarrow t \to \infty$

$x \to \infty \Rightarrow t \to -\infty$

$$= + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ist} dt$$

$$= F(-s)$$

$$\Rightarrow F[\overline{f(-x)}] = \overline{F(-s)}$$

5. $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

proof:

Diff w.r.t 's', o.b.s, n times

$= \frac{1}{\sqrt{2\pi}} \frac{d^n}{ds^n} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$\frac{d^n F(s)}{ds^n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty} \frac{\partial^n}{\partial s^n} [f(x) e^{isx}] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (ix)^n e^{isx} dx$$

$$= i^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) x^n e^{isx} dx = (i)^n F[x^n f(x)] \Rightarrow F[x^n f(x)] = \frac{1}{(i)^n} \frac{d^n}{ds^n}$$

1. Show that $f(x) = 1,\ 0 < x < \infty$ Cannot be represented by a Fourier integral.

$$\int_0^\infty |f(x)|\,dx = \int_0^\infty 1\,dx = (x)_0^\infty = \infty.$$

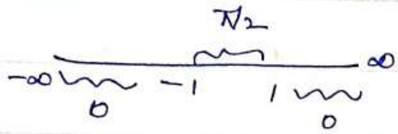
(ie) $\int_0^\infty |f(x)|\,dx$ is not cgt. Hence $f(x) = 1$ Cannot be represented by a Fourier integral.

2. If $f(x) = \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ Show that $f(x) = \int_0^\infty \frac{\cos sx \sin s}{s}\,ds$

Hence Show that $\int_0^\infty \frac{\sin s}{s}\,ds = \frac{\pi}{2}$

We know that the Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t) \cos (x-t)s\,dt\,ds$$



Here $f(t) = \begin{cases} \frac{\pi}{2} & \text{for } |t| < 1 \ (-1 < t < 1) \\ 0 & \text{for } |t| > 1 \text{ (ie) } t > 1 \ \& \ -t > 1 \Rightarrow t < -1 \end{cases}$ This

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-1}^1 \frac{\pi}{2} \cos (x-t)s\,dt\,ds$$

$$= \frac{1}{2}\int_0^\infty \left(\frac{\sin (x-t)s}{-s}\right)_{-1}^1 ds$$

$$= \frac{1}{2}\int_0^\infty \left(-\frac{\sin (x-1)s}{s} + \frac{\sin (x+1)s}{s}\right) ds$$

$F(s) = (-i)^n \frac{d^n F(s)}{ds^n}$

$(i)^n(-i)^n = (-1)^n(i^2)^n$
$= (-1)^n(-1)^n = (-1)^{2n} = 1.$

$$= \frac{1}{2}\int_0^\infty \frac{1}{s}\left\{-\sin(x-1)s + \sin(x+1)s\right\} ds$$

$$= \frac{1}{2}\int_0^\infty \frac{1}{s}\{-\sin \cancel{sx}\cos s + \cos sx \sin s$$
$$+ \sin \cancel{sx}\cos x + \cos sx \sin s\} ds$$

$$= \frac{1}{2}\int_0^\infty \frac{2\cos sx \sin s}{s} ds$$

$$= \int_0^\infty \frac{\cos sx \sin s}{s} ds$$

(ie) $\displaystyle\int_0^\infty \frac{\cos sx \sin s}{s} ds = f(x)$

$$= \begin{cases} \frac{\pi}{2}, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

Putting $x=0$ we get $\displaystyle\int_0^\infty \frac{\sin s}{s} dx = \frac{\pi}{2}$.

3. Express the function $f(x) = \begin{cases} 1 & \text{for } |x|\le 1 \\ 0 & \text{for } |x|>1 \end{cases}$ as a Fourier integral.

Hence evaluate $\displaystyle\int_0^\infty \frac{\sin s \cos sx}{s} ds$ and find the value of $\displaystyle\int_0^\infty \frac{\sin s}{s} ds$.

Sol: The Fourier integral formula is

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos(x-t)s\, dt\, ds.$$

Here $f(t) = 1, |t|<1$ ie $-1 \le t \le 1$

$= 0, |t|>1$ (ie) $t>1, \; -t>1 \Rightarrow t<-1$

$t<1, -t<1$
$\Rightarrow t>-1$
$-1<t<1$

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_{-1}^{1} \cos(x-t)s \, dt \, ds$$

$$= \frac{1}{\pi}\int_0^\infty \left(\frac{\sin(x-t)s}{-s}\right)_{-1}^{1} \cancel{dt} \, ds$$

$$= \frac{1}{\pi}\int_0^\infty \left(-\frac{\sin(x-1)s}{s} + \frac{\sin(x+1)s}{s}\right) \cancel{dt} \, ds$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{2\cos sx \sin s}{s} \cancel{dt} \, ds$$

$$\Rightarrow \int_0^\infty \frac{\cos sx \sin s}{s} ds = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2}, & |x| \le 1 \\ 0, & |x|>1 \end{cases}$$

Putting $x=0$, we get $\int_0^\infty \frac{\sin s}{s} ds = \frac{\pi}{2}$.

Find the Fourier integral of the fun $f(t) = \begin{cases} e^{at}, & t<0 \\ e^{-at}, & t>0. \end{cases}$

$$\int_0^\infty \frac{\cos \lambda x}{a^2+\lambda^2} d\lambda = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2a} e^{ax}, & x<0 \\ \frac{\pi}{2a} e^{-ax}, & x>0 \end{cases}$$

Find the Fourier integral of the function $f(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$

verify the representation directly at the pt $x=0$.

The Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{+\infty} f(t)\cos(x-t)s\,dt\,ds$$

Here $f(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}, & t = 0 \\ e^{-t}, & t > 0 \end{cases}$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} e^{-t}\cos(x-t)s\,dt\,ds$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} e^{-t}\left[\cos sx \cos st + \sin sx \sin st\right] dt\,ds$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\cos sx\int_0^{\infty} e^{-t}\cos st\,dt + \sin sx\int_0^{\infty} e^{-t}\sin st\,dt\right] ds$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\cos sx\left(\frac{1}{1+s^2}\right) + \sin sx\left(\frac{s}{1+s^2}\right)\right] ds$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{\cos sx + s\sin sx}{1+s^2}\,ds$$

$$\int_0^{\infty}\frac{\cos sx + s\sin sx}{1+s^2}\,ds = \pi f(x)$$

$$= \pi\begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$$

Putting $x = 0$, $\int_0^{\infty}\frac{1}{1+s^2}\,ds = \pi f(0)$

$$\left[\tan^{-1}(s)\right]_0^\infty = \pi f(0)$$

$$\frac{\pi}{2} = \pi f(0) \Rightarrow f(0) = \frac{1}{2}.$$

The value of the given fun at $x=0$ is $\frac{1}{2}$.

Fourier sine & Cosine integrals:

The Fourier sine integral is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t \, dt \, d\lambda$$

The Fourier Cosine integral is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t \, dt \, d\lambda$$

1. Use the appropriate Fourier integral formula to prove that

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos x\lambda}{\lambda^2+a^2}\, d\lambda.$$

(Here λ is used instead of s. Presence of $\cos\lambda x$ shows that F. Cosine integral formula to be used)

Fourier Cosine integral of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t \, dt \, d\lambda.$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty e^{-at}\cos\lambda t \, dt \, d\lambda$$

Using Fourier integral formula, p.t. $e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{(\lambda^2+2)\cos x\lambda}{\lambda^4+4}\,d\lambda$.

(Cosine integral)

<u>Property</u>:

i) $F[f'(x)] = -is\,F(s)$ if $f(x)\to 0$ as $x\to\pm\infty$.

ii) $F[f^{(n)}(x)] = (-is)^n F(s)$ if $f(x), f'(x), \ldots, f^{(n-1)}(x)\to 0$ as $x\to\pm\infty$.

We know that

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f'(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\,d[f(x)]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[e^{isx}f(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{isx}(is)f(x)\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{(0-0) - is\int_{-\infty}^{\infty} f(x)e^{isx}\,dx\right\}$$

$$= -is\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx = -is\,F(s)$$

Similarly, $F[f^{(n)}(x)] = (-is)^n F(s)$ if $f(x), f'(x), \ldots, f^{(n-1)}(x)\to 0$ as $x\to\pm\infty$

$$F\left[\int_a^x f(x)\,dx\right] = \frac{F(s)}{(-is)}$$

Let $\phi(x) = \int_a^x f(x)\,dx$.

$\Rightarrow \phi'(x) = f(x)$

$F[\phi'(x)] = -is\,F[\phi(x)]$ by ~~pre~~ previous property.

$= -is\,F\left[\int_a^x f(x)\,dx\right]$

$$F\left[\int_a^x f(x)\,dx\right] = \frac{+1}{-is}\,F[\phi'(x)]$$

$$= \frac{+1}{(-is)}\,F[f(x)]$$

$$= \frac{F(s)}{(-is)}$$

1. Find the Complex Fourier transform of $f(x) = \begin{cases} x, & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$x \le a,\ -x \le a$
$\Rightarrow x \ge -a$

$x > a,\ -x > a$
$\Rightarrow x < -a$

$-\infty \quad -a \quad a \quad \infty$

$x < -a \qquad x > a$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ x\left(\frac{e^{isx}}{is}\right) - 1.\left(\frac{e^{isx}}{(is)^2}\right) \right\}_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \left\{ x\left(\frac{e^{isx}}{is}\right) + \frac{e^{isx}}{s^2} \right\}_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{a\,e^{isa}}{is} + \frac{e^{isa}}{s^2} - \frac{(-a)\,e^{-isa}}{is} - \frac{e^{-isa}}{s^2} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{2a \cos sa}{is} + \frac{1}{s^2}\left[e^{isa} - e^{-isa}\right] \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{2a \cos sa}{is} + \frac{1}{s^2}\, 2i \sin sa \right\}$$

$$= \frac{2i}{s^2}\frac{1}{\sqrt{2\pi}}\left[\sin sa - s a\cos sa\right] \qquad \left\{ \frac{1}{i}\times\frac{(-i)}{-i} = \frac{-i}{-i^2} = -i \right.$$

2. Find the Complex F.T. of $f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a \text{ \& } x > b. \end{cases}$

$-\infty$ to a

a to b

b to ∞

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_a^b e^{ikx} e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_a^b e^{i(k+s)x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{e^{i(k+s)x}}{i(k+s)}\right\}_a^b$$

$$= \frac{-i}{(k+s)\sqrt{2\pi}}\left\{e^{i(k+s)b} - e^{i(k+s)a}\right\}$$

$$= \frac{i}{\sqrt{2\pi}\,(k+s)}\left[e^{i(k+s)a} - e^{i(k+s)b}\right]$$

3. Find the Fourier Transform of $f(x) = \begin{cases} 1-x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ Hence, evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx$

Inversion problem

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{(1-x^2)\left(\frac{e^{isx}}{is}\right) - (-2x)\left(\frac{e^{isx}}{(is)^2}\right) + (-2)\left(\frac{e^{isx}}{(is)^3}\right)\right\}_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ -\cancel{4}\frac{2e^{is}}{s^2} - \frac{2e^{is}}{-is^3} + \frac{2e^{-is}}{i^2 s^2} + \frac{e^{-is}}{i^3 s^3} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{-2e^{is}}{s^2} + \frac{2e^{is}}{is^3} - \frac{2e^{-is}}{s^2} - \frac{e^{-is}}{is^3} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{-2}{s^2} \left[e^{is} + e^{-is} \right] + \frac{2}{is^3} \left[e^{is} - e^{-is} \right] \right\}$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ \frac{-2}{s^2} \cos s + \frac{2i \sin s}{is^3} \right\}$$

$$= \frac{4}{\sqrt{2\pi}} \left\{ -\frac{\cos s}{s^2} + \frac{\cancel{2}\sin s}{s^3} \right\} = \frac{-4}{\sqrt{2\pi}} \left\{ \frac{\cos s}{s^2} - \frac{\sin s}{s^3} \right\}$$

$$= \frac{-4}{\sqrt{2\pi}} \left\{ \frac{s\cos s - \sin s}{s^3} \right\}$$

Using inversion formula;

