POLLACHI INSTITUTE OF ENGINEERING AND TECHNOLOGY

(Approved by AICTE and affiliated to Anna University,Coimbatore.)

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QUESTION BANK

MATHEMATICS – 1

UNIT IV

FUNCTIONS OF SEVERAL VARIABLES

PART - A

1. If $z=e^{ax+by}$ f(ax-by) show that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

Solution:

Given $z=e^{ax+by}$ f(ax-by)

$\frac{\partial z}{\partial x} = e^{ax+by}$. f'(ax-by).a + f(ax-by). e^{ax+by}.a

$b\frac{\partial z}{\partial x}$ =ab (e^{ax+by}) [f'(ax-by) + f(ax-by). e^{ax+by}] ---------------------(1)

$\frac{\partial z}{\partial y} = e^{ax+by}$. f'(ax-by)(-b) + f(ax-by). e^{ax+by}.b

$a\frac{\partial z}{\partial y}$ =ab(e^{ax+by}) [-f'(ax-by) + f(ax-by)] ---------------------(2)

(1)+(2) =>

$$\mathbf{b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab(e^{ax+by})\, f(ax\text{-}by) = 2abz}$$

2. If z = yf(x^2 -y^2) show that that $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = \frac{xz}{y}$

Solution:

Given z = yf(x^2 -y^2)

$\frac{\partial z}{\partial x}$ = y f'(x^2 -y^2) .2x

$y\frac{\partial z}{\partial x}$ = 2 y^2 x f'(x^2 -y^2) -------------------------------(1)

$\frac{\partial z}{\partial y}$ = y f'(x^2 -y^2)(-2y) + f(x^2 -y^2)

$x\frac{\partial z}{\partial y}$ = -2 xy^2 f'(x^2 -y^2) +x f(x^2 -y^2) -------------------------------(2)

(1)+(2) =>

$$\mathbf{y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x.\, f(x^2 - y^2) = \frac{xz}{y}}$$

3. $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$

Solution:

Here u is a homogeneous function of degree n=0.

(x=kx, y=ky , z=kz)

$u = \frac{k}{k}\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)$

(Power of k= degree =n=0)

Hence by Euler's theorem,

$$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 0.}$$

4. If u= $tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$

Solution:

Let z =tan u = $\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

Here z is a homogeneous function of degree n=1/2.

Therefore , z satisfies the Euler's theorem.

$$\mathbf{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz} \qquad \text{-------------(1)}$$

$$\frac{\partial z}{\partial x} = \frac{dz}{du}.\frac{\partial u}{\partial x} = sec^2 u\,.\frac{\partial u}{\partial x} \qquad \text{-------------(2)}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du}.\frac{\partial u}{\partial y} = sec^2 u\,.\frac{\partial u}{\partial y} \qquad \text{-------------(3)}$$

Put (2) & (3) in (1) we get,

$$x\, sec^2 u\frac{\partial u}{\partial x} + y\, sec^2 u\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

Therefore, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$

5. If u= $\log\left(\frac{x^3+y^3}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$

Solution:

Given u= $\log\left(\frac{x^3+y^3}{x+y}\right)$

$z = e^u = \frac{x^3+y^3}{x+y}$

As z is a homogeneous function of order n=2, it satisfies the Euler's theorem.

$$\mathbf{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz} = 2e^u \qquad \text{------------------(1)}$$

$\frac{\partial z}{\partial x} = \frac{dz}{du} . \frac{\partial u}{\partial x} = e^u . \frac{\partial u}{\partial x}$

$\frac{\partial z}{\partial y} = \frac{dz}{du} . \frac{\partial u}{\partial y} = e^u . \frac{\partial u}{\partial y}$

(1) => x $e^u . \frac{\partial u}{\partial x}$ + y $e^u . \frac{\partial u}{\partial y} = 2e^u$

$\boldsymbol{x \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2}$

6. If u = e^{x^2} f(y/x), show that $x \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ = 2u log u.

Solution:

z = log u = x^2 f(y/x)

Here z is a homogeneous function of order n=2, it satisfies the Euler's equation.

