POLLACHI INSTITUTE OF ENGINEERING AND TECHNOLOGY

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QUESTION BANK

MATHEMATICS – 1

UNIT - 1

MATRICES

PART – A

1. Define linear dependence and independence of vectors.

Soln:

The vectors $x_{1,}\ x_{2\ ,......,}x_n$ are said to be linearly dependent if there exists scalars $\lambda_{1,}$ $\lambda_{2,}$ …….. $\lambda_{n,}$ such that

$$\lambda_1\ x_1 + \lambda_2 x_2 + \ldots\ldots\ \lambda_n\ x_n = 0$$

The vectors $x_{1,}\ x_{2\ ,......,}x_n$ are said to be linearly independent if λ_1 $x_1 + \lambda_2 x_2 + \ldots\ldots\ \lambda_n\ x_n = 0$. Such that $\lambda_1 = 0$, $\lambda_2 = \ldots\ldots = \lambda_n$.

2. Verify whether the vectors (1,-1,1) (2,-3,0) (0,1,5) are linearly independent

Soln:

Consider the given matrix as a rows of the matrix $A=\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

Now, $|A| = 1(-15) + 2(10) + 1(2)$

$= 7 \neq 0$

Thus the given vectors are linearly independent.

3. For what values of k the vectors (-1,2,4) (2,-3,1) and (1,-1,k) are linearly independent

Soln:

Given $A=\begin{bmatrix} -1 & 2 & 4 \\ 2 & -3 & 1 \\ 1 & -1 & k \end{bmatrix}$

Now, $|A| = -1(-3k+1) - 2(2k-1) + 4(-2+3)$

$= -k+5$

We know that, the vectors are linearly dependent if $|A| = 0$

$\Rightarrow -k+5=0$

$\therefore k = 5$

4. Find the eigen values of A=$\begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 4 \\ 3 & 1 & -5 \end{bmatrix}$ also find the eigen values of 3A.

Soln:

Since A is upper triangular matrix, the eigen values are its leading diagonal elements

∴ The eigan values of A are 2, -1 and 3 and

The eigen values of -3A are -6, 3 and -9.

5. Find the sum and product of the eigen values of the matrix. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Soln:

Sum of the eigen values = sum of the leading diagonal elements

= -2 + 1+0 = -1

Product of the eigen values

$$= \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

= -2(0-12) -2 (0-6) -3(-4+1)

= 45

6. If two eigen values of the matrix A=. $\begin{bmatrix} 1 & 0 & -4 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$ are 3 and 2. Find the third eigen value.

Soln:

We know that, the sum of the eigen values of a given matrix A is equal to its sum of the diagonal elements

∴ Sum of the eigen value of A = 6

we have the two eigen values of A are 3 and 2

⇒ The sum of two eigen values = 5

∴ The third eigen value is 1

7. Show that the $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ matrix satisfies its own characteristic equation

Soln:

Let A= $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)^2 + 4 = 0$

$\Rightarrow \lambda^2 - 2\lambda + 5 = 0$

We have to prove that $A^2-2A+5I = 0$

$A^2 = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$

$A^2-2A+5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - 2\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus the given matrix satisfies its own characteristic equation.

8. Using Cayley- Hamilton theorem find the inverse of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}$

Soln:

The characteristic equation of A is

$| A - \lambda I | = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & -5-\lambda \end{vmatrix} = 0$

$(2-\lambda)(-5-\lambda) - 1 = 0$

$\Lambda^2 + 3\lambda - 11 = 0$

By Cayley-Hamilton theorem, we have

$A^2+3A-11I = 0$

Pre- multiplying by A^{-1}, we get,

$A^{-1} A^2 + 3A^{-1}A - 11A^{-1}I = 0$

$A^{-1} = 1/11(A+3I)$

$= 1/11\left[\begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$

$= 1/11\begin{pmatrix} 5 & 1 \\ 1 & -2 \end{pmatrix}$

9. If $A = \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}$ express A^3 in terms of A and I using Cayley-Hamilton theorem.