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, \cancel{dx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{-4}{\sqrt{2\pi}} \left\{ \frac{s\cos s - \sin s}{s^3} \right\} e^{-isx} \, ds$$

$$= \frac{-4}{2\pi} \int_{-\infty}^{\infty} \left(\frac{s\cos s - \sin s}{s^3} \right) e^{-isx} \, \cancel{dx} \, ds$$

$$= -\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{s\cos s - \sin s}{s^3} \right) e^{-isx} \, \cancel{dx} \, ds$$

$$\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)(\cos sx - i\sin sx)\,ds = -\frac{\pi}{2}f(x)$$

$$= \begin{cases} -\frac{\pi}{2}(1-x^2), & |x|<1 \\ 0, & |x|>1 \end{cases}$$

Equating the real parts,

$$\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sx\,ds = \begin{cases} -\frac{\pi}{2}(1-x^2), & |x|<1 \\ 0, & |x|>1 \end{cases}$$

Set $x=\frac{1}{2}$.

$$\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos\frac{s}{2}\,ds = -\frac{\pi}{2}\left(1-\frac{1}{4}\right)$$

$$2\int_{0}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos\frac{s}{2}\,ds = -\frac{3\pi}{8}$$

$$\therefore \int_{0}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos\frac{s}{2}\,ds = \frac{-3\pi}{16}$$

Changing the dummy variable s into x, we get

$$\int_{0}^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}.$$

4. Show that the transform of $e^{-\frac{x^2}{2}}$ is $e^{-\frac{s^2}{2}}$ by finding the Fourier transform of $e^{-a^2x^2}$, $a>0$.

$$F\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2+\frac{s^2}{4a^2}-\frac{s^2}{4a^2}+isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2 + \frac{i^2s^2}{4a^2} - isx\right) - \frac{s^2}{4a^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} . e^{-\frac{s^2}{4a^2}} dx$$

$$= e^{-\frac{s^2}{4a^2}} . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

Let $ax - \frac{is}{2a} = t$

$$= e^{-\frac{s^2}{4a^2}} . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$a\,dx = dt$

$dx = \frac{dt}{a}$

$$= \frac{1}{a\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{1}{a\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} . \sqrt{\pi}$$

$$= \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} .$$

Setting $a = \frac{1}{\sqrt{2}}$,

$$F\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{s^2}{2}}$$

If a transformation of a function $f(x)$ is equal to $f(s)$ then the function $f(x)$ is called self reciprocal.

Note: $e^{-\frac{x^2}{2}}$ is self-reciprocal.

Find the Fourier transform of $f(x)$ given by

Parsevals

$$f(x) = \begin{cases} 1, & |x| < 2 \\ 0, & |x| > 2 \end{cases}$$ and hence evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$

& $\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} 1.\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-2}^{2}$$

$$= \frac{1}{is\sqrt{2\pi}} \left[e^{is2} - e^{-is2} \right]$$

$$= \frac{1}{is\sqrt{2\pi}}\, 2i \sin s2 \quad = \frac{2\sin 2s}{s\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\, \frac{\sin 2s}{s}.$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{\sin 2s}{s}\, e^{-isx}\, ds.$$

$$\begin{cases} 1, & |x|<2 \\ 0, & |x|>2 \end{cases} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin 2s}{s} \left\{ \cos sx - i \sin sx \right\} ds$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin 2s \cos sx}{s}\, ds = \begin{cases} 1, & |x|<2 \\ 0, & |x|>2 \end{cases}$$

Putting $x=0$,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin 2s}{s}\, ds = 1$$

$$\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin 2s}{s}\, ds = 1$$

Let $2s = x$

$\Rightarrow s = \frac{x}{2}$

$$\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin x}{\frac{x}{2}} \cdot \frac{dx}{2} = 1$$

$$\frac{2}{\pi}\int_0^\infty \frac{\sin x}{x}\,dx = 1$$

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Using Parseval's identity, $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$

$$\int_{-2}^{2} dx = \int_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{\sin 2s}{s}\right)^2 ds$$

$$(x)_{-2}^{2} = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin 2s}{s}\right)^2 ds \quad \text{Put } 2s = t \quad = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{\frac{t}{2}}\right)^2 \frac{dt}{2}$$

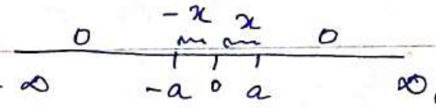
$$4 = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{\frac{t}{2}}\right)^2 \frac{dt}{2} \qquad = \frac{4}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$$

$$4 = \frac{4}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt \Rightarrow 1 = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$$

$$1 = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt \Rightarrow \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

S.t. the FT of $f(x) = \begin{cases} |x|, & |x| < a \\ 0, & |x| > a,\ a > 0 \end{cases}$ is $\sqrt{\frac{2}{\pi}}\left[\frac{sa\sin sa + \cos sa - 1}{s^2}\right]$

Here $f(x) = \begin{cases} |x|, & -a < x < a \\ 0, & |x| > a \end{cases}$



$$= \begin{cases} -x, & -a < x < 0 \\ x, & 0 < x < a \\ 0 & \text{for } x < -a \ \& \ x > a \end{cases} \qquad \frac{1}{\sqrt{2\pi}}\left\{\int_{-a}^{a} |x|\cos sx + i\int_{-a}^{a} |x|\sin sx\,dx\right\}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{0} -x e^{isx}\,dx + \int_0^a x e^{isx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[(-x)\left(\frac{e^{isx}}{is}\right) - (-1)\left(\frac{e^{isx}}{i^2 s^2}\right)\right]_{-a}^{0} + \left[x\left(\frac{e^{isx}}{is}\right) - 1\cdot\left(\frac{e^{isx}}{i^2 s^2}\right)\right]_0^a\right.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ -\frac{1}{s^2} + \frac{a e^{isa}}{is} + \frac{e^{isa}}{s^2} - \frac{1}{s^2} \right\}$$

$$- \frac{a e^{-isa}}{is} + \frac{e^{-isa}}{s^2} \Big\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ -\frac{2}{s^2} + \frac{a}{is} \left[e^{isa} - e^{-isa} \right] + \frac{1}{s^2} \left[e^{isa} + e^{-isa} \right] \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ -\frac{2}{s^2} + \frac{a}{is} 2i \sin sa + \frac{1}{s^2} 2 \cos sa \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ -\frac{2}{s^2} + \frac{2a \sin sa}{s} + \frac{2 \cos sa}{s^2} \right\}$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ -\frac{1}{s^2} + \frac{a \sin sa}{s} + \frac{\cos sa}{s^2} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{sa \sin sa + \cos sa - 1}{s^2} \right]$$

Find the Fourier transform of f(x) given by

$$f(x) = \begin{cases} x^2, & |x| \le a \\ 0, & |x| > a \end{cases} \qquad \sqrt{\frac{2}{\pi}} \left[\frac{a^2 s^2 \sin sa + 2sa \cos sa - 2 \sin sa}{s^3} \right]$$

Find the Fourier transform of f(x) given by

$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$ and $\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s} ds$.

Inversion problem

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{isx}}{is} \right)_{-a}^{a}$$

$$= \frac{1}{is\sqrt{2\pi}} \left[e^{isa} - e^{-isa} \right]$$

$$= \frac{2i \sin sa}{is\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}$$

Using inversion formula,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{\sin as}{s} \right) e^{-isx} ds = f(x)$$

$$= \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

Equating real parts,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin as}{s} \cos sx \, ds = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin as \cos sx}{s} ds = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$\int_{0}^{\infty} \frac{\sin as \cos sx}{s} ds = \begin{cases} \frac{\pi}{2}, & |x| < a \\ 0, & |x| > a \end{cases}$$

Setting $x = 0$, $\displaystyle\int_{0}^{\infty} \frac{\sin as}{s} ds = \frac{\pi}{2}$ $\qquad \displaystyle\int_{0}^{\infty} \frac{\sin x}{x} dx = \pi/2$

Let $as = t \Rightarrow ds = \frac{dt}{a}$.

$$\therefore \int_{0}^{\infty} \frac{\sin t}{\frac{t}{a}} \frac{dt}{a} = \pi/2 \Rightarrow \int_{0}^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2} \Rightarrow$$ (by changing dummy variable t to x)

complete argument or statement in a complete way

Convolution theorem or Faltung theorem.

Def: The Convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f*g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt .$$

Theorem: The Fourier transform of the Convolution of $f(x)$ and $g(x)$ is the product of their Fourier Transforms.

(ie) $F[f(x) * g(x)] = F(s) . G(s)$

$$= F[f(x)] . F[g(x)]$$

$$F[f*g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)\, e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt \right) e^{isx} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left(\int_{-\infty}^{\infty} g(x-t)\, e^{isx} dx \right) dt$$ (by Changing the order of integration)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \sqrt{2\pi}\, F[g(x-t)]\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{its}\, G(s) dt$$ (by Shifting property $F[f(x-a)] = e^{isa} F(s)$)

$$= G(s) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{its} dt$$

$$= G(s)\; F(s)$$

$$= F(s)\; G(s)$$

By inversion,

$$F^{-1}[F(s)G(s)] = f*g$$
$$= F^{-1}[F(s)] * F^{-1}[G(s)]$$

Parseval's identity: If $F(s)$ is the Fourier transform of $f(x)$, then
$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

First let us prove
$F[\overline{f(-x)}] = \overline{F(s)}$ where $\overline{F(s)}$ denotes the conjugate of $F(s)$.
$$\overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{-isx} dx$$

Let $x = -v$
$x \to -\infty \Rightarrow v \to \infty$
$x \to \infty \Rightarrow v \to -\infty$

$$= \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \overline{f(-v)}\, e^{+isv} dv$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-v)}\, e^{isv} dv$$
$$= F[\overline{f(-v)}] = F[\overline{f(-x)}] \quad \text{by changing the dummy variable.} \quad —(1)$$

By Convolution theorem,
$$F[f(x) * g(x)] = F(s)\,G(s)$$
$$f*g = F^{-1}[F(s)\,G(s)]$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, G(s)\, e^{-isx} ds$$

Putting $x=0$, we get
$$\cancel{\frac{1}{\sqrt{2\pi}}} \int_{-\infty}^{\infty} f(t)\, g(-t)\, dt = \cancel{\frac{1}{\sqrt{2\pi}}} \int_{-\infty}^{\infty} F(s)\, G(s)\, ds \quad —(2)$$

~~Since it is true for all g(t), take g(t) = $\overline{f(-t)}$~~

~~Let~~ $g(-t) = \overline{f(t)}$. $\therefore g(t) = \overline{f(-t)}$

Now $G(s) = F[g(t)] = F[\overline{f(-t)}] = \overline{F(s)}$ by ①

② becomes $\int_{-\infty}^{\infty} f(t)\overline{f(t)}\,dt = \int_{-\infty}^{\infty} F(s)\overline{F(s)}\,ds$

$$\text{(ie)} \quad \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

1. Using Parseval's identity, prove $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ where

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{isx}}{is}\right)_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}\, is}\left[e^{isa} - e^{-isa}\right]$$

$$= \frac{2i\sin sa}{is\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\,\frac{\sin sa}{s}$$