$\boldsymbol{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz =}$ 2 log u ----------------(1)

$\frac{\partial z}{\partial x} = \frac{dz}{du} . \frac{\partial u}{\partial x} = \frac{1}{u} . \frac{\partial u}{\partial x}$ (Order: Put x=kx, y=ky , z=$x^2 k^2 \, f\left(\frac{ky}{kx}\right)$

$\frac{\partial z}{\partial y} = \frac{dz}{du} . \frac{\partial u}{\partial y} = \frac{1}{u} . \frac{\partial u}{\partial y}$ n=2.)

Put in (1)

$x \frac{1}{u}\frac{\partial u}{\partial x} + y \frac{1}{u}\frac{\partial u}{\partial y}$ =2 log u

$\boldsymbol{x \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}}$ **=2u log u.**

7. If $z = sin^{-1}(x - y)$, x=3t, y=4t^3, find $\frac{dz}{dt}$.

Solution :

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} . \frac{dx}{dt} + \frac{\partial z}{\partial y} . \frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}} (3) + \frac{1}{\sqrt{1-(x-y)^2}} (-1) (12t^2)$$

$$= \frac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}}$$

$$= \frac{3}{\sqrt{1-t^2}}$$

Therefore, $\boldsymbol{\frac{dz}{dt} = \frac{3}{\sqrt{1-t^2}}}$

8. If z = f(x,y) and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$ show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}.e^u + \frac{\partial z}{\partial y}.e^{-u}(-1)$$

$$\frac{\partial z}{\partial u} = e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \quad \text{----------------- (1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(e^{-v})(-1) + \frac{\partial z}{\partial y}(-e^v) \quad \text{----------------- (2)}$$

(1) − (2) ⇒

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^v)$$

$$\boldsymbol{\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}}$$

9. If u = ϕ(x,y), x=r cosѳ, y=r sinѳ, show that $\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2$

Solution:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{\partial \phi}{\partial x}.\cos\Theta + \frac{\partial \phi}{\partial y}.\sin\Theta \quad \text{--------------------(1)}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial \phi}{\partial x}.r(-\sin\Theta) + \frac{\partial \phi}{\partial y}(r\cos\Theta) \quad \text{--------------------(2)}$$

L.H.S $\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial \phi}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial \phi}{\partial y}\right)^2(\cos^2\theta + \sin^2\theta)$

$$\boldsymbol{\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}$$

10. If w=f(r) , x=r cosѳ, y=r sinѳ, show that $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$ = f''(r) + $\frac{1}{r}$ f'(r) .

Solution:

w= f(r) , r= $\sqrt{x^2 + y^2}$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}.\frac{\partial r}{\partial x} = f'(r).\frac{x}{r} = f'(r).xr^{-1}$$

$$\frac{\partial^2 w}{\partial x^2} = xr^{-1}f''\frac{\partial r}{\partial x} + f'r^{-1}(1) + f'(r)x(-1)\,r^{-2}\frac{\partial r}{\partial x}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{x^2}{r^2}f'' + \frac{f'}{r} - \frac{x^2}{r^3}f' \quad \text{-----------------------(1)}$$

As r is symmetry in x,y

$\frac{\partial^2 w}{\partial y^2}$ $= \frac{y^2}{r^2}$ f'' + $\frac{f'}{r}$ - $\frac{y^2}{r^3}$f' ------------------------(2)

(1) + (2)=>

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \mathbf{f''(r)} + \frac{1}{r}\mathbf{f'(r)}\ .$$

11. f(x,y) = $4x^3$ + 3xy - $6x^2y^2$ $+2y^3$ =2, find $\frac{dy}{dx}$.