Soln:

The characteristic equation of A is $|A-\lambda I|$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(5-\lambda) = 0$

$\lambda^2 - 6\lambda + 5I = 0$

By Cayley-Hamilton theorem, we have

$A^2 - 6A + 5 = 0$

Pre multiplying on bothsides by A, we get,

$A^3 + 6A^2 + 5A = 0$

$A^3 = 6A^2 - 5A$

$= 6(6A-5I) - 5A$

$= 31A - 30I$

$$= 31\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix} - 30\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 124 & 125 \end{bmatrix}$$

10. If 3 and 15 are two eigen values of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, what is the third eigen value?

Soln:

The sum of the eigen values = sum of the leading diagonal elements
= 8+7+3 =18

But two eigen values 3 and 15 are given.

Thus the third eigen value is 0.

11. The characteristic equation of a matrix A is $\lambda^2 - 2 = 0$. What is A^3?

Soln:

The characteristic equation of A is $\lambda^2 - 2 = 0$.

By Cayley-Hamilton theorem, $A^2 - 2 = 0$.

$A^2 A - 2A = 0$.

$A^3 = 2A$

12. Show that the matrix A= $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

Soln:

A= $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$A^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$A\,A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

Thus A is orthogonal

13. Form the matrix whose eigen values are α-5, β-5, γ-5 where α, β, γ are the eigen values of A= $\begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$.

Soln:

If the matrix A has the eigen values λ_1, $\lambda_2$and λ_3.

Then the matrix A-kI has the eigen values $\lambda_1 - k$, $\lambda_2 - k$, $\lambda_3 - k$,

Hence the required matrix is

$$A-5I = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} -6 & -2 & -3 \\ 4 & 0 & -6 \\ 7 & -8 & 4 \end{bmatrix}.$$

14. If 2 is an eigen values of A= $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.find the other two.

Soln:

Let λ_1, $\lambda_2$and λ_3 are the eigen values ofA.

Then, $\lambda_1 + \lambda_2 + \lambda_3$ = 2+1-1 =2

$\lambda_2 + \lambda_3 = 0$ → (1)

The product of the eigen values are

$$\lambda_1\,\lambda_2\,\lambda_3 = \begin{vmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

= 2(-4)+2(-2)+2(2)

$=8$

$\lambda_2 \lambda_3 = -4 \qquad \rightarrow (2)$

$\lambda_2 = -\lambda_3 \qquad \rightarrow (3)$

Substituting (3) in(2), we get,

$-\lambda_3^2 = -4$

$\Rightarrow \lambda_3 = 2$

$\therefore \lambda_2 = -2$

Thus the eigen values are 2, 2 and -2.

15. State Cayley- Hamilton theorem.

Soln:

Every square matrix satisfies its own characteristics equation.

16. Write the matrix of the quadratic form $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$.

Soln:

The matrix corresponding to the quadratic form is

$$= \begin{bmatrix} co.eff\ of\ x^2 & \frac{1}{2}co.eff\ of\ xx & \frac{1}{2}co.eff\ of\ xx \\ \frac{1}{2}co.eff\ of\ xx & co.eff\ of\ x^2 & \frac{1}{2}co.eff\ of\ xx \\ \frac{1}{2}co.eff\ of\ xx & \frac{1}{2}co.eff\ of\ xx & co.eff\ of\ x^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \frac{1}{2}(2) & -\frac{1}{2}(6) \\ \frac{1}{2}(2) & -2 & \frac{1}{2}(6) \\ \frac{1}{2}(-6) & \frac{1}{2}(6) & 4 \end{bmatrix}$$

$= \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$, which is the required matrix.

17. Write the quadratic form corresponding to the following matrix $\begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$.

Soln:

The general form of quadratic form for a given matrix is

Q.F = $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2(a_{12})x_1x_2 + 2(a_{23})x_2x_3 + 2(a_{13})x_1x_3$ ------------- (1)

Let $\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$.

Equation (1) becomes $0x_1^2 + x_2^2+3x_3^2 - 2x_1x_2 + 8x_2x_3+ 4x_1x_3,$ which is the required canonical form of given matrix.