By Parseval's identity,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-a}^{a} dx = \int_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{\sin sa}{s}\right)^2 ds$$

$$2a = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds$$

$$a = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin sa}{s}\right)^2 ds.$$

Let $sa = t \Rightarrow s = t/a$

$$= \frac{2}{\pi}\int_0^\infty \left(\frac{\sin t}{\frac{t}{a}}\right)^2 \cdot \frac{dt}{a}$$

$$= \frac{2a}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

2. Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$ and hence find the value of

$$\int_0^\infty \frac{\sin^4 t}{t^4}\, dt$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1-|x|)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_{-1}^{1} (1-|x|)(\cos sx + i\sin sx)\, dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_{-1}^{1} (1-|x|)\cos sx\, dx + i\int_{-1}^{1} (1-|x|)\sin sx\, dx\right\}$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^1 (1-x)\cos sx\, dx.$$

$$= \sqrt{\frac{2}{\pi}}\left\{(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(-\frac{\cos sx}{s^2}\right)\right\}_0^1$$

$2\theta = s$

$2\sin^2\theta = 1-\cos 2\theta$

$2\sin^2 s/2$

$$= \sqrt{\frac{2}{\pi}}\left\{-\frac{\cos s}{s^2} + \frac{1}{s^2}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right] = \sqrt{\frac{2}{\pi}}\ \frac{2\sin^2 s/2}{s^2}.$$

Using Parseval's identity,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1-|x|)^2 dx = \int_{-\infty}^{\infty} \frac{2}{\pi} \cdot 4 \frac{\sin^4 \frac{s}{2}}{s^4} ds$$

$$2\int_{0}^{1} (1-x)^2 dx = \frac{8}{\pi} \int_{-\infty}^{\infty} \frac{\sin^4 \frac{s}{2}}{s^4} ds$$

$$2\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{16}{\pi}\int_0^{\infty} \frac{\sin^4 \frac{s}{2}}{s^4} ds$$

$$\frac{2}{3} = \frac{16}{\pi}\int_0^{\infty} \frac{\sin^4 \frac{s}{2}}{s^4} ds$$

$$\int_0^{\infty} \frac{\sin^4 \frac{s}{2}}{s^4} ds = \frac{2\pi}{16.3}$$

$$= \frac{\pi}{24}$$

Setting $\frac{s}{2} = x$,

$$\int_0^{\infty} \frac{\sin^4 x}{16x^4} 2dx = \frac{\pi}{24}$$

$$\int_0^{\infty} \frac{\sin^4 x}{8x^4} dx = \frac{\pi}{24}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}.$$

Replacing x by t, $\int_0^{\infty} \frac{\sin^4 t}{t^4} dt = \frac{\pi}{3}$

Infinite Fourier Cosine and Sine transform:

Infinite Fourier Cosine transform of $f(x)$ is defined by

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

The inversion formula for infinite Cosine transform is given by,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

The Fourier Sine transform is given by

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$ and its inversion formula is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

Properties regarding Cosine and Sine Transforms:

1. Cosine and Sine transforms are linear.

$$F_c[af(x) + bg(x)] = a F_c[f(x)] + b F_c[g(x)]$$

$$F_s[af(x) + bg(x)] = a F_s[f(x)] + b F_s[g(x)]$$

lation
roperty

2. $F_s[f(x) \sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$
3. $F_s[f(x) \cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$
4. $F_c[f(x) \sin ax] = \frac{1}{2}[F_s(a+s) + F_s(a-s)]$
5. $F_c[f(x) \cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$
6. $F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{s}{a}\right)$
7. $F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right)$

Identities If $F_c(s), G_c(s)$ are the Fourier Cosine transforms and $F_s(s), G_s(s)$ are the Fourier Sine transforms of $f(x)$ and $g(x)$ resp, then

1. $\int_0^\infty f(x) g(x)\, dx = \int_0^\infty F_c(s)\, G_c(s)\, ds$

2. $\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_s(s)\, G_s(s)\, ds$

3. $\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(s)|^2\, ds = \int_0^\infty |F_s(s)|^2\, ds.$

1. Find the Fourier Cosine and Sine transforms of e^{-ax}, $a>0$ and hence deduce the inversion formula.

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-ax}}{a^2+s^2} \left[-a\cos sx + s \sin sx \right] \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}, \quad a>0.$$

Using inversion formula,

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} \cos sx\, ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{a^2+s^2}\, ds$$

$$\int_0^{\infty} \frac{\cos sx}{a^2+s^2}\, ds = \frac{\pi}{2a} e^{-ax}, \quad a>0$$

Changing the variables (s by x & x by α)

$$\int_0^{\infty} \frac{\cos \alpha x}{a^2+x^2}\, dx = \frac{\pi}{2a} e^{-\alpha a}, \quad a>0.$$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-ax}}{a^2+s^2} \left[-a \sin sx - s\cos sx\right] \right\}_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{s}{s^2+a^2}$$

Using the inversion formula,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \sin sx\, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \sin sx\, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{s}{s^2+a^2} \sin sx\, ds$$

$$\int_0^{\infty} \frac{s}{s^2+a^2} \sin sx\, ds = \frac{\pi}{2} e^{-ax}$$

$$\int_0^{\infty} \frac{x \sin \alpha x}{x^2+a^2}\, dx = \frac{\pi}{2} e^{-\alpha a}, \quad a>0$$

2. Find the Fourier sine Transform of $\frac{x}{a^2+x^2}$ and Fourier Cosine Transform of $\frac{1}{a^2+x^2}$.

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x}{a^2+x^2} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2} e^{-as} \right) \text{ (by previous problem)}$$

$$= \sqrt{\frac{\pi}{2}} e^{-as}$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{a^2+x^2} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2a} e^{-as} \right] \text{ (by previous problem).}$$

$$= \sqrt{\frac{\pi}{2}} \frac{e^{-as}}{a}$$

3. Find the Fourier Sine & Cosine Transform of e^{-x} and hence find the Fourier sine Transform of $\frac{x}{1+x^2}$ & Fourier Cosine Transform of $\frac{1}{1+x^2}$.

$$F_c[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-x}}{1+s^2} \left[-\cos sx + s \sin sx \right] \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}.$$

Using inversion formula,

$$e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+s^2} \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\cos sx}{1+s^2} \, ds.$$

$$\int_0^\infty \frac{\cos sx}{1+s^2} \, ds = \frac{\pi}{2} e^{-x}.$$

$$F_c\left(\frac{1}{1+x^2}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2} e^{-s}\right) = \sqrt{\frac{\pi}{2}} e^{-s}.$$

$$F_s[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-x}}{1+s^2} [-\sin sx - s\cos sx] \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{s}{1+s^2} \right\}$$

Using inversion formula,

$$e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2} \sin sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sx \, ds$$

$$\int_0^\infty \frac{s}{1+s^2} \sin sx \, ds = \frac{\pi}{2} e^{-x}$$

$$F_s\left(\frac{x}{1+x^2}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x}{1+x^2} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\pi}{2} e^{-s}$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-s}$$

4. Find the Fourier Cosine transform of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^a \cos x \cos sx\, dx + \int_a^\infty 0 \cos sx\, dx \right\}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^a [\cos(s+1)x + \cos(s-1)x]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1} \right]$$

5. Find the Fourier sine transform of $\frac{1}{x}$

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x}\, dx$$

$sx = \theta$

$dx = \frac{d\theta}{s}$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\frac{\theta}{s}} \frac{d\theta}{s}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta}\, d\theta$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}}$$

6. Using Parseval's identity, evaluate $\int_0^\infty \frac{dx}{(a^2+x^2)^2}$ & $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}dx$, if $a>0$

Let $f(x) = e^{-ax}$. Then $F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$.

$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$.

Using Parseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F_c(s)|^2 ds \qquad \int_{-\infty}^{\infty} (e^{-ax})^2 dx = \int_{-\infty}^{\infty} \frac{2}{\pi} \frac{a^2}{(s^2+a^2)^2} ds$$

$$2\int_0^\infty |f(x)|^2 dx = 2\int_0^\infty |F_c(s)|^2 ds$$

Using the identity,

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c(s)|^2 ds$$

$$\int_0^\infty e^{-2ax} dx = \int_0^\infty \frac{2}{\pi}\left(\frac{a}{s^2+a^2}\right)^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{ds\, a^2}{(s^2+a^2)^2}$$

$$\frac{1}{2a} = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2}$$

$$\int_0^\infty \frac{ds}{(s^2+a^2)^2} = \frac{\pi}{2a\,2a^2} = \frac{\pi}{4a^3}, \; a>0$$

Changing the parameter s by x,

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

Also $\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_s(s)|^2 ds$

$$\frac{1}{2a} = \int_0^\infty \frac{2}{\pi} \frac{s^2}{(s^2+a^2)^2} ds$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds = \frac{\pi}{4a}, \text{ if } a>0$$

Changing the parameter s by x, $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\, dx = \frac{\pi}{4a}$.

Evaluate $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$ using transform methods.

Let $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-ax}}{a^2+s^2} \left[-a\cos sx + s\sin sx\right] \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}$$

Similarly, $G_c(s) = \sqrt{\frac{2}{\pi}} \frac{b}{b^2+s^2}$.

Using the identity,

$$\int_0^\infty F_c(s)\, G_c(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{a}{a^2+s^2} \cdot \frac{b}{b^2+s^2}\, ds = \int_0^\infty e^{-ax} e^{-bx}\, dx$$

$$= \int_0^\infty e^{-(a+b)x}\, dx$$

$$\frac{2ab}{\pi} \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{1}{a+b}$$

$$\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2ab(a+b)}, \text{ if } a,b>0$$

$$\therefore \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)} \text{ if } a,b>0$$

Show that i) $F_s[x f(x)] = -\frac{d}{ds} F_c(s)$ ii) $F_c[x f(x)] = \frac{d}{ds} F_s(s)$ and hence find FCT and FST of xe^{-ax} Also evaluate $\int_0^\infty \frac{(x^2-a^2)^2}{(x^2+a^2)^4} dx$ & $\int_0^\infty \frac{x^2 dx}{(x^2+a^2)^4}$ (Refer back)

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$\frac{d}{ds} F_c(s) = -\sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \sin sx \, dx$$

$$= -F_s[x f(x)]$$

$$F_s[x f(x)] = -\frac{d}{ds} F_c(s). \quad \text{Hence i)}$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \cos sx \, dx$$

$$= F_c[x f(x)]$$

Hence ii)

$$F_c[xe^{-ax}] = +\frac{d}{ds} F_s(s)$$

$$= \frac{d}{ds} F_s[e^{-ax}]$$

$$= \frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{\frac{(s^2+a^2).1 - s(2s)}{(s^2+a^2)^2}\right\} = \sqrt{\frac{2}{\pi}} \left\{\frac{a^2-s^2}{(s^2+a^2)^2}\right\}$$

$$F_s[xe^{-ax}] = -\frac{d}{ds} F_c[e^{-ax}]$$

$$= -\frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}\right\}$$

$$= -\sqrt{\frac{2}{\pi}} \left\{\frac{-a(2s)}{(s^2+a^2)^2}\right\} = \sqrt{\frac{2}{\pi}} \cdot \frac{2as}{(s^2+a^2)^2}$$

Fourier Transform

Find the Fourier Transform of $e^{-|x|}$ and hence find the Fourier Transform of $e^{-|x|}\cos 2x$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty} e^{-|x|}\cos sx\,dx + i\int_{-\infty}^{\infty} e^{-|x|}\sin sx\,dx\right\}$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+s^2}\right)$$