    Solution:

    f(x,y) = $4x^3$ + 3xy - $6x^2y^2$ $+2y^3 - 2$

    $\frac{\partial f}{\partial x} = 12x^2 + 3y - 12xy^2$

    $\frac{\partial f}{\partial y} = 3x - 12x^2y + 6y^2$

    $$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-(4x^2+y-4xy^2)}{x-4x^2y+2y^2}$$

12. If $x^3 + y^3 - 3axy = 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

    Solution:

    Let f(x,y) = $x^3 + y^3 - 3axy = 0$

    $\frac{\partial f}{\partial x}$ = p = $3x^2$-3ay

    $\frac{\partial^2 f}{\partial x^2}$ = r =6x

    $\frac{\partial^2 f}{\partial x \partial y}$ = s = -3a

    $\frac{\partial f}{\partial y}$ = q = $3y^2 - 3ax$

    $\frac{\partial^2 f}{\partial y^2}$ = t =6y

    $$\frac{dy}{dx} = -\frac{p}{q} = -\frac{3x^2-3ay}{3y^2-3ax}$$

    $$\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}$$

    $$\frac{d^2y}{dx^2} = -\left(\frac{q^2r-2pqs+p^2t}{q^3}\right)$$

    $$\frac{d^2y}{dx^2} = \frac{2a^3xy}{(ax-y^2)^3}$$

13. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ if x+y =u and y=uv

    Solution:

    Given: x+y =u ; y=uv

    x = u-uv = u(1-v)

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

$$\boldsymbol{\frac{\partial(x,y)}{\partial(u,v)} = u}$$

14. If x=r cosθ, y=r sinθ , prove that the Jacobian $J=\frac{\partial(x,y)}{\partial(r,\theta)} = r$

Solution:

Given : x=r cosθ ; y=r sinθ

$\frac{\partial x}{\partial r} = \cos\theta$; $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$; $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$J=\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$\boldsymbol{J = \frac{\partial(x,y)}{\partial(r,\theta)} = r}$. Hence the proof.

15. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ find the Jacobian of u,v,w with respect to x, y & z.

Solution:

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{bmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{bmatrix}$$

$$= -\frac{yz}{x^2}\left[\frac{x^2}{yz} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[-\frac{x}{z} - \frac{x}{z}\right] + \frac{y}{x}\left[\frac{x}{y} + \frac{x}{y}\right]$$

$$= 2+2 = 4$$

16. Give the Maclaurins series expansion of f(x,y)= xy^2

Solution:

At the origin f=0 , f_x=0 , f_{xx}= 0 , f_{xxx}= 0, $f_y = 0$, f_{yy}= 0, f_{yyy} =0, $f_{xy} = 0$, f_{xxy} =0 & f_{xyy} =0.

Therefore the required series is xy^2 = xy^2

17. Using the definition of total derivative, find the value of $\frac{du}{dt}$, given u=$y^2 - 4ax$, x=at^2 , y=2at.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

= (-4a)(2at) + (2a)(4at) = 0.

$$\boldsymbol{\frac{du}{dt} = 0}$$

18. Expand $e^x \log(1 + y)$ as the Taylors series in the neighbourhood of (0,0).

Solution:

f(x,y) = $e^x \log(1 + y)$	;	(a,b) =(0,0)
f(x,y) = $e^x \log(1 + y)$	;	f(a,b) =f(0,0)=0
$f_x(x,y)$= $e^x \log(1 + y)$	;	$f_x(a,b)$= f_x(0,0)= 0
$f_{xx}(x,y)$= $e^x \log(1 + y)$	;	$f_{xx}(a,b)$= f_{xx}(0,0)= 0
$f_{xy}(x,y) = e^x \frac{1}{1+y}$	;	$f_{xy}(a,b)$= f_{xy}(0,0)= 1
$f_y(x,y) = e^x \frac{1}{1+y}$	;	$f_y(a,b)$= f_y(0,0)= 1
$f_{yy}(x,y) = -e^x \frac{1}{(1+y)^2}$	;	$f_{yy}(a,b)$= f_{yy}(0,0)= -1

Taylor's Series is f(x,y) = y + xy $-\frac{y^2}{2}$ + …..

19. Find the Stationary points for *f(x,y)* = $x^3 + y^3 - 12x - 3y + 20$

Solution:

Given: *f(x,y)* = $x^3 + y^3 - 12x - 3y + 20$

p =f_x = $3x^2 - 12$; q = $f_y = 3y^2 - 3$

r = f_{xx}= 6x ; s = $f_{xy} = 0$, t = f_{yy}= 6y

At maximum point minimum point : f_x=0 , $f_y = 0$.