18. Determine the nature of the following quadratic form $f(x_1,x_2,x_3.) = x_1^2+2x_2^2$.

Soln:

The matrix corresponding to the quadratic form is

$$= \begin{bmatrix} co.eff\ of\ x^2 & \frac{1}{2}co.eff\ of\ xx & \frac{1}{2}co.eff\ of\ xx \\ \frac{1}{2}co.eff\ of\ xx & co.eff\ of\ x^2 & \frac{1}{2}co.eff\ of\ xx \\ \frac{1}{2}co.eff\ of\ xx & \frac{1}{2}co.eff\ of\ xx & co.eff\ of\ x^2 \end{bmatrix}$$

$$\text{Matrix } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now $D_1= 1$ (+ve)

$D_2= \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2$ (+ve)

$D_3 = \det(A) = 0$

∴The quadratic form is positive semi definite.

19. Determine λ so that $\lambda (x^2+y^2+z^2) +2xy-2yz+2zx$ is positive definite

Soln:

The Symmetric matrix of the given quadratic form can be given as

$$A=\begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & -1 \\ 1 & -1 & \lambda \end{bmatrix}$$

$D_1=|\lambda| > 0$

$D_2= \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} = \lambda^2-1 > 0$

$D_3 = \det(A) = \lambda^3-3\lambda-2$

$= (\lambda + 1)^2 (\lambda-2) > 0$

$\lambda > 0,\ \lambda>2,\ \lambda>-1,\ \lambda>1$ (or) $\lambda > 0,\ \lambda>2, \lambda^2>1, \lambda>1.$

∴ The given quadratic form is positive definite only if $\lambda>2$.

20. Determine the nature of the following quadratic form $x_1^2+2x_2^2+3x_3^2+2x_1x_2+2x_1x_3-2x_2x_3$.

Soln:

The matrix corresponding to the quadratic form is

$$= \begin{bmatrix} co.eff\ of\ x^2 & \frac{1}{2}co.eff\ of\ xx & \frac{1}{2}co.eff\ of\ xx \\ \frac{1}{2}co.eff\ of\ xx & co.eff\ of\ x^2 & \frac{1}{2}co.eff\ of\ xx \\ \frac{1}{2}co.eff\ of\ xx & \frac{1}{2}co.eff\ of\ xx & co.eff\ of\ x^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Now, $D_1=|1| = 1 > 0$

$D_2= \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2-1 = 1 > 0$

$D_3 = \det(A) = 1(6-1) - 1(3+1) - 1(1+2)$

$= 1(5)-4-3 = -2 < 0$

$\therefore$ The quadratic form is indefinite.

PART-B

1. The eigenvectors of a 3×3 real symmetric matrix corresponding to the eigenvalues 2, 3, 6 are $[1\ \ 0\ \ -1]^T$, $[1\ \ 1\ \ 1]^T$, $[-1\ \ 2\ \ -1]^T$ respectively. Find the matrix A.
2. Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$
3. Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$
4. Using Cayley-Hamilton theorem find A^4 and A^{-1} when $A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$
5. Verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 7 & 3 \\ 2 & 6 \end{pmatrix}$ and hence find A^{-1} & A^3.
6. Verify Cayley-Hamilton theorem and hence find A^{-1} if $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

7. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ then show that $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$ using Cayley-Hamilton theorem.

8. Use Cayley-Hamilton theorem to find the value of the matrix given by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ if the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

9. Use Cayley- Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ to express

(i) $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

(ii) $A^4 - 4A^3 - 5A^2 + A + 2I$

10. Diagonalise the matrix A = $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ by means of an orthogonal transformation.

11. Reduce the matrix $\begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$ to diagonal form.

12. Reduce the given quadratic form Q to its canonical form using orthogonal transformation Q = $x^2 + 3y^2 + 3z^2 - 2yz$.

13. Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to the canonical form through an orthogonal transformation and hence show that it is positive semi definite. Also give a non zero set of values (x_1, x_2, x_3) which makes this quadratic form zero.

14. Obtain an orthogonal transformation which will transform the quadratic form

Q = $2x_1x_2 + 2x_2x_3 + 2x_3x_1$ into sum of squares.

15. Reduce the quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$ to a canonical form through an orthogonal transformation and hence find rank, index, signature, nature and also give non- zero set of values for $x_{1,} x_{2,} x_3$, that will make the quadratic form zero.