To find $F[e^{-|x|}\cos 2x]$

By Modulation property,

$$F[f(x)\cos ax] = \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

$$F[e^{-|x|}\cos 2x] = \frac{1}{2}\left[F(s+2) + F(s-2)\right]$$

$F(s+2)$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+2)x}\,dx.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\cos(s+2)x\,dx$$

$$= \frac{2}{\sqrt{2\pi}}\,\frac{1}{(s+2)^2+1}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{(s+2)^2+1}$$

$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\frac{1}{(s+2)^2+1} + \sqrt{\frac{2}{\pi}}\frac{1}{(s-2)^2+1}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{s^2+4s+5} + \frac{1}{s^2-4s+5}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{s^2-4s+5+s^2+4s+5}{(s^2+4s+5)(s^2-4s+5)}\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{2(s^2+5)}{s^4 - \cancel{4s^3} + 5s^2 + \cancel{4s^3} - 16s^2 + \cancel{20s} + 5s^2 - \cancel{20s} + 25} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{2(s^2+5)}{s^4 - 6s^2 + 25} \right\}$$

Inverse FCT

Solve for $f(x)$ from the integral equation $\int_0^\infty f(x)\cos\alpha x\,dx = e^{-\alpha}$

By changing α to s,

$$\int_0^\infty f(x)\cos sx\,dx = e^{-s}$$

Multiplying by $\sqrt{\frac{2}{\pi}}$,

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\, e^{-s}$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\, e^{-s}$$

$$f(x) = F_c^{-1}\left[\sqrt{\frac{2}{\pi}}\, e^{-s}\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\, e^{-s}\cos sx\,ds$$

$$= \frac{2}{\pi} \int_0^\infty e^{-s}\cos sx\,ds$$

$$= \frac{2}{\pi} \left\{ \frac{e^{-s}}{1+x^2}\left[-\cos sx + \overset{x}{\cancel{s}}\sin sx\right] \right\}_0^\infty$$

$$= \frac{2}{\pi} \left\{ \frac{1}{1+x^2} \right\}$$

Solve for $f(x)$ from the integral equation

$$\int_0^\infty f(x)\sin sx\,dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2 \end{cases}$$

Inverse Fourier Sine

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2 \end{cases}$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\left\{\int_0^1 1\cdot\sin sx\,ds + \int_1^2 2\sin sx\,ds + \int_2^\infty 0\sin sx\,ds\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{\cos sx}{x}\right]_0^1 + 2\left[-\frac{\cos sx}{x}\right]_1^2\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\cos x - 2\cos 2x + 1}{x}\right\}$$

Find $f(x)$ if its sine transform is e^{-as}

$$F_s(s) = e^{-as}$$

Inverse Fourier Sine

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\sin sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{e^{-as}}{a^2+x^2}\left[-a\sin sx - x\cos sx\right]\right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{x}{a^2+x^2}\right)$$

Find the Fourier Sine transform of $\frac{e^{-ax}}{x}$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx\, dx.$$

$$F_s(s) = $$

$$\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x e^{-ax}}{x} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right)$$

Integrating we get

$$F_s(s) = a\sqrt{\frac{2}{\pi}} \int \frac{ds}{s^2+a^2}$$

$$= a\sqrt{\frac{2}{\pi}} \cdot \frac{1}{a} \tan^{-1}\left(\frac{s}{a}\right) + c$$

$$= \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) + c$$

when $s=0$, $F_s(s)=0$. $\therefore c=0$.

$$\Rightarrow F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right).$$

Use transform ~~ex~~ methods to evaluate $\int_0^\infty \frac{x^2\, dx}{(x^2+4)(x^2+9)}$.

Let $f(x) = e^{-2x}$, $g(x) = e^{-3x}$.

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-2x}}{s^2+4} \left[-2\sin sx - s\cos sx\right] \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+4}\right)$$

Similarly $G_s(s) = \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+9} \right)$

Using the identity, $\int_0^\infty F_s(s)\, G_s(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$.

$$\frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+4)(s^2+9)}\, ds = \int_0^\infty e^{-(2+3)x}\, dx$$

$$= \left(\frac{e^{-5x}}{-5} \right)_0^\infty$$

$$\int_0^\infty \frac{s^2}{(s^2+4)(s^2+9)}\, ds = \frac{1}{5} \cdot \frac{\pi}{2} = \frac{\pi}{10}.$$

(ie) $$\int_0^\infty \frac{x^2\, dx}{(x^2+4)(x^2+9)} = \frac{\pi}{10}.$$

Find the Fourier Sine Transform of xe^{-ax} & hence evaluate $\int_0^\infty \frac{x^2\, dx}{(x^2+a^2)^4}$

$$F_s(xe^{-ax}) = \sqrt{\frac{2}{\pi}} \frac{2as}{(s^2+a^2)^2}$$

Using the identity,

$$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_s(s)|^2\, ds.$$

$$\int_0^\infty (xe^{-ax})^2\, dx = \frac{8a^2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)^4}\, ds$$

$$\int_0^\infty \frac{s^2\, ds}{(s^2+a^2)^4}\, ds = \frac{\pi}{8a^2} \int_0^\infty x^2 e^{-2ax}\, dx$$

$$= \frac{\pi}{8a^2} \left\{ x^2 \left(\frac{e^{-2ax}}{-2a} \right) - 2x \left(\frac{e^{-2ax}}{4a^2} \right) + 2 \left(\frac{e^{-2ax}}{-8a^3} \right) \right.$$

$$= \frac{\pi}{8a^2} \left\{ \frac{1}{4a^3} \right\} = \frac{\pi}{32a^5}$$

Replacing s by x,

$$\int_0^\infty \frac{x^2\, dx}{(x^2+a^2)^4}\, dx = \frac{\pi}{32a^5}$$

Find the Fourier Cosine transform of xe^{-ax} & hence find the value of $\int_0^\infty \frac{(x^2-a^2)^2}{(x^2+a^2)^4}dx$

$$F_c(xe^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{a^2-s^2}{(s^2+a^2)^2}\right)$$

Using the identity,

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c(s)|^2 ds$$

$$\int_0^\infty x^2 e^{-2ax} dx = \frac{2}{\pi}\int_0^\infty \frac{(a^2-s^2)^2}{(s^2+a^2)^4} ds$$

$$\left\{x^2\left(\frac{e^{-2ax}}{-2a}\right) - 2x\left(\frac{e^{-2ax}}{4a^2}\right) + 2\left(\frac{e^{-2ax}}{-8a^3}\right)\right\}_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{(a^2-s^2)^2}{(s^2+a^2)^4} ds$$

$$\frac{1}{4a^3} = \frac{2}{\pi}\int_0^\infty \frac{(s^2-a^2)^2}{(s^2+a^2)^4} ds$$

$$\int_0^\infty \frac{(s^2-a^2)^2}{(s^2+a^2)^4} ds = \frac{1}{4a^3}\cdot\frac{\pi}{2} = \frac{\pi}{8a^3}$$

(ie) $\int_0^\infty \frac{(x^2-a^2)^2}{(x^2+a^2)^4} dx = \frac{\pi}{8a^3}$

Find the Fourier Cosine transform of $e^{-x^2/2}$

$$F_c(e^{-\frac{x^2}{2}}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} \cos sx \, dx$$

$$= R.P. \text{ of } \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} e^{isx} \, dx$$

$$= R.P. \text{ of } \sqrt{\frac{2}{\pi}} \int_0^\infty e^{\frac{-x^2}{2} + isx + \frac{s^2}{2} - \frac{s^2}{2}} \, dx$$

$$= R.P. \text{ of } \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\left(\frac{x^2}{2} - isx - \frac{s^2}{2}\right) - \frac{s^2}{2}} \, dx$$

$$= R.P. \text{ of } \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\left(\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right)^2} \cdot e^{-\frac{s^2}{2}} \, dx$$

$$= R.P. \text{ of } \sqrt{\frac{2}{\pi}} e^{-\frac{s^2}{2}} \int_0^\infty e^{-\left(\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right)^2} \, dx$$

$$t = \frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}$$

$$= R.P. \text{ of } \sqrt{\frac{2}{\pi}} e^{-\frac{s^2}{2}} \frac{2}{2} \int_0^\infty e^{-\left(\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right)^2} \, dx$$

$$dt = \frac{dx}{\sqrt{2}}$$

$$= R.P. \text{ of } \sqrt{\frac{2}{\pi}} e^{-\frac{s^2}{2}} \cdot \frac{1}{2} \int_{-\infty}^\infty e^{-\left(\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right)^2} \, dx$$

$$= R.P. \text{ of } \sqrt{\frac{2}{\pi}} e^{-\frac{s^2}{2}} \frac{1}{2} \int_{-\infty}^\infty e^{-t^2} \sqrt{2} \, dt$$

$$= R.P. \text{ of } \frac{1}{\sqrt{\pi}} e^{-\frac{s^2}{2}} \int_{-\infty}^\infty e^{-t^2} \, dt$$

$$= R.P. \text{ of } \frac{1}{\sqrt{\pi}} e^{-\frac{s^2}{2}} \sqrt{\pi}$$

$$= e^{-\frac{s^2}{2}}$$

UNIT-5

Z-transforms

The development of Communication branch is based on discrete analysis. Z-transform plays the same role in discrete analysis as Laplace transform in Continuous systems. The main difference is that the z-transform operates not on functions of the Continuous arguments but on sequences of the discrete integer-valued arguments $n = 0, \pm 1, \pm 2, \ldots$

(Z-transform has many properties similar to those of the Laplace transform)

Defn of Z-transform: Let $\{x(n)\}$ be a sequence defined for $n = 0, \pm 1, \pm 2, \ldots$ Then the 2 sided or bilateral Z-transform of the sequence $x(n)$ is defined as

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

If $f(t)$ is defined for discrete values of 't' where $t = nT$, $n = 0, 1, 2, \ldots$ T being sampling period, then

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} = F(z)$$

If $\{x(n)\}$ is a causal sequence (ie $x(n) = 0$ for $n < 0$), then Z-transform reduces to one-sided or unilateral Z-transform & is given by

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

Unit Sample Sequence: The unit sample sequence $\delta(n)$ is defined as the sequence with values

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Unit step sequence: The unit step sequence $u(n)$ is defined as $u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases}$

Linearity property

$$Z[a\{x(n)\} + b\{y(n)\}] = a Z\{x(n)\} + b Z\{y(n)\}$$
$$= \sum_{n=0}^{\infty} [a x(n) + b y(n)] z^{-n}$$

$$= a \sum x(n) z^{-n} + b \sum y(n) z^{-n}$$

$$= a\, Z\{x(n)\} + b\, Z\{y(n)\}$$

$$Z[af(t) + bg(t)] = \sum_{n=0}^{\infty} [af(nT) + bg(nT)] z^{-n}$$

$$= a \sum f(nT) z^{-n} + b \sum g(nT) z^{-n}$$

$$= a\, Z[f(t)] + b\, Z[g(t)]$$

<u>First Shifting property</u>

$$Z\{a^n x(n)\} = X\left(\frac{z}{a}\right)$$

$$Z[a^n f(t)] = F\left(\frac{z}{a}\right)$$

$$Z[a^n x(n)] = \sum_{n=0}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n}$$

$$= X\left(\frac{z}{a}\right)$$

$$Z[a^n f(t)] = \sum_{n=0}^{\infty} a^n f(nT) z^{-n} = \sum_{n=0}^{\infty} f(nT) \left(\frac{z}{a}\right)^{-n} = F\left(\frac{z}{a}\right)$$