⇨ **The points (2,1) , (2,-1), (-2,1) , (-2,-1) are the stationary points.**

20. A flat circular plate is heated so that the temperature at any point (x,y) is u(x,y)= $x^2 + 2y^2 - x$. Find the coldest point on the plate.

Solution:

Given : u(x,y)= $x^2 + 2y^2 - x$. ---------------------(1)

p= u_x = 2x-1 ; q = u_y=4y

r =$u_{xx} = 2$; t = $u_{yy=4}$

s = u_{xy} = 0

p = q= 0 ⇨ The point is (1/2 , 0)

At (1/2, 0) ⤇ rt-s^2 = 8 >0

Therefore the point (1/2 , 0) is a minimum point.

Hence the point (1/2 , 0) is the coldest point.

21. Find the shortest distance from the origin to the curve $x^2 + 8xy + 7y^2 = 225$.
Solution:

$f = x^2 + y^2$

$\phi = x^2 + 8xy + 7y^2 - 225$

f+λϕ ⤇ $(x^2 + y^2)$ + λ$(x^2 + 8xy + 7y^2 - 225)$

f_x + λϕ_x= 0 ⤇ (λ+1)x +4λy = 0 ---------------------(1)

f_y + λϕ_y= 0 ⤇4λx + (1+7λ)y = 0 ---------------------(2)

Solving (1) & (2), we get λ=1, λ=-1/9

If λ=1 ⤇ x=-2y and $-5y^2$= 225

(No real value of y)

If λ=-1/9 ⤇ y=2x ⤇ x=$\sqrt{5}$, y=$\sqrt{20}$

Therefore shortest distance = $\sqrt{x^2 + y^2}$ = $\sqrt{5 + 20}$ = $\sqrt{25}$ =5 units.

shortest distance =5 units.

<u>PART –B</u>

1. If z=f(x,y), where x=$u^2 - v^2$ y=2uv, prove that $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4(u^2 + v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$.
2. Find the Taylor's series expansion of $x^2y^2 + 2x^2y + 3xy^2$ in powers of (x+2) and (y-1) up to 3rd degree terms.
3. If x+y+z =u, y+z=uv, z=uvw, prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.
4. Find the extreme values of the function *f(x,y)* $= x^3 + y^3 - 3x - 12y + 20$.
5. Expand $e^x \cos y$ in powers of x and y as far as the terms of the 3rd degree.
6. A rectangular box open at the top is to have a volume of 32cc. Find the dimensions of the box, that requires the least material for its construction.
7. If u= f($x^2 + 2yz$, $y^2 + 2zx$) , prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - yx)\frac{\partial u}{\partial z}$ =0.
8. Find $\frac{dy}{dx}$ if u=$\tan^{-1}\left(\frac{y}{x}\right)$, where $x^2 + y^2 = a^2$.
9. Examine u(x,y) = $x^4 - y^4 - 2x^2 + 4xy - 2y^2$ for extreme values.
10. A rectangular box without a top is to be made from $12m^2$ of cardboard. Find the maximum volume of a such box.
11. Expand $f(x, y) = e^y \log(1 + x)$ in powers of x and y and verify the result by direct expansion.

12. Given the transformation $u = e^x cosy\ and\ v = e^x siny$ and that f is a function of u and v and also of x and y, prove that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{2x}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$.
13. Using Taylor's series, expand e^{x+y} e^{xy} and in powers of x and y up to third degree terms.
14. Find the minimum value of $x^2 + y^2 + z^2$, when xyz= a^3.
15. If w=f(y-z, z-x, x-y), prove that $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$.
16. If u = $(x^2 + y^2 + z^2)^{-1/2}$.Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \mathbf{0}$.
17. Find the minimum value of $x^2 + y^2 + z^2$,when x+y+z=3a
18. If u=$\tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ = ½ sin 2u.
19. The temperature T at any point (x,y,z) in space is T= $cxyz^2$, where c is constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = 1$.
20. Discuss the maxima and minima of $f(x, y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$.