If $Z\{f(t)\} = F(z)$ then $Z\{e^{at} f(t)\} = F[z e^{-aT}]$

If $Z\{f(t)\} = F(z)$ then $Z\{e^{-at} f(t)\} = F[z e^{aT}]$

$$Z\{e^{at} f(t)\} = \sum_{n=0}^{\infty} e^{anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) (z e^{-aT})^{-n}$$

$$= Z[f(t)]_{z \to z e^{-aT}}$$

$$= F[z e^{-aT}]$$

$$Z\{e^{-at} f(t)\} = \sum e^{-anT} f(nT) z^{-n}$$

$$= \sum f(nT) (z e^{aT})^{-n}$$

$$= Z[f(t)]_{z \to z e^{aT}} = F[z e^{aT}]$$

<u>Differentiation in the z-domain</u>

$$Z[n x(n)] = -z \frac{d}{dz} X(z)$$

$$X(z) = Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$\frac{d}{dz}X(z) = -\sum_{n=0}^{\infty} n\,x(n)\,z^{-n-1}$$

$$= -\frac{\sum n\,x(n)\,z^{-n}}{z}$$

$$z\frac{d}{dz}X(z) = -\sum_{n=0}^{\infty} n\,x(n)\,z^{-n} = -Z\{n\,x(n)\}$$

$$F(z) = Z\{f(t)\}$$

$$= \sum_{n=0}^{\infty} f(nT)\,z^{-n}$$

$$\frac{d}{dz}F(z) = -\sum f(nT)\,n\,z^{-n-1}$$

$$= \frac{-\sum f(nT)\,n\,z^{-n}}{z}$$

$$z\frac{d}{dz}F(z) = -\sum_{n=0}^{\infty} n f(nT)\,z^{-n} = -Z[n f(t)]$$

$$* \quad Z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n)\,z^{-n}$$

$$= 1$$

$$Z\{u(n)\} = \frac{z}{z-1}, \quad |z|>1$$

proof: $$Z\{u(n)\} = \sum_{n=0}^{\infty} u(n)\,z^{-n} \qquad u(n) = \begin{cases} 1, & n\geq 0 \\ 0, & n<0 \end{cases}$$

$$= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{z}{z-1}, \quad |z|>1$$

$$(1+x)^{n} = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots\cdots$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots\cdots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

$$Z(t) = \sum_{n=0}^{\infty} nTz^{-n}$$

$$= T\left\{\frac{1}{z}+\frac{2}{z^2}+\cdots\right\}$$

$$= T\cdot\frac{1}{z}\left\{1+2\left(\frac{1}{z}\right)+3\left(\frac{1}{z^2}\right)+\cdots\right\}$$

$$= \frac{T}{z}\left(1-\frac{1}{z}\right)^{-2} = \frac{T}{z}\left(\frac{z^2}{(z-1)^2}\right) = \frac{Tz}{(z-1)^2}$$

$$Z[e^{-at}t] = \left\{\frac{Tz}{(z-1)^2}\right\}_{z\to ze^{aT}}$$

$$= \frac{Tze^{aT}}{(ze^{aT}-1)^2}$$

$$Z[e^{t}\sin 2t] = Z(\sin 2t)_{z\to ze^{-T}}$$

$$= Z(\sin n(2T))$$

$$= \left\{\frac{z\sin 2T}{z^2-2z\cos 2T+1}\right\}_{z\to ze^{-T}}$$

$$= \frac{ze^{-T}\sin 2T}{z^2e^{-2T}-2ze^{-T}\cos 2T+1}$$

$$Z[na^{n-1}] = Z[na^n a^{-1}]$$

$$= \frac{1}{a}Z[na^n]$$

$$= \frac{1}{a}\{Z(n)\}_{z\to \frac{z}{a}}$$

$$= \frac{1}{a}\left\{\frac{z}{(z-1)^2}\right\}_{z\to\frac{z}{a}}$$

$$= \frac{1}{a}\left\{\frac{\frac{z}{a}}{\left(\frac{z-a}{a}\right)^2}\right\} = \frac{z}{a^2}\cdot\frac{a^2}{(z-a)^2} = \frac{z}{(z-a)^2}$$

$$\mathcal{Z}\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots$$

$$= z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= -z\log\left(1-\frac{1}{z}\right) = -z\log\left(\frac{z-1}{z}\right) = +z\log\left(\frac{z}{z-1}\right)$$

$$z\left(\frac{1}{n+2}\right) = \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$$

$$= \frac{1}{2}\cancel{\left(\frac{1}{z}\right)^2} + \frac{1}{3}\left(\frac{1}{z}\right)^{2} + \frac{1}{4}\left(\frac{1}{z}\right)^{4} + \cdots$$

$$= z^2\left[\frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \frac{1}{4}\left(\frac{1}{z}\right)^4 + \cdots\right]$$

$$= z^2\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots - \frac{1}{z}\right]$$

$$= z^2\left[\log\left(1-\frac{1}{z}\right) - \frac{1}{z}\right]$$

$$= z^2\left[\log\left(\frac{z}{z-1}\right) - \frac{1}{z}\right] = z^2\log\left(\frac{z}{z-1}\right) - z.$$

$$Z(r^n\cos n\theta) = \frac{z(z-r\cos\theta)}{z^2 - 2rz\cos\theta + 1}$$

$$Z(r^n\sin n\theta) = \frac{rz\sin\theta}{z^2 - 2rz\cos\theta + 1}$$

$$z(a^{n-1}) = z^{-1}z(a^n)$$

$$\cancel{z(a^{-1}a^n)}$$

$$= \frac{1}{z}\,\frac{z}{z-a}$$

$$= \frac{1}{z-a}$$

$$\mathcal{Z}(\cos\omega t) = z(\cos\omega nT) = \mathcal{Z}[\cos n(\omega T)]$$

$$= \frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}$$

$e^{-at}t$

We know that $\mathcal{Z}[e^{-at}f(t)] = Z[f(t)]_{z \to ze^{aT}}$

$$\mathcal{Z}[e^{-at}t] = Z[t]_{z\to ze^{aT}}$$

1. $Z(k) = \sum_{n=0}^{\infty} k z^{-n} = k\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right]$

$= k\left[1-\frac{1}{z}\right]^{-1} = \frac{k.z}{z-1}, \ |z|>1.$

Note: $Z(1) = \frac{z}{z-1}$

2) $Z(a^n) = \sum_{z=0}^{\infty} a^n z^{-n} = 1+\left(\frac{a}{z}\right)+\left(\frac{a}{z}\right)^2+\cdots$

$= \left(1-\frac{a}{z}\right)^{-1} = \frac{z}{z-a}.$

$Z\left[\frac{a^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$

$= \frac{1}{n!} \quad 1+\frac{a}{z}+\frac{1}{2!}\left(\frac{a}{z}\right)^2+\cdots$

$= e^{\frac{a}{z}}$

when $a=1$, $Z\left(\frac{1}{n!}\right] = e^{\frac{1}{z}}$

$Z(-1)^n = \frac{z}{z+1}$

3) $Z(na^n) = \sum_{n=0}^{\infty} n a^n z^{-n}$

$= \frac{a}{z}+2\left(\frac{a}{z}\right)^2+3\left(\frac{a}{z}\right)^3+\cdots$

$= \frac{a}{z}\left[1+2\left(\frac{a}{z}\right)+3\left(\frac{a}{z}\right)^2+\cdots\right]$

$= \frac{a}{z}\left[1-\frac{a}{z}\right]^{-2}$

$= \frac{a}{z}\frac{z^2}{(z-a)^2} = \frac{az}{(z-a)^2}$

4) $Z[n^2] = Z[n n]$

$= -z\frac{d}{dz}Z[n] = -z\frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\} = \frac{z^2+z}{(z-1)^3}$

5) $Z[3.2^n+4(-1)^n] = z\left\{\frac{3}{z-2}+\frac{4}{z+1}\right\}$

6) $Z\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty}\frac{1}{n}z^{-n}$

$= \frac{1}{z}+\frac{1}{2}\left(\frac{1}{z}\right)^2+\frac{1}{3}\left(\frac{1}{z}\right)^3+\cdots$

$= -\log\left(1-\frac{1}{z}\right) = +\log\left(\frac{z}{z-1}\right)$

7) $Z(e^{an}) = z\left[(e^a)^n\right] = \frac{z}{z-e^a}$

$z\log\left(\frac{z}{z-1}\right)$

$Z\left[\frac{1}{n(n+1)(n+2)}\right] = \frac{1}{2}Z\left[\frac{1}{n}\right]-Z\left[\frac{1}{n+1}\right]+\frac{1}{2}Z\left[\frac{1}{n+2}\right]$

8) $Z(e^{-at}) = \sum_{n=0}^{\infty} e^{-anT} z^{-n} = \sum_{n=0}^{\infty} (e^{-aT})^n z^{-n} = Z[(e^{+aT})^n] = \dfrac{z}{z-e^{+aT}}$

$Z(e^{2n}_{2}\sinh 3n)$ $2^n\left(\frac{e^{3n}-e^{-3n}}{2}\right)$ $\rightarrow \dfrac{2z\sin h3}{z^2-4z\cosh 3+z}$

$z\left(\cos\frac{n\pi}{2}\right) = \sum_{n=0}^{\infty} \cos\frac{n\pi}{2} z^{-n}$
$= 1-\frac{1}{z^2}+\frac{1}{z^4}-\cdots$
$= 1-\frac{1}{z^2}+\left(\frac{1}{z^2}\right)^2\cdots$
$= \left(1+\frac{1}{z^2}\right)^{-1} = \dfrac{z^2}{z^2+1}$

$Z(e^{3t-5}) = e^{-5} Z(e^{3t}) = e^{-5}\dfrac{z}{z-e^{-3T}}$.

$Z(r^n\cos n\theta)$ $Z(r^n\sin n\theta)$

We know that $Z(a^n) = \dfrac{z}{z-a}$. Putting $a = re^{i\theta}$,

$Z[(re^{i\theta})^n] = \dfrac{z}{z-re^{i\theta}}$

$Z[r^n(\cos n\theta + i\sin n\theta)] = \dfrac{z}{z-r(\cos\theta+i\sin\theta)}$

$= \dfrac{z}{(z-r\cos\theta) - ir\sin\theta}$

$= \dfrac{z[(z-r\cos\theta)+ir\sin\theta]}{(z-r\cos\theta)^2+r^2\sin^2\theta}$

$= \dfrac{z(z-r\cos\theta)+irz\sin\theta}{z^2-2rz\cos\theta+r^2}$

$Z(na^n\cos n\theta)$
$= \{Z(n\cos n\theta)\}_{z\to\frac{z}{a}}$

$Z(n\cos n\theta)$
$= -z\dfrac{d}{dz}Z(\cos n\theta)$

$\dfrac{z^3\cos\theta-2z^2+z\cos\theta}{(z^2-2z\cos\theta+1)^2}$

$\dfrac{a(z^3\cos\theta-2az^2+a^2z\cos\theta)}{(z^2-2az\cos\theta+a^2)^2}$

Equating real parts, $Z(r^n\cos n\theta) = \dfrac{z(z-r\cos\theta)}{z^2-2rz\cos\theta+r^2}$

$Z(r^n\sin n\theta) = \dfrac{rz\sin\theta}{z^2-2rz\cos\theta+r^2}$

$Z(\sin^2 2t) = Z\left[\dfrac{1-\cos 4t}{2}\right]$

$= \frac{1}{2}[Z(1) - Z(\cos 4t)]$ $Z(\cos 4nT)$

$= \frac{1}{2}\left[\dfrac{z}{z-1} - \dfrac{z(z-\cos 4T)}{z^2-2z\cos 4T+1}\right]$

$Z(t^k) = -Tz\dfrac{d}{dz}[Z(t)^{k-1}]$

$\sum nT z^{-n}$
$T\left(\frac{1}{z}+\frac{2}{z^2}+\cdots\right)$
$\frac{T}{z}\left(1+\frac{2}{z}+\cdots\right)$
$\frac{T}{z}\left(1-\frac{1}{z}\right)^{-2}$
$\frac{Tz}{\cdots}$

$Z(e^{2t}t) = \{Z[t]\}_{z\to ze^{2T}} = \dfrac{e^{aT}}{(ze^{aT}-1)^2}$

Second Shifting property

1) $Z\{f(n-n_0)\} = z^{-n_0} Z\{f(n)\}$.

2) $Z\{f(t+T)\} = z[Z[f(t)] - f(0)]$

$Z\left[e^{2(t+T)}\right]$

We k.t. $Z\{f(t+T)\} = z[Z(f(t)) - f(0)]$

Here $f(t) = 2t$, $f(0) = e^{0} = 1$.

$$Z(e^{2t}) = \frac{z}{z - e^{2T}}$$

$$Z\left[e^{2(t+T)}\right] = z\left\{\frac{z}{z-e^{2T}} - 1\right\} = \frac{ze^{2T}}{z - e^{2T}}$$

Initial value theorem: If $Z\{f(n)\} = F(z)$, then $f(0) = \lim_{z\to\infty} F(z)$.

ii) If $Z\{f(t)\} = F(z)$, then $f(0) = \lim_{z\to\infty} F(z)$.

Final value theorem

1. If $Z[f(n)] = F(z)$, then $\lim_{n\to\infty} f(n) = \lim_{z\to 1} (z-1)F(z)$.

2) If $Z[f(t)] = F(z)$, then $\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)F(z)$.

1) If $F(z) = \frac{2z}{z - e^{-T}}$, find $\lim_{t\to\infty} f(t)$ & $f(0)$

By IVT, $f(0) = \lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{2z}{z - e^{-T}}$

$$= \lim_{z\to\infty} \frac{2z}{z\left(1 - \frac{e^{-T}}{z}\right)} = 2$$

By FVT, $\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)F(z)$

$$= \lim_{z\to 1} (z-1)\cdot\frac{2z}{z - e^{-T}} = 0.$$

Table of z-transforms

$f(n)$	$F(z)$
1	$\frac{z}{z-1}$
$(-1)^n$	$\frac{z}{z+1}$
$a^n U(n)$	$\frac{z}{z-1}$
$U(n-m)$	$z^{-m}\frac{z}{z-1}$
n	$\frac{z}{(z-1)^2}$
n^2	$\frac{z^2+z}{(z-1)^3}$
na^n	$\frac{az}{(z-a)^2}$
$U(n)\cos n\theta$	$\frac{z^2-z\cos\theta}{z^2-2z\cos\theta+1}$
$U(n)\sin n\theta$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$
$r^n\cos n\theta$	$\frac{z^2-rz\cos\theta}{z^2-2rz\cos\theta+r^2}$
$r^n\sin n\theta$	$\frac{rz\sin\theta}{z^2-2rz\cos\theta+r^2}$
$\delta(n)$	1
$\delta(n-k)$	$\frac{1}{z^k}$
$na^n U(n)$	$\frac{az}{(z-a)^2}$
$a^n\cos\frac{n\pi}{2}$	$\frac{z^2}{z^2+a^2}$
$a^n\sin\frac{n\pi}{2}$	$\frac{az}{z^2+a^2}$
e^{-at}	$\frac{z}{z-e^{-aT}}$
e^{at}	$\frac{z}{z-e^{aT}}$

~~Table of z-transforms~~ Inverse z-Transform

The inverse z-transform of $Z[f(n)] = F(z)$ is defined as $Z^{-1}[F(z)] = f(n)$

Method 1: Expansion mtd If $F(z)$ can be expanded in a series of ascending powers of z^{-1} (ie) in the form $\sum_{n=0}^{\infty} f(n) z^{-n}$, by binomial, exponential & logarthmic theorems, the coefficient of z^{-n} in the expansion gives $Z^{-1}[F(z)]$

1) $Z^{-1}\left(\frac{z}{z-a}\right)$

$$F(z) = \frac{z}{z-a} = \frac{1}{1-\frac{a}{z}} = \left(1-\frac{a}{z}\right)^{-1}$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots = \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \sum_{n=0}^{\infty} a^n z^{-n}$$

Coefficient of z^{-n} is a^n

$$\therefore Z^{-1}[F(z)] = a^n \text{ (or) } Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

2) $Z^{-1}\left(e^{\frac{a}{z}}\right)$

$$F(z) = e^{\frac{a}{z}} = 1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a}{z}\right)^n$$

coefficient of z^{-n} is $\frac{a^n}{n!}$ ie, $Z^{-1}\left(e^{\frac{a}{z}}\right) = \frac{a^n}{n!}$

3) $Z^{-1}\left[\log\left(\frac{z-a}{z+b}\right)\right]$

$$F(z) = \log\left(\frac{z-a}{z+b}\right) = \log\left(\frac{1-\frac{a}{z}}{1+\frac{b}{z}}\right)$$

$$= \log\left(1-\frac{a}{z}\right) - \log\left(1+\frac{b}{z}\right)$$

$$= -\left\{\frac{a}{z} + \frac{1}{2}\left(\frac{a}{z}\right)^2 + \cdots\right\}$$

$$- \left\{\frac{b}{z} - \frac{1}{2}\left(\frac{b}{z}\right)^2 + \frac{1}{3}\left(\frac{b}{z}\right)^3 - \cdots\right\}$$

$$= -\left\{\frac{a}{z} - \frac{1}{2}\left(\frac{a}{z}\right)^2 + \frac{1}{3}\left(\frac{a}{z}\right)^3 - \cdots\right\}$$
$$+ \left\{-\frac{b}{z} + \frac{1}{2}\left(\frac{b}{z}\right)^2 - \frac{1}{3}\left(\frac{b}{z}\right)^3 + \cdots\right\}$$
$$= -\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{a}{z}\right)^n + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{b}{z}\right)^n$$

$$Z^{-1}(\;) = -\frac{a^n}{n} + \frac{(-1)^n b^n}{n}$$

Method 2: Long Division mtd when the usual methods of expansion of $F(z)$ fail & if $F(z) = \frac{g(z^{-1})}{h(z^{-1})}$, then $g(z^{-1})$ is divided by $h(z^{-1})$ & hence the expansion $\sum_{n=0}^{\infty} f(n) z^{-n}$ is obtained in the quotient.

$$\frac{4z}{(z-1)^2}.$$

Let $F(z) = \frac{4z}{(z-1)^2} = \frac{4z}{z^2\left(1-\frac{1}{z}\right)^2} = \frac{4z^{-1}}{(1-z^{-1})^2}$

$$= \frac{4z^{-1}}{1-2z^{-1}+z^{-2}}$$

By actual division,

$$\begin{array}{r|l} & 4z^{-1} + 8z^{-2} + 12z^{-3} + 16z^{-4} + \cdots \\ 1-2z^{-1}+z^{-2} & 4z^{-1} \\ & 4z^{-1} - 8z^{-2} + 4z^{-3} \\ \hline & 8z^{-2} - 4z^{-3} \\ & 8z^{-2} - 16z^{-3} + 8z^{-4} \\ \hline & 12z^{-3} - 8z^{-4} \\ & 12z^{-3} - 24z^{-4} + 12z^{-5} \\ \hline \end{array}$$

$$= \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\frac{4z}{(z-1)^2} = 4z^{-1} + 8z^{-2} + 12z^{-3} + 16z^{-4} + \cdots$$

$f(0) = 0$, $f(1) = 4$, $f(2) = 8$, $f(3) = 12$, $f(4) = 16 \ldots$

2) $\frac{2z^2+4z}{(z-2)^3} \rightarrow \frac{2z^2+4z}{z^3\left(z-\frac{2}{z}\right)^3} = \frac{2+4z^{-1}}{z\,(1-2z^{-1})}$

$= \frac{(2+4z^{-1})\,z^{-1}}{(1-2z^{-1})^3} = \frac{2z^{-1}+4z^{-2}}{1-6z^{-1}+12z^{-2}-8z^{-3}}$

$f(0)=0,\ f(1)=2,\ f(2)=16,\ f(4)=72 \ldots$

3) $\frac{2z(z^2-1)}{(z^2+1)^2} = \frac{2z^3\left(1-\frac{1}{z^2}\right)}{z^4\left(1+\frac{1}{z^2}\right)^2} = \frac{2z^{-1}(1-z^{-2})}{(1+z^{-2})^2}$

$= \frac{2z^{-1}-2z^{-3}}{1+2z^{-2}+z^{-4}}$, $f(0)=0,\ f(1)=2,\ f(2)=0,\ f(3)=-6 \ldots$

Method 3 Partial fraction mtd.

1. $\frac{z}{(z-1)^2(z+1)}$

Let $F(z) = \frac{z}{(z-1)^2(z+1)}$.

$\frac{F(z)}{z} = \frac{1}{(z-1)^2(z+1)}$.

$\frac{1}{(z-1)^2(z+1)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+1}$

$A = -\frac{1}{4},\ B = \frac{1}{2},\ C = \frac{1}{4}$.

$\frac{F(z)}{z} = \frac{-\frac{1}{4}}{z-1} + \frac{\frac{1}{2}}{(z-1)^2} + \frac{\frac{1}{4}}{z+1}$.

$z^{-1}(F(z)) = -\frac{1}{4}z^{-1}\left[\frac{z}{z-1}\right] + \frac{1}{2}z^{-1}\left[\frac{z}{(z-1)^2}\right] + \frac{1}{4}z^{-1}\left(\frac{z}{z+1}\right)$

$= -\frac{1}{4}\cdot 1 + \frac{1}{2}n + \frac{1}{4}(-1)^n$.

$\frac{z+2}{z^2-5z+6}$

$n=4,\ 3=5$

$-4\,2^{n-1}+5\,3^{n-1}$

$\frac{10z}{z^2-3z+2}$

$\frac{z^2}{(z+2)(z^2+4)}$

2) $\frac{z^2+z}{(z-1)(z^2+1)}$

$\frac{F(z)}{z} = \frac{z+1}{(z-1)(z+i)(z-i)}$

$A=1,\ B=-\frac{1}{2},\ C=-\frac{1}{2}$.

$z^{-1}\left(\frac{z}{z-1}\right) - \frac{1}{2}z^{-1}\left(\frac{z}{z+i}\right)$

$-\frac{1}{2}z^{-1}\left(\frac{z}{z-i}\right)$

$= 1 - \frac{1}{2}(-i)^n - \frac{1}{2}(i)^n$

H.W $\dfrac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$. $\dfrac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})} \to 2\left(\frac{1}{2}\right)^n-\left(\frac{1}{4}\right)^n$.

Mtd:4 Inverse integral mtd (Cauchy's residue thm).

By Cauchy's residue thm,

$\int_C F(z)\, z^{n-1}\, dz = 2\pi i \times$ Sum of the Residues of $F(z)\, z^{n-1}$ at the isolated Singularities.

$f(n)$ = sum of the residues of $F(z)\, z^{n-1}$ at the isolated Singularities.

Evaluation of Residues

1 Residue at a simple pole

$$[\text{Res } f(z)]_{z=a} = \lim_{z\to a} (z-a)\, F(z).$$

Residue at a pole of order m is given by,

$$[\text{Res } f(z)]_{z=a} = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m F(z)\right]$$

1) $\dfrac{z-4}{z^2+5z+6}$ Let $F(z) = \dfrac{z-4}{z^2+5z+6}$.

$$f(z) = F(z)\, z^{n-1} = \frac{z^n-4z^{n-1}}{(z+2)(z+3)}$$

The poles are $z=-2,-3$.

$R_1 = [\text{Res } f(z)]_{z=-2}$

$$= \lim_{z\to(-2)} (z+2) \frac{z^n-4z^{n-1}}{(z+2)(z+3)} = \frac{(-2)^n-4(-2)^{n-1}}{1}$$

$$= (-2)^n + 2(-2)^n = 3(-2)^n.$$

$R_2 = [\text{Res } f(z)]_{z=-3}$

$$= \lim_{z\to -3} (z+3) \frac{z^n-4z^{n-1}}{(z+2)(z+3)} = \frac{(-3)^n-4(-3)^{n-1}}{-1}$$

$$= +(-3)^n\left[-1-\frac{4}{3}\right] = -\frac{7}{3}(-3)^n \quad (f(n) = R_1+R_2$$

2) $\dfrac{4z^2-2z}{(z-1)(z-2)^2}$

$\dfrac{z^2-3z}{(z-5)(z+2)}$

Let $F(z) = \dfrac{4z^2-2z}{(z-1)(z-2)^2}$

$f(n) = F(z)z^{n-1} = \dfrac{4z^{n+1}-2z^n}{(z-1)(z-2)^2}$

$\frac{2}{7}(5)^n + \frac{5}{7}(-2)^n$

$z=1$, $z=2$ is of order 2.

$R_1 = [\operatorname{Res} f(z)]_{z=1}$

$= \lim_{z\to 1} (z-1)\dfrac{4z^{n+1}-2z^n}{(z-1)(z-2)^2} = \dfrac{4-2}{1} = 2$

$R_2 = \lim_{z\to 2} \dfrac{1}{(2-1)!}\dfrac{d}{dz}\left\{(z-2)^2\left[\dfrac{4z^{n+1}-2z^n}{(z-1)(z-2)^2}\right]\right\}$

$= \lim_{z\to 2}\dfrac{d}{dz}\left\{\dfrac{4z^{n+1}-2z^n}{z-1}\right\}$

$= \lim_{z\to 2}\left\{\dfrac{(z-1)[(n+1)4z^n - 2nz^{n-1}] - (4z^{n+1}-2z^n)\cdot 1}{(z-1)^2}\right\}$

$= (n+1)4(2)^n - 2n(2)^{n-1} - 4(2)^{n+1} + 2(2)^n$

$= 4n2^n + 4\cdot 2^n - n2^n - 2^{n+3} + 2^{n+1}$

$= n\cdot 2^{n+2} + 2^{n+2} - n2^n - 2^{n+3} + 2^{n+1}$

$= 4n2^n + 4\cdot 2^n - n2^n - 8\cdot 2^n + 2\cdot 2^n$

$= 3n2^n - 2\cdot 2^n = 2^n(3n-2)$

P. fraction

$\dfrac{z^2}{(z+2)(z+4)}$

Convolution $f(n) = R_1 + R_2$

(3) $\dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}$

$\dfrac{z^2}{(z-a)(z-b)} = \dfrac{1}{a-b}\left[a^{n+1} - b^{n+1}\right]$

Convolution thm:

The Convolution of the 2 sequences $\{f(n)\}$ & $\{g(n)\}$ is defined as

i) $\{f(n) * g(n)\} = \sum_{k=0}^{\infty} f(k)\,g(n-k)$ if the sequence are

iii) $\{f(n) * g(n)\} = \sum_{k=0}^{n} f(k)\, g(n-k)$ if the sequences are Causal.

The Convolution of 2 functions $f(t)$ & $g(t)$ is defined as $f(t) * g(t) = \sum_{k=0}^{n} f(kT)\, g(n-k)T$, T is the sampling period.

Convolution thm:

i) If $Z\{f(n)\} = F(z)$ & $Z\{g(n)\} = G(z)$ then
$Z\{f(n) * g(n)\} = F(z)\, G(z)$

ii) If $Z\{f(t)\} = F(z)$ & $Z\{g(t)\} = G(z)$, then
$Z\{f(t) * g(t)\} = F(z)\, G(z)$

note: $Z^{-1}[F(z)\,G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$

1) $Z^{-1}\left[\dfrac{z^2}{(z-a)^2}\right]$

$= Z^{-1}\left[\dfrac{z}{z-a} \cdot \dfrac{z}{z-a}\right]$

$= Z^{-1}\left[\dfrac{z}{z-a}\right] * Z^{-1}\left[\dfrac{z}{z-a}\right]$

$= a^n * a^n$

$= \sum_{k=0}^{n} a^k\, a^{n-k} = \sum_{k=0}^{n} a^n = a^n \sum_{k=0}^{n} 1 = a^n (n+1)$

=

2) $Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right] = \sum_{k=0}^{n} a^k\, b^{n-k} = b^n \sum_{k=0}^{n} \left(\dfrac{a}{b}\right)^k$

$= b^n \left\{ 1 + \dfrac{a}{b} + \left(\dfrac{a}{b}\right)^2 + \left(\cdot\dfrac{a}{b}\right)^n \right\}$

$\cancel{= b^n \left[\dfrac{1}{1-\frac{a}{b}}\right] = \dfrac{b^n (b-a)}{b}}$

$n > 1$
$\dfrac{a}{b} > 1$

$= b^n \left\{ \dfrac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1} \right\}$

$= b^n \left\{ \dfrac{a^{n+1} - b^{n+1}}{b^{n+1}} \cdot \dfrac{b}{a-b} \right\} = \dfrac{a^{n+1} - b^{n+1}}{a-b}.$

Using convolution thm, find $Z^{-1}\left(\frac{z^2}{(z-1)(z-3)}\right)$.

$$Z^{-1}\left[\frac{z}{(z-1)}\cdot\frac{z}{(z-3)}\right] = Z^{-1}\left(\frac{z}{z-1}\right) * Z^{-1}\left(\frac{z}{z-3}\right)$$

$$= 1 * 3^n$$

$$= \sum_{k=0}^{n} 1^k\, 3^{n-k}$$

$$= 3^n \sum_{k=0}^{n} \frac{1}{3^k}$$

$$= 3^n\left[1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\dots+\left(\frac{1}{3}\right)^n\right]$$

$1+r+r^2+\dots+r^n$

$r<1$ ↓ $\frac{1-r^{n+1}}{1-r}$

$r>1 \to \frac{r^{n+1}-1}{r-1}$

$$= 3^n\left[\frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}}\right]$$

$$= 3^n\left[\left(1-\frac{1}{3^{n+1}}\right)\cdot\frac{3}{2}\right]$$

$$= 3^n\left[\frac{3^{n+1}-1}{3^{n+1}}\cdot\frac{3}{2}\right]$$

$$= \frac{1}{2}\left(3^{n+1}-1\right)$$

* $Z^{-1}\left[\frac{14z^2}{(4z-1)(2z-1)}\right]$

$$= Z^{-1}\left[\frac{14z^2}{4\left(z-\frac{1}{4}\right)2\left(z-\frac{1}{2}\right)}\right]$$

$$= Z^{-1}\left[\frac{z}{z-\frac{1}{4}}\cdot\frac{z}{z-\frac{1}{2}}\right]$$

$$= \left(\frac{1}{4}\right)^n * \left(\frac{1}{2}\right)^n$$

$$= \sum^{n} \left(\frac{1}{4}\right)^k\left(\frac{1}{2}\right)^{n-k}$$

$$= \left(\frac{1}{4}\right)^n \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$$

$$= \left(\frac{1}{4}\right)^n \left(\frac{1}{2}\right)^n \left\{ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots \right\}$$

$$= \left(\frac{1}{4}\right)^n \left(\frac{1}{2}\right)^n \left[\frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} \right]$$

$$= \left(\frac{1}{4}\right)^n \left(\frac{1}{2}\right)^n \left[\frac{2^{n+1} - 1}{2^{n+1}} \right]$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(\frac{1}{7}\right)^k \cdot \frac{1}{\left(\frac{1}{2}\right)^k}$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(\frac{2}{7}\right)^k$$

$$= \left(\frac{1}{2}\right)^n \left\{ 1 + \frac{2}{7} + \left(\frac{2}{7}\right)^2 + \cdots + \left(\frac{2}{7}\right)^n \right\}$$

$$= \left(\frac{1}{2}\right)^n \left\{ \frac{1 - \left(\frac{2}{7}\right)^{n+1}}{1 - \frac{2}{7}} \right\}$$

$$= \left(\frac{1}{2}\right)^n \cdot \frac{7}{5} \cdot \cancel{7^{n+1}} \cdot \left[\frac{7^{n+1} - 2^{n+1}}{7^{n+1}} \right]$$

$$= \left(\frac{1}{2}\right)^n \frac{7}{5} - \frac{1}{\cancel{2^n}} \cdot \frac{7}{5} \cdot \frac{\cancel{2^n} \cdot 2}{7^n \cdot 7}$$

$$= \frac{7}{5}\left(\frac{1}{2}\right)^n - \frac{2}{5}\left(\frac{1}{7}\right)^n$$

3) $z^{-1}\left[\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}\right]$

$= z^{-1}\left[\frac{1}{\left(1-\frac{1}{2z}\right)\left(1-\frac{1}{4z}\right)}\right]$

$= z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right] = z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right]$

$\left(\frac{1}{2}\right)^{2n}\frac{1}{2^{-n}}\frac{1}{2^{-1}} - \left(\frac{1}{4}\right)^n$

$\left(\frac{1}{2}\right)^{2n-n-1} - \left(\frac{1}{4}\right)^n$

$= \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{4}\right)^n$

$\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{4}\right)^n$

$\left(\frac{1}{2^2}\right)^n[2^n.2] - \left(\frac{1}{4}\right)^n$

$\left(\frac{1}{4}\right)^n(1+2+2^2+\dots+2^n)$

$= \left(\frac{1}{4}\right)^n\frac{2^{n+1}-1}{2-1}$

Difference equations DE's arise in all situations in which sequential relation exists at various discrete values of the independent variable.

Application to D.E's z-transforms are used for solving linear Difference equations

Working procedure: To solve a linear difference equation with constant coefficients by z-transforms

1. Take the z-transform of both sides of the difference equations using the formulae & the given conditions.
2) Transpose all terms without $F(z)$ to the right.
3. Divide by the coefficient of $F(z)$ getting $F(z)$ as a function of z.
4. Express this function in terms of the z-transform of known functions & take the inverse z-transform of both sides. This gives y_n as a function of n which is the desired solution.

Results:

$Z[f(n-m)] = z^{-m} Z[f(n)]$

$Z[f(n+k)] = z^k[F(z) - f(0) - f(1)z^{-1} \dots - f(k-1)z^{-(k-1)}]$

$Z[f_{n+2}] = z^2[F(z) - f_0 - f_1 z^{-1}]$

$Z[f_{n+3}] = z^3[F(z) - f_0 - f_1 z^{-1} - f_2 z^{-2}]$

* Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given $y_0 = y_1 = 0$.

Taking z-transform on both sides,

$$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = Z[2^n]$$

$$z^2[Y(z) - y_0 - y_1 z^{-1}] + 6z[Y(z) - y_0] + 9Y(z) = \frac{z}{z-2}$$

$$(z^2 + 6z + 9)Y(z) = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2}$$

$$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{(z+3)^2} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + \cancel{C(z+3)}\ C(z-2)$$

$z = 2 \Rightarrow 1 = A(5)^2 \Rightarrow A = \frac{1}{25}$

$z = -3 \Rightarrow 1 = C(-5) \Rightarrow C = -\frac{1}{5}$

$z^2;\quad 0 = A + B \quad B = -\frac{1}{25}$

$$Y(z) = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

Taking inverse z-transform,

$Z[na^n] = \frac{az}{(z-a)^2}$

$$y(n) = \frac{1}{25}Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25}Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5}Z^{-1}\left[\frac{z}{(z+3)^2}\right]$$

$$= \frac{1}{25}2^n - \frac{1}{25}(-3)^n - \frac{1}{5}\frac{1}{(-3)}Z^{-1}\left[\frac{-3z}{(z+3)^2}\right]$$

$$= \frac{2^n}{25} - \frac{1}{25}(-3)^n + \frac{1}{15}n(-3)^n.$$

* $y_{n+2} + y_n = n2^n$

Taking z-transform on both sides,

$$Z[y_{n+2}] + Z[y_n] = Z[n2^n]$$

$$z^2[Y(z) - y_0 - y_1 z^{-1}] + Y(z) = \frac{\cancel{2z} \; 2z}{(z-2)^2}$$

$$z^2 Y(z) - z^2 A - zB + Y(z) = \frac{2z}{(z-2)^2}$$

$$(z^2+1) Y(z) - z^2 A - z^2 B = \frac{2z}{(z-2)^2}$$

$$(z^2+1) Y(z) = \frac{2z}{(z-2)^2} + z^2 A + zB$$

$$Y(z) = \frac{2z}{(z-2)^2(z^2+1)} + \frac{Az^2 + Bz}{(z^2+1)}$$

$$Y(z) z^{n-1} = \frac{2z^n}{(z-2)^2(z^2+1)} + \frac{Az^{n+1} + Bz^n}{z^2+1}$$

$$= I_1 + I_2$$

Consider I_1 :

$z=2$ is a double pole.

$z = \pm i$ are simple poles.

$$[\text{Res } I_1]_{z=2} = \lim_{z\to 2} \frac{d}{dz}(z-2)^2 \cdot \frac{2z^n}{(z-2)^2(z^2+1)}$$

$$= \lim_{z\to 2} \frac{d}{dz} \frac{2z^n}{z^2+1}$$

$$= \lim_{z\to 2} \left\{ \frac{(z^2+1) \cdot 2nz^{n-1} - 2z^n(2z)}{(z^2+1)^2} \right\}$$

$$= \frac{5 \cdot 2n2^{n-1} - 2 \cdot 2^n \cdot 2 \cdot 2}{5^2}$$

$$= \frac{2^n}{25}[5n-8]$$

$y(n) = $ sum of the residues

$$= \frac{2^n}{25}[5n-2] + \frac{i^{n-1}}{(i-2)^2} + \frac{(-i)^{n-1}}{(-i+2)^2}$$

$$+ \frac{A}{2}\left[i^n + (-i)^n\right] + \frac{B}{2}\left[i^{n-1} + (-i)^{n-1}\right].$$

* $y_{n+2} + y_n = 2, \quad y_0 = y_1 = 0$

$$Z[y_{n+2}] + Z[y_n] = Z(2)$$

$$z^2\left[Y(z) - y_0 - y_1 z^{-1}\right] + Y(z) = \frac{2z}{z-1}$$

$$(z^2+1)\,Y(z) = \frac{2z}{z-1}$$

$$\frac{Y(z)}{z} = \frac{2}{(z-1)(z^2+1)}$$

$$\frac{2}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1}$$

$$2 = A(z^2+1) + (Bz+C)(z-1)$$

$$z=1 \Rightarrow 2 = A(2) \Rightarrow A = 1$$

$$z^2; \quad 0 = A + B \Rightarrow B = -1$$

$$z; \quad 0 = C - B \quad C = -1$$

$$Z\left[a^n \sin\frac{n\pi}{2}\right] = \frac{az}{z^2+a^2}$$

$$Z\left[a^n \cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+a^2}$$

$$\frac{Y(z)}{z} = \frac{1}{z-1} + \frac{-z-1}{z^2+1}$$

$$Y(z) = \frac{z}{z-1} - \frac{z^2+z}{z^2+1}$$

$$y(n) = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z^2}{z^2+1}\right] - Z^{-1}\left[\frac{z}{z^2+1}\right]$$

$$= 1 - \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}.$$

* $f(n) + 3f(n-1) - 4f(n-2) = 0, \; n \geq 2$ given that $f(0) = 3$, $f(1) = -2$.

Changing n into $n+2$,

$f(n+2) + 2f(n+1) - 4f(n) = 0, \; n \geq 0, \; f(0) = 3, \; f(1) = -2.$

$$Z[f(n+2)] + 2Z[f(n+1)] - 4Z[f(n)] = Z(0)$$

$z^2[F(z) - f(0) - f(1)z^{-1}] + 3z[F(z) - f(0)] - 4F(z) = 0$

$z^2[F(z) - 3 - \cancel{f(1)}\ (-2)z^{-1}] + 3z[F(z) - 3] - 4F(z) = 0$

$(z^2 + 3z - 4)F(z) - 3z^2 + 2z - 9z = 0$

$(z^2 + 3z - 4)F(z) = 3z^2 + 7z$

$$\frac{F(z)}{z} = \frac{3z+7}{z^2+3z-4}$$

$4 \times -1 = -4$
$4 - 1 = 3$

$$\frac{3z+7}{(z+4)(z-1)} = \frac{A}{z+4} + \frac{B}{z-1}$$

$3z+7 = A(z-1) + B(z+4)$

$z=1 \Rightarrow 10 = B(5) \Rightarrow \boxed{B=2}$

$z=-4 \Rightarrow -12+7 = A(-5) \Rightarrow \boxed{A=1}$

$$\frac{F(z)}{z} = \frac{1}{z+4} + \frac{2}{z-1}$$

$$F(z) = \frac{z}{z+4} + \frac{2z}{z-1}$$

$$f(n) = z^{-1}\left[\frac{z}{z+4}\right] + 2z^{-1}\left[\frac{z}{z-1}\right]$$

$$= (-4)^n + 2$$

Difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

ex: $\Delta y_{n+1} + y_n = 2$

$\Delta y_{(n+1)} = y_{(n+2)} - y_{(n+1)}$ are Difference equation

* $y_{n+2} - 4y_{n+1} + 4y_n = 0$, given $y_0 = 1$ & $y_1 = 0$

$$Y(z) = \frac{z^2 - 4z}{(z-2)^2}$$

$$y(n) = 2^n(1-n)$$

* $y(n+2) - 5y(n+1) + 6y(n) = n(n-1)$, $y(0) = 0$, $y(1) = 0$

$= n^2 - n$

$$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = 2^n, \quad y_0 = y_1 = 0$$

$$z^2\left[Y(z) - y_0 - y_1 z^{-1}\right] + 6z\left[Y(z) - f_0\right] + 9Y(z) = Z(2^n)$$

$$\cancel{z^2 Y} \quad z^2 Y(z) + 6zY(z) + 9Y(z) = \frac{z}{z-2}$$

$$(z^2 + 6z + 9)Y(z) = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)\ (z+3)^2}$$

$$\frac{Y(z)}{z} = \frac{1}{(z-2)\ (z+3)^2}$$

$$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

$$z = 2 \Rightarrow 1 = A(5)^2 + 0 + 0 \Rightarrow A = \frac{1}{25}.$$

$$z = -3 \Rightarrow 1 = C(-5) \Rightarrow C = -1/5.$$

$$z^2; \quad 1 = A + B \rightarrow 0 = \frac{1}{25} + B \Rightarrow B = \cancel{\frac{24}{25}}. \ -\frac{1}{25}.$$

$$\frac{Y(z)}{z} = \frac{1}{25}\frac{1}{z-2} - \frac{1}{25}\frac{1}{z+3} - \frac{1}{5}\frac{1}{(z+3)^2}$$

$$Y(z) = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

$$Z^{-1}[Y(z)] = \frac{1}{25}Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25}Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5}Z^{-1}\left[\frac{z}{(z+3)^2}\right]$$

$$= \frac{1}{25}2^n - \frac{1}{25}(-3)^n - \frac{1}{5(-3)} \cancel{n(-3)^n}\ Z^{-1}\left[\frac{-3z}{(z+3)^2}\right]$$

$$= \frac{1}{25}2^n - \frac{1}{25}(-3)^n + \frac{1}{15}n(-3)^n$$

Initial value theorem.

If $Z[f(t)] = F(z)$, then $f(0) = \lim_{z\to\infty} F(z)$

Final value theorem:

If $Z[f(t)] = F(z)$, then $\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1) F(z)$

1. If $F(z) = \dfrac{5z}{(z-2)(z-3)}$, find $f(0)$ & $\lim_{t\to\infty} f(t)$

By IVT,

$$f(0) = \lim_{z\to\infty} F(z)$$

$$= \lim_{z\to\infty} \frac{5\cancel{z}}{\cancel{z}\left(1-\frac{2}{z}\right) z\left(1-\frac{3}{z}\right)} = 0$$

By FVT, $\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1) F(z)$

$$= \lim_{z\to 1} (z-1) \frac{5z}{(z-2)(z-3)} = 0$$

2. If $U(z) = \dfrac{2z^2+5z+14}{(z-1)^4}$, find u_2 & u_3.

By IVT,

$$u_0 = \lim_{z\to\infty} U(z)$$

$$u_1 = \lim_{z\to\infty} [zU(z) - u_0]$$

$$u_2 = \lim_{z\to\infty} \left[z^2 [U(z) - u_0 - u_1 z^{-1}]\right]$$

$$u_3 = \lim_{z\to\infty} \left[z^3 (U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2})\right]$$

$$U(z) = \frac{2z^2+5z+14}{(z-1)^4}$$

$$= \frac{z^2\left[2+\frac{5}{z}+\frac{14}{z^2}\right]}{z^4\left(1-\frac{1}{z}\right)^4}$$

$$= \frac{1}{z^2}\frac{\left[2+5z^{-1}+14z^{-2}\right]}{(1-z^{-1})^4}$$

$$u_1 = \underset{z\to\infty}{Lt}\ U(z) = 0$$

$$u_1 = \underset{z\to\infty}{Lt}\left[z\left(U(z)-u_0\right)\right]$$

$$= \underset{z\to\infty}{Lt}\ z\left\{\frac{1}{z^2}\frac{(2+5z^{-1}+14z^{-2})}{(1-z^{-1})^4}\right\}$$

$$= 0$$

$$u_2 = \underset{z\to\infty}{Lt}\left[z^2\left(U(z)-u_0-u_1z^{-1}\right)\right]$$

$$= \underset{z\to\infty}{Lt}\ \frac{2+5z^{-1}+14z^{-2}}{(1-z^{-1})^4} = 2$$

$$u_3 = \underset{z\to\infty}{Lt}\left[z^3\left(U(z)-u_0-u_1z^{-1}-u_2z^{-2}\right)\right]$$

$$= \underset{z\to\infty}{Lt}\ z^3\ \cancel{\frac{1}{z^2}}\left[\frac{2z^2+\frac{5z}{\cancel{z}}+\frac{14}{\cancel{z^2}}}{(z-1)^4}-\frac{2}{z^2}\right]$$

$$= 13